Thus the set of functions $f(x)$ satisfying condition (3) for every $x$ is a residual set in space $H_{0}$ not only for $\omega_\beta(h)$, but also for $\omega_\alpha(h)$.

The same applies to space $C$.

We note also that the supposition

$$
\lim_{\beta \to 0} \omega_\beta(h) = 0
$$

is evidently satisfied in view of (6) in the case of Theorem 1, i.e. in the case of accepting supposition (5). In the case of the contrary supposition, i.e. $\lim_{\beta \to 0} \omega_\beta(h) > 0$, space $H_{0}$ is a set of functions, all of which satisfy Lipschitz's condition $\beta$.

References


Reçu par la Rédaction le 19.1.1944

A generalization of maximal ideals method of Stone and Gelfand

by

W. Słowiński and W. Zawadowski (Warszawa)

It is well known that the ring $\mathcal{C}(T)$ of all real-valued continuous functions defined on a bicom pact Hausdorff space $T$ characterizes topologically the space $T$. More exactly, two bicom pact Hausdorff spaces $T$ and $T'$ are homeomorphic if and only if the rings $\mathcal{C}(T)$ and $\mathcal{C}(T')$ are algebraically isomorphic. This theorem is usually referred to as Stone's theorem (see [1] and also [2]).

The analysis of the proof of Stone's theorem shows that instead of considering the whole ring $\mathcal{C}(T)$ it suffices to consider some weaker classes of functions, e.g. the class $\mathcal{C}^+(T)$ of all non-negative continuous functions on $T$. The class $\mathcal{C}^+(T)$ can be considered as an abstract algebra with the usual operations of addition and multiplication. Two algebras $\mathcal{C}^+(T)$ and $\mathcal{C}^+(T')$ are algebraically isomorphic, if and only if the bicom pact Hausdorff spaces $T$ and $T'$ are homeomorphic.

On the other hand, it is known that the bicom pact Hausdorff spaces $T$ are also topologically characterized by the lattices of all open subsets of $T$.

The usual methods of proofs are similar in both characterizations: by real functions or by open subsets. This method may be called the method of maximal ideals. In both cases we consider certain abstract algebras with two operations, addition and multiplication, and we define the notion of maximal ideals. This notion is the algebraic analogue of the notion of a point in a space.

The purpose of this paper is to develop the common idea of both characterizations of bicom pact spaces. We introduce a general notion of a semi-ring with two operations, addition and multiplication, characterized by a set of simple axioms. Algebras $\mathcal{C}^+(T)$ and some lattices (in particular, Boolean algebras and lattices of open subsets which ap-
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(x) \(\mathcal{A}\) contains all characteristic functions of open sets in \(T\).

(y) \(x \in \mathcal{A}, y \in \mathcal{A}\) implies \(x + y \in \mathcal{A}\) and \(xy \in \mathcal{A}\).

(z) \(x \in \mathcal{A}, x(t) \neq 0\) for all \(t \in T\) and \(y(t) = 1/x(t)\) implies that \(y \in \mathcal{A}\).

The set \(\mathcal{A}\) with the usual operations is a semi-ring.

Let \(\mathcal{A}\) be a semi-ring. We say that the element \(x \in \mathcal{A}\) has an inverse if there exists an element \(x^{-1} \in \mathcal{A}\), called an inverse of \(x\), such that \(xx^{-1} = 1\).

We denote by \(\Omega(\mathcal{A})\), or simply by \(\Omega\), the set of all elements which have inverses. The set \(\Omega\) is not empty, as it contains the element 1. Evidently, if \(x \in \Omega\) and \(y \in \Omega\), then \(xy \in \Omega\).

In the semi-ring of all open sets of a topological space \(T\) the set \(\Omega\) consists of only one element, namely the whole space \(T\). In examples 3 and 4 the set \(\Omega\) consists of functions \(x(t)\) for which \(x(t) \neq 0\) for all \(t \in T\).

Definition 2. A semi-ring \(\mathcal{A}\) is positive, if for every element \(x \in \mathcal{A}\), we have \(1+x \in \Omega\). i.e. 1+x has an inverse.

In examples 2, 3 and 4 all semi-rings are positive.

