Thus the set of functions $f(x)$ satisfying condition (3) for every x is a residual set in space H_ω not only for $\omega_1^*(h)$, but also for $\omega_1(h)$.

The same applies to space C.

We note also that the supposition

$$\lim_{h\to+0}\frac{h}{\omega(h)}=0$$

is evidently satisfied in view of (6) in the case of Theorem 1, *i. e.* in the case of accepting supposition (5). In the case of the contrary supposition, *i. e.* $\lim_{h\to+0}\Lambda(h)>0$, space H_ω is a set of functions, all of which satisfy Lipschitz's condition [6]).

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A generalization of maximal ideals method of Stone and Gelfand

by

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It is well known that the ring $C(T)$ of all real-valued continuous functions defined on a bicompact Hausdorff space T characterizes topologically the space T. More exactly, two bicompact Hausdorff spaces T and T' are homeomorphic if and only if the rings $C(T)$ and $C(T')$ are algebraically isomorphic. This theorem is usually referred to as Stone's theorem (see [4] and also [2]).

The analysis of the proof of Stone's theorem shows that instead of considering the whole ring $C(T)$ it suffices to consider some weaker classes of functions, *e. g.* the class $C^+(T)$ of all non-negative continuous functions on T. The class $C^+(T)$ can be considered as an abstract algebra with the usual operations of addition and multiplication. Two algebras $C^+(T)$ and $C^+(T')$ are algebraically isomorphic, if and only if the bicompact Hausdorff spaces T and T' are homeomorphic.

On the other hand, it is known that the bicompact Hausdorff spaces T are also topologically characterized by the lattices off all open subsets of T [1]).

The usual methods of proofs are similar in both characterizations: by real functions or by open subsets. This method may be called the method of maximal ideals. In both cases we consider certain abstract algebras with two operations, addition and multiplication, and we define the notion of maximal ideals. This notion is the algebraic analogue of the notion of a point in a space.

The purpose of this paper *) is to develop the common idea of both characterizations of bicompact spaces. We introduce a general notion of a semi-ring with two operations, addition and multiplication, characterized by a set of simple axioms. Algebras $C^+(T)$ and some lattices (in particular, Boolean algebras and lattices of open subsets which ap-

*) Presented to the Mathematical Institute of the Polish Academy of Sciences Group of Topology, in October 1952.

[1]) This is valid for more general spaces. See [5] and [6].

pear in the characterization theorems) are semi-rings. Then we develop the theory of maximal ideals in an important class of semi-rings called in this paper positive semi-rings. Specifying the kind of semi-rings, we obtain the above mentioned theorems on characterization of bicompact Hausdorff spaces by means of functions or of open subsets. We obtain also two representation theorems for certain kinds of distributive lattices and Stone's representation theorem for Boolean algebras.

The developed theory enables us also to translate easily all topological notions and theorems about bicompact T_1-spaces into the language of semi-rings.

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§ 1. Semi-rings

Definition 1. A *semi-ring* is a set $\mathfrak{A}$ of elements which is closed under two binary operations, addition $+$ and multiplication $\cdot$, with the following properties:

(a) both the addition and the multiplication are associative and commutative;

(b) the addition is distributive under the multiplication: $a(x+y) = ax+by$ for $a, x, y \in \mathfrak{A}$;

(c) there exist two distinct elements 0 and 1 in $\mathfrak{A}$ such that for every $x \in \mathfrak{A}$ we have

$$x+0=x, \qquad 1x=x\ ^{2)}.$$

Examples. 1. The ring of all real valued functions on an abstract set T with the usual operations of addition and multiplication is a semi-ring. The same is true of every commutative ring with identity.

2. Every distributive lattice containing maximal and minimal elements is a semi-ring. In particular the class of all open sets in a certain topological space T with $+$ as union and $\cdot$ as intersection is a semi-ring. Dually the class of all closed sets with $+$ as intersection and $\cdot$ as union is a semi-ring.

3. The set $C^+(T)$, of all real-valued non-negative continuous functions on a topological space T with the usual operations is a semi-ring.

4. The set of all real-valued, non-negative functions on an abstract set T with $+$ and $\cdot$ defined as usual is a semi-ring.