Definition 3. A non-empty subset \(I\) of a semi-ring \(\mathcal{A}\) is an ideal if

(a) \(a \in I\) and \(b \in I\) implies \(a+b \in I\);

(b) \(a \in I\) and \(x \in \mathcal{A}\) implies \(ax \in I\);

(c) \(I \neq \emptyset\).

It is easy to see that condition (c) may be replaced by

(c') \(1 \in I\),

and that an element which has inverse cannot belong to any ideal.

Theorem 1. Let \(\mathcal{A}\) be a semi-ring. A set \(E = \{x \in \mathcal{A}\} \) is an ideal, if and only if an element \(a\) has no inverse.

Proof. If \(a\) has no inverse, the equation \(ax = 1\) is not satisfied by any \(x\). Therefore \(1 \in E\), and, being different from the whole \(\mathcal{A}\), and satisfying conditions (a) and (b), \(E\) is an ideal.

On the other hand if \(a\) has an inverse, then \(ax^{-1} = 1\). This implies that \(1 \in E\) in contradiction to (c').

Definition 4. An ideal \(I\) is maximal, if it is not a proper subset of any other ideal.

The class of all maximal ideals is denoted by \(\mathfrak{M}\). Maximal ideals are denoted by \(M\) or \(N\).

Theorem 2. Every ideal is a subset of at least one maximal ideal.

1) This is an abstract form of the axiom formulated by I. Gelfand in [1] for normed rings (Banach algebras) namely: "element \(x+1\) has an inverse".
Proof. For a given semi-ring $\mathbb{A}$ and an ideal $I\subseteq\mathbb{A}$ let $I$ be the class of all ideals contained in $\mathbb{A}$ and containing $I$. The class $I$ is partially ordered by inclusion. If $I$ is a simply ordered subset of $\mathbb{A}$, the union of all ideals contained in $I$ belongs to $\mathbb{A}$. Consequently, by Zorn's lemma there exists a maximal element $M$ in $I$. An ideal $M$ is not properly contained in any ideal, and therefore $M$ is maximal.

**COROLLARY 1.** An element $x \in \mathbb{A}$ has an inverse if and only if $x \in M$ for all $M \subseteq \mathbb{A}$.

**LEMMA.** If an ideal $M$ is maximal, then for $x \in M$ the set

$$E = \bigcap_{m \in M} (y = m + x, m \in M, y \in \mathbb{A})$$

is identical with the whole $\mathbb{A}$, and therefore for some $m_0 \in M$ and $z_0 \in \mathbb{A}$ we have $m_0 + z_0 = 1$.

Proof. In the opposite case the set $E$ would be an ideal containing $M$ properly. This is impossible, for $M$ is maximal.

**THEOREM 3.** If $M$ is a maximal ideal, then for all $x, y \in M$ $xy \in M$ is equivalent to $x \in M$ or $y \in M$.

Proof. Implication to the left is evident. Let us suppose that implication to the right is not true, i.e., that there are an ideal $M$ and two elements $x$ and $y$ with $xy \in M, x \notin M,$ and $y \notin M$. By the lemma there are elements $m_0 \in M$ and $z_0 \in \mathbb{A}$ such that $1 = m_0 + z_0 x$. We have $y = m_0 y + z_0 y$, and as the left side is a sum of two elements of $M$, it follows that $y \in M$. This contradicts our supposition.

**THEOREM 4.** A semi-ring $\mathbb{A}$ is positive if and only if for all $M \subseteq \mathbb{A}$ and $x, y \in M$

$$(1) \quad x + y \in M \iff x \in M \text{ and } y \in M.$$  

Proof. a) Suppose that $\mathbb{A}$ is positive and that there are elements $x, y \in \mathbb{A}$ and a maximal ideal $M$, with

$$x + y \in M \text{ and } x \in M.$$  

By the lemma there are elements $m_0 \in M$ and $z_0 \in \mathbb{A}$ such that $1 = m_0 + z_0 x$. The supposition that $\mathbb{A}$ is positive implies that $1 + z_0 y$ has an inverse. This is impossible for $1 + z_0 y = m_0 + z_0 y = m_0 + z_0 (x + y) \in M$.

Therefore if $\mathbb{A}$ is positive condition (1) is satisfied.

b) Now let the semi-ring $\mathbb{A}$ satisfy condition (1). It may be formulated in an equivalent form:

$$x \in M \text{ or } y \in M \implies (x + y) \in M.$$  

Assuming that $y = 1$ we have $1 + x \in M$ for all $M \subseteq \mathbb{A}$ and all $x \in \mathbb{A}$. By corollary 1 $\mathbb{A}$ is positive.

**COROLLARY 2.** If $\mathbb{A}$ is a positive semi-ring, then for $M \subseteq \mathbb{A}$ and $x, y \in M$

$$x + y \in M \text{ is equivalent to } x \in M \text{ and } y \in M.$$  

**COROLLARY 3.** In a positive semi-ring $\mathbb{A}$ element zero belongs to all the maximal ideals.

For a given maximal ideal $M$ there exists at least one element $x \in M$. Then $x + 0 = M, 0(x + 0) = 0x + 0 \in M$ and by corollary 2 $0 \in M$.

**COROLLARY 4.** In a positive semi-ring $\mathbb{A}$, for all $x, y \in \mathbb{A}$ the condition $x \in M$ implies $x + y \in M$.

**Definition 5.** The intersection of all maximal ideals of a positive semi-ring $\mathbb{A}$ we call its radical. We denote it by $\text{Rad } \mathbb{A}$.

**Definition 6.** A semi-ring $\mathbb{A}$ is said to be without radical if $\text{Rad } \mathbb{A} = \{0\}$.

**Examples.** 1. The semi-ring of all real-valued non-negative functions on an abstract set $T$ is without radical.

2. The semi-ring of all open sets of a topological $T_1$-space under usual operations is without radical.

3. In the positive semi-ring of all open sets of a $T_1$-space the radical may be different from 0. An example is the set of all real numbers with $U = \{a < x\}$ as open sets.

**Theorem 3.** Let $\mathbb{A}$ be a positive semi-ring. A necessary and sufficient condition for $\text{Rad } \mathbb{A} = \{0\}$ is that

$$(2) \quad \text{for every } x \neq 0 \text{ there exist an } y \in \mathbb{A} \text{ such that } x + y \in \mathbb{A}.$$  

Proof. a) Let $\mathbb{A}$ satisfy (2) and suppose that there is an element $x \neq 0, x \neq \text{Rad } \mathbb{A}$. By (2) there is an element $y \in \mathbb{A}$ such that $x + y \in \mathbb{A}$. By theorems 1 and 2 $y$ belongs to a certain maximal ideal $M$. At the same time $x \in M$, because otherwise $x + y \in M$, and this is impossible.

b) Suppose now that the radical contains only element 0. If $x \neq 0$, then the element $y = 0$ has the desired properties: $y \in \mathbb{A} \text{ and } x + y \in \mathbb{A}$. If $x \in \mathbb{A}$ and $x \neq 0$, then there is a maximal ideal $M$ with $x \in \mathbb{A}$. By the lemma there are elements $m_0 \in M$ and $z_0 \in \mathbb{A}$ such that $1 = m_0 + z_0 x$. If $x + m_0 \in \mathbb{A}$, there would exist a maximal ideal $N$ with $x + m_0 \in \mathbb{A}$. By theorem 3 we should have $x \in \mathbb{A}$ and $m_0 \in \mathbb{A}$, which together with $z_0 x \in \mathbb{A}$ would give $z_0 x + m_0 \in \mathbb{A}$. It follows that $x + m_0 \in \mathbb{A}$ and that element $m_0 \in \mathbb{A}$ satisfies condition (2).

**Corollary 5.** In a positive semi-ring $\mathbb{A}$ without radical $0x = 0$ for all $x \in \mathbb{A}$.

Indeed, from definition 5 and corollary 3 element 0 belongs to the radical.
Definition 7. In a positive semi-ring \( \mathcal{H} \) a set of elements \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) is a dual finite covering if \( x_1, x_2, \ldots, x_n \in \mathcal{H} \).

A set of elements \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) is a finite covering if \( x_1 + x_2 + \ldots + x_n \in \mathcal{O} \).

Examples. 1. In the case of the semi-ring \( \mathcal{H} \) of all open sets of a topological space \( T \), a finite class of sets \( \{E_1, \ldots, E_k\} \mathcal{H} \) is a dual finite covering if \[ T \setminus \bigcup_{i} (E_i - E_i) \]

The class \( \{E_1, \ldots, E_k\} \mathcal{H} \) is a finite covering if \[ T \setminus \bigcup_{i} (E_i - E_i) = \emptyset. \]

2. In the case of a semi-ring of all real-valued, non-negative functions on a set \( T \), the set of functions \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) is a finite covering if \[ T F_1 + F_2 + \ldots + F_k, \]

with \( F_i = \{ x_i | T, x_i(\emptyset) = \emptyset \}. \)

The set of functions \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) is a finite covering if \[ T C E_1 + E_2 + \ldots + E_k, \]

where \( E_i = \{ x_i | T, x_i(\emptyset) = \emptyset \}. \)

Theorem 6. In a positive semi-ring \( \mathcal{H} \) a set of elements \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) is a dual finite covering if and only if for every maximal ideal \( M \) at least one of the elements \( x_1, x_2, \ldots, x_n \) belongs to \( M \).

(b) A finite covering if and only if there is no maximal ideal \( M \) such that \( x_i \in M \) for \( i = 1, 2, \ldots, k \).

Proof. (a) By Corollary 3 the condition \[ x_1, x_2, \ldots, x_n \in \mathcal{H} \]

ensures that for every \( M \in \mathcal{H} \) we have \( x_1, x_2, \ldots, x_n \in M \), which by Theorem 3 is equivalent to the fact that for every \( M \) at least one of the elements \( x_i \) \( i = 1, 2, \ldots, k \) belongs to \( M \).

(b) By Corollary 2 the condition \[ x_1 + x_2 + \ldots + x_n \in \mathcal{O} \]

is equivalent to the fact that for every \( M \in \mathcal{H} \), \( x_1 + x_2 + \ldots + x_n \not\in M \).

Definition 8. A set of elements \( \{x_1, x_2, \ldots, x_n\} \mathcal{H} \) belonging to a certain set of indices \( A \), is a covering in a positive semi-ring \( \mathcal{H} \) if for every maximal ideal \( M \in \mathcal{H} \) there exists \( x_a \in A \) with \( x_a \not\in M \).
is possible by (a). Let us take a set $A$ containing $x_0$ and all elements $x$ which are taken from dual finite coverings corresponding to maximal ideals $N$. The set $A$ is a covering. Indeed, for every maximal ideal $M$, either $x_0 \in M$ or there exists a dual finite covering $(x,y)$ corresponding to $M$, such that $x \notin M$. By theorem 7, $A$ contains a finite covering, say $A_2$. By theorem 3 and because of $ax \in \text{Rad} \mathfrak{A}$, for all $x \neq x_0$ in $A$. Therefore the covering $A_2$ must contain element $x_0$, and can be expressed as

$$A_0 = \{ax_0, a_1, \ldots, a_k\}$$

We put

$$a = a_1 + a_2 + \ldots + a_k, \quad b = y_1 + y_2 + \ldots + y_k$$

where $(y_1, y_2, \ldots, y_k)$, $i = 1, 2, \ldots, k$, is a dual finite covering corresponding to maximal ideal $N$. By theorem 3 condition (a), $y_1, y_2, \ldots, y_k \in M_1$. Since $a_1 y_1, a_2 y_2, \ldots, a_k y_k$ are dual finite covering $(a,b)$ satisfies condition (b).

II. (b) implies (c). Let us take the covering $(x,y)$. If $x$ or $y$ has an inverse, condition (c) is satisfied by the elements 0 and 1. Suppose now that $x$ and $y$ have no inverses. Let $N$ be a maximal ideal containing $y$; $x + y \in \Omega$ implies that $x \notin N$. Making use of condition (b) we associate with every ideal $N$ containing $y$ a dual finite covering $(a,b)$ in such a way that $x + a \in \Omega$ and $b \notin N$. A set $A$ which consists of $y$ and all elements $b$ from dual finite coverings corresponding to maximal ideals $N$ is a covering. By theorem 7, $A$ contains a finite covering $A_2$.

As $x$ has no inverse, there exists a maximal ideal $M$ containing it. Since $x + a \in \Omega$, no element of $A$ belongs to $M$. On the other hand $ab \in \text{Rad} \mathfrak{A}$, and by theorem 3 every element $b$ belongs to $M$. This implies that the covering $A_0$ must contain the element $y$, and can be expressed as

$$A_0 = \{y_1, y_2, y_3, \ldots, y_k\}$$

We put

$$a_1 y_1 + a_2 y_2 + \ldots + a_k y_k, \quad a = a_1 a_2 \ldots a_k$$

where $(a_1, a_2, \ldots, a_k)$, $i = 1, 2, \ldots, k$, is a dual finite covering corresponding to $N_1$.

We must show that $x + u \in \Omega$. If it were not so, there would exist a maximal ideal $M'$ containing $x + u$. By corollary 2 $x \in M'$, $u \in M'$ and by theorem 3 for some $i$, $a_i \in M'$, and this implies that $x + a_i \in M'$, in spite of $x + a_i \in \Omega$.

III. (c) implies (a). Let us take two maximal ideals $M \neq N$. The set

$$F = \{z = x + y, x \in M, y \in N\}$$

is a whole $\mathfrak{A}$. Therefore there exist elements $x_0 \in M$ and $y_0 \in N$ with $x_0 + y_0 = 1$. By condition (c) there exist $u$ and $v$ such that

$$x_0 + u \in \Omega, \quad y_0 + v \in \Omega \quad \text{and} \quad u \in \text{Rad} \mathfrak{A}.$$

Since $x_0 \in M$ and $y_0 \in N$, we have $u \in M$ and $v \in N$. Therefore the dual finite covering $(u,v)$ satisfies condition (a).

Definition 9. A positive semi-ring which satisfies condition (c) of theorem 8 we call a normal semi-ring.

Definition 10. (a). A dual finite covering $K$ is conjugate to a covering $A$ if for every element $x$ in $K$ there exists an element $y$ in $A$ such that $x + y \in \Omega$.

(b) The order of a dual finite covering $K$ is the largest integer $n$ for which there exists a subset $(x_1, x_2, \ldots, x_n) \subset K$ which is not a covering.

(c). The order of a covering $A$ is the largest integer $n$ for which there exists a subset $(a_1, a_2, \ldots, a_n) \subset A$, which is not a dual finite covering.

Definition 11. The algebraic dimension of a positive semi-ring $\mathfrak{A}$ is the least number $n$ for which to every covering there corresponds a conjugate dual finite covering with the order $\approx n + 1$. We denote it by $\text{dim} \mathfrak{A} = n$.

Let $\mathfrak{A}$ be a semi-ring. We define a certain relation of equivalence in $\mathfrak{A}$. We write $x \equiv y$, if and only if for every maximal ideal $M \in \mathfrak{A}$

$$x \in M \quad \text{is equivalent to} \quad y \in M.$$

It is easy to see that relation $\equiv$ is reflexive, symmetric and transitive. The abstraction classes are denoted by $[x], [y], z, y \in \mathfrak{A}$, and the sets of these classes by $[\mathfrak{A}]$. If we define, in a natural way, operations $\cdot$ and $+$ in $[\mathfrak{A}]$ as

$$[x] + [y] = [x + y], \quad [x][y] = [xy],$$

then $[\mathfrak{A}]$ is a semi-ring. Addition and multiplication in $[\mathfrak{A}]$ are idempotent, i.e.

$$[x] + [x] = [x] \quad \text{and} \quad [x][x] = [x].$$

If $\mathfrak{A}$ is positive, then so is $[\mathfrak{A}]$.

Example. Let $\mathfrak{A}$ be the semi-ring of real-valued, non-negative, continuous functions $x(t)$ on a certain bicompact Hausdorff space $T$. It is easy to verify that in this case all maximal ideals $M \in \mathfrak{A}$ have the form

$$M = \{x \in \mathfrak{A}, x(t_0) = 0\} \quad t_0 \in T.$$
§ 2. Topology in the space $\mathcal{M}$ of maximal ideals

Let $\mathcal{M}$ be a positive semi-ring. We introduce topology in the set $\mathcal{M}$ of maximal ideals of $\mathcal{M}$ in the following way. As a basis of open sets in $\mathcal{M}$ we take all the sets of the form

$$\Gamma_{x}^{+} = \{ y \in \mathcal{M} \mid x + y \in \mathcal{M} \}.$$  

Dually the sets

$$\Lambda_{x}^{-} = \{ y \in \mathcal{M} \mid x - y \in \mathcal{M} \}$$

form a basis of closed sets in the same topology.

It is easy to see that $\mathcal{M}$ is a topological $\mathcal{T}_{1}$-space and satisfies four axioms of Kuratowski. By theorem 1, $\mathcal{M}$ is bicomplete.

We have seen that it is possible to identify the set $\mathcal{M}$ of all maximal ideals in $\mathcal{M}$ and such a set for $[\mathcal{M}]$ by a one-to-one correspondence

$$\mathcal{M} \rightarrow [\mathcal{M}].$$

It is a direct consequence of the definition of neighbourhoods that $[\mathcal{M}]$ yields in $\mathcal{M}$ the same topology as $\mathcal{M}$.

The $\Gamma$-sets have the following properties:

(a) $\Gamma_{x}^{+} \cap \Gamma_{y}^{+} = \Gamma_{x+y}^{+}$,

(b) $\Gamma_{x}^{+} \cap \Gamma_{y}^{-} = \Gamma_{x+y}^{-}$,

(c) $\Gamma_{x}^{+} = \mathcal{M}$ if and only if $x \in \mathcal{M}$,

(d) $\Gamma_{x}^{-} = \mathcal{M}$ if and only if $x \in \mathcal{M}$.

The first three properties can be derived from theorem 3, the last from theorem 5.

Dually, for $\Lambda$-sets, we have

(a') $\Lambda_{x}^{-} = \Lambda_{y}^{-} \cap \Lambda_{x+y}^{-}$,

(b') $\Lambda_{x}^{-} = \Lambda_{x+y}^{-} \cap \Lambda_{y}^{-}$,

(c') $\Lambda_{x}^{-} = \mathcal{M}$ if and only if $x \in \mathcal{M}$,

(d') $\Lambda_{x}^{+} = \mathcal{M}$ if and only if $x \in \mathcal{M}$.

Theorem 13. In the sense of topology defined, as above, in $\mathcal{M}$, for every maximal ideal $\mathcal{M}$ and each of the sets $\mathcal{E} \in \mathcal{M}$ we have

$$\mathcal{M} \in \mathcal{E}$$

if and only if \( \bigcap \mathcal{N} \in \mathcal{M} \).

Proof. $\mathcal{M} \in \mathcal{E}$ is equivalent to: $\mathcal{M} \in \Gamma_{x}^{+}$ implies $\mathcal{M} \in \mathcal{E}$. The latter statement is equivalent to: for every $x \in \mathcal{M}$ there exists a maximal ideal $\mathcal{N} \in \mathcal{E}$ such that $x \in \mathcal{N}$. This is further equivalent to:

$$x \in \bigcap \mathcal{N} \in \mathcal{M}$$

implies $x \in \mathcal{M}$.
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In terms of $\Delta$-sets $x \in M$ passes into $M \in \Delta_x$, $x \in M$ passes into $M \in \Delta_x$.

With this in mind we see that theorems 3 and 4 as well as corollary 4 characterize trivial properties of union and intersection of sets in terms of positive semi-rings.

**Theorem 15.** The space $\mathcal{R}$ is normal if and only if the semi-ring $\mathcal{R}$ is normal.

**Proof.** (a) Let $\mathcal{R}$ be a normal space. For every two distinct points $M \in \mathcal{R}, N \in \mathcal{R}, M \neq N$ there exist disjoint neighbourhoods $\Gamma_M$ and $\Gamma_N$ such that $M \in \Gamma_M$ and $N \in \Gamma_N$. In terms of semi-rings it means that there are elements $x, y \in \mathcal{R}$ such that $xy = 0$ and $x \in M, y \in N$. By theorem 8 $\mathcal{R}$ is thus normal.

(b) If $\mathcal{R}$ is normal, then, translating for example condition (a) of theorem 8 into the language of $\Gamma$-sets in $\mathcal{R}$, we see that $\mathcal{R}$ is a Hausdorff space. Being bicom pact and a Hausdorff space, $\mathcal{R}$ is a normal space.

**Theorem 16.** The space $\mathcal{R}$ has the (Lebesgue) dimension $n$ if and only if the corresponding positive semi-ring has the algebraic dimension $n$.

**Proof.** It is a direct consequence of definitions 7 and 11.

As an example we deduce the following representation theorem of M. H. Stone:

Every Boolean algebra is isomorphic with a class of all open-closed sets of a certain topological bicom pact space which is totally disconnected.

**Proof.** Every Boolean algebra $\mathcal{B}$ is a reduced semi-ring. This follows at once from theorem 12.

Since for given $x$ and $y$ with $x + y = 1$ the condition (c) of theorem 8 is satisfied by the elements $u = 1 - x$ and $v = 1 - y$, it follows that $\mathcal{B}$ is a normal semi-ring. Therefore by theorem 15 $\mathcal{B}$ is a Hausdorff space.

It may also be directly deduced from the fact that in this case a complement of a $\Gamma$-set is a $\Gamma$-set

$\mathcal{R} - \Gamma = \Gamma_{1 - \mathcal{R}}$.

It suffices to demonstrate that every open-closed set may be expressed as a $\Gamma$-set. This follows from the fact that in bicom pact Hausdorff spaces every open basis, closed under formation of finite unions, contains every open-closed set.

As another example we prove the following

**Theorem 17.** Every distributive lattice $\mathcal{H}$ with 0 and 1 such that for every $x, y \in \mathcal{H}$ with $x + 1 = 1$, there exist $u, v \in \mathcal{H}$ with $u + v = 0$, such that

$x + u = 1$ and $y + v = 1$,

is isomorphic with a basis of open sets for a certain bicom pact Hausdorff space.

**Proof.** This is an immediate consequence of definition 8 and theorems 12, 14 and 15.
§ 3. $\Gamma$-homomorphisms

Given two semi-rings $\mathcal{A}$ and $\mathcal{A}'$ we say that $\mathcal{A}'$ is a $\Gamma$-homomorphic image of $\mathcal{A}$ if and only if $\mathcal{A}'$ is a homomorphic image of $\mathcal{A}$ in the usual algebraical sense and if the inverse image of the set $\Omega'$ is identical with $\Omega$. In other words, the mapping $F$ of $\mathcal{A}$ onto $\mathcal{A}'$ is a $\Gamma$-homomorphism if and only if

(a) $F(x+y) = F(x) + F(y)$,
(b) $F(xy) = F(x)F(y)$,
(c) $F^{-1}(\Omega') \supset \Omega$.

If $\mathcal{A}'$ is a $\Gamma$-homomorphic image of $\mathcal{A}$, we write

$\mathcal{A} \xrightarrow{\Gamma} \mathcal{A}'$.

Conditions (a) and (b) imply that

$F^{-1}(\Omega') \supset \Omega$,

so that from (c) we have

$F^{-1}(\Omega') = \Omega$.

The importance of $\Gamma$-homomorphisms is due to the fact that the property of having an inverse is invariant under this type of mappings. The structure of the sets of elements which have no inverses of two $\Gamma$-homomorphic semi-rings is for topological purposes the same.

Examples. 1. For positive semi-rings the natural mapping $F(x) = [x]$ is a $\Gamma$-homomorphism of $\mathcal{A}$ onto $[\mathcal{A}]$. In particular:

2. Every positive semi-ring $\mathcal{A}'$ is $\Gamma$-homomorphic with a basis of open sets $I_*, x \subseteq \mathcal{A}$ in the space of its maximal ideals $F(x) = I_x$.

3. The semi-ring $C^*(T)$ of all real-valued, non-negative continuous functions on a certain topological bicompact Hausdorff space $T$ with the usual operations of addition and multiplication is $\Gamma$-homomorphic with the semi-ring $\gamma$ of all open sets in $T$ if we put for $x \in C^*(T)$

$F(x) = \int x(t) > 0$, \quad t \in T.

4. The semi-ring $\mathcal{R}$ (see example 4 to definition 1) generated by characteristic functions of open sets in $T$ is $\Gamma$-homomorphic with the semi-ring of open sets in $T$ if we put, as before,

$F(x) = \int x(t) > 0$, \quad t \in T.$

Theorem 18. Let $\mathcal{A}$ be a positive semi-ring, $T$ a bicompact topological $T_1$-space, $\mathcal{A}'$ a semi-ring of some open sets in $T$. If $\mathcal{A}$ is $\Gamma$-homomorphic with $\mathcal{A}'$, then every maximal ideal $M$ in $\mathcal{A}$ has the form

$M = F^{-1}(\{ t \neq F(x) \})$;

here $F$ is a $\Gamma$-homomorphism of $\mathcal{A}$ onto $\mathcal{A}'$, and $t$ a certain point of $T$, chosen for $M$.

Proof. If there is a maximal ideal $M$ not having the form (7), then for every point $t \in T$ there exists an element $x \in M$ such that $t \neq F(x)$. It means that the class of open sets $F(M)$ is a covering of $T$.

Since $T$ is bicompact, there are elements $x_1, x_2, ..., x_n$ in $\mathcal{A}$ such that

$T = F(x_1) + F(x_2) + ... + F(x_n) = F(x_1 + x_2 + ... + x_n)$,

$T$ as identity of $\mathcal{A}'$ has an inverse, and thus

$x_1 + x_2 + ... + x_n \in M$.

In spite of $x_1 + x_2 + ... + x_n \in M$.

If to different points $t \neq t'$ correspond different sets (7), then we have a one-to-one correspondence between the sets $\mathcal{A}$ and $T$. It is so if $\mathcal{A}'$ is an open basis in $T$.

It is true that if every maximal ideal in $\mathcal{A}$ has the form (7), then the topology which is determined in $\mathcal{A}$ by taking $\mathcal{A}'$ as an open basis is bicompact. We prove a somewhat more general

Theorem 19. Let $\mathcal{A}$ and $\mathcal{A}'$ be two positive semi-rings, $\mathcal{R}$ and $\mathcal{R}'$ the corresponding spaces of maximal ideals. Then

$\mathcal{A} \xrightarrow{\Gamma} \mathcal{A}'$ implies $\mathcal{R} \cong \mathcal{R}'$

i.e. a $\Gamma$-homomorphism of positive semi-rings implies a homeomorphism of the corresponding spaces of maximal ideals.

Proof. Let us assume that $F$ is a $\Gamma$-homomorphism such that $F(\mathcal{R}) = \mathcal{R}'$.

If $M' \subseteq \mathcal{R}'$ then $F^{-1}(M')$ is a maximal ideal in $\mathcal{A}$. We define the mapping $f$ of $\mathcal{R}'$ in $\mathcal{R}$ as follows:

$f(M') = F^{-1}(M')$.

We have to show that $f$ maps $\mathcal{R}'$ onto $\mathcal{R}$, that it is one-to-one and bicompact.

a) Let $M = F^{-1}(M')$, i.e. $M = \bigcap_{x \in M} F(x) \subseteq M'$. Suppose there is an $M \in \mathcal{R}$ which is not an inverse image of any $M' \subseteq \mathcal{R}'$. Then, for every
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It must only be noted that isomorphism of the semi-rings of all non-negative functions induces an isomorphism between rings of all continuous functions.

2. Assuming in corollary 7 that $\mathcal{H}$ and $\mathcal{W}$ are open bases in $T$ and $T'$, we obtain another well known theorem:

A necessary and sufficient condition for two topological $T_1$-spaces to be homeomorphic is the existence of bases of open sets in these spaces which are algebraically isomorphic (considered as lattices).

References


Institute Mathematyczny Polskiej Akademii Nauk
Mathematical Institute of the Polish Academy of Sciences

Received for publication 19.5.1954