5. Let T be a topological space. We denote by $\mathfrak{R}$ the least set of real non-negative functions $x(t)$ on T with the following properties:

(α) $\mathfrak{R}$ contains all characteristic functions of open sets in T.

(β) $x \in \mathfrak{R}$, $y \in \mathfrak{R}$ implies $x+y \in \mathfrak{R}$ and $xy \in \mathfrak{R}$.

(γ) $x \in \mathfrak{R}$, $x(t) \neq 0$ for all $t \in T$ and $y(t) = 1/x(t)$ implies that $y \in \mathfrak{R}$.

The set $\mathfrak{R}$ with the usual operations is a semi-ring.

Let $\mathfrak{A}$ be a semi-ring. We say that the element $x \in \mathfrak{A}$ *has an inverse* if there exists an element $x^{-1} \in \mathfrak{A}$, called an *inverse* of x, such that $xx^{-1}=1$. We denote by $\Omega(\mathfrak{A})$, or simply by Ω, the set of all elements which have inverses. The set Ω is not empty, as it contains the element 1. Evidently, if $x \in \Omega$ and $y \in \Omega$, then $xy \in \Omega$.

In the semi-ring of all open subsets of a topological space T the set Ω consists of only one element, namely the whole space T. In examples 3 and 4 the set Ω consists of functions $x(t)$ for which $x(t) \neq 0$ for all $t \in T$.

Definition 2. A *semi-ring* $\mathfrak{A}$ is *positive*, if for every element $x \in \mathfrak{A}$, we have $1+x \in \Omega$ *i. e.* $1+x$ has an inverse [3]).

In examples 2, 3 and 4 all semi-rings are positive.

Definition 3. A non-empty subset I of a semi-ring $\mathfrak{A}$ is an *ideal* if

(a) $a \in I$ and $b \in I$ implies $a+b \in I$;

(b) $a \in I$ and $x \in \mathfrak{A}$ implies $ax \in I$;

(c) $I \neq \mathfrak{A}$.

It is easy to see that condition (c) may be replaced by

(c′) $1 \notin I$,

and that an element which has inverse cannot belong to any ideal.

THEOREM 1. *Let $\mathfrak{A}$ be a semi-ring. A set $E = \underset{ax}{E}\,(x \in \mathfrak{A})$ is an ideal, if and only if an element a has no inverse.*

Proof. If a has no inverse, the equation $ax=1$ is not satisfied by any x. Therefore $1 \notin E$, and, being different from the whole $\mathfrak{A}$, and satisfying conditions (a) and (b), E is an ideal.

On the other hand if a has an inverse, then $aa^{-1}=1$. This implies that $1 \in E$ in contradiction to (c′).

Definition 4. An *ideal* is *maximal*, if it is not a proper subset of any other ideal.

The class of all maximal ideals is denoted by $\mathfrak{M}$. Maximal ideals are denoted by M or N.

THEOREM 2. *Every ideal is a subset of at least one maximal ideal.*

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[2]) Evidently 0 and 1 are the only elements with these properties.

[3]) This is an abstract form of the axiom formulated by I. Gelfand in [1] for normed rings (Banach algebras) namely: "element x^2+1 has an inverse".

Proof. For a given semi-ring $\mathfrak{A}$ and an ideal $I \subset \mathfrak{A}$ let Σ be the class of all ideals contained in $\mathfrak{A}$ and containing I. The class Σ is partially ordered by inclusion. If Σ_0 is a simply ordered subset of Σ, the union of all ideals contained in Σ_0 belongs to Σ. Consequently, by Zorn's lemma there exists a maximal element M in Σ. An ideal M is not properly contained in any ideal, and therefore M is maximal.

COROLLARY 1. *An element $x \in \mathfrak{A}$ has an inverse if and only if $x \notin M$ for all $M \in \mathfrak{M}$.*

LEMMA. *If an ideal M is maximal, then for $x \notin M$ the set*

$$E = \underset{y}{E}\,(y = m + zx,\ m \in M,\ z \in \mathfrak{A})$$

is identical with the whole $\mathfrak{A}$, and therefore for some $m_0 \in M$ and $z_0 \in \mathfrak{A}$ we have $m_0 + z_0 x = 1$.

Proof. In the oppositive case the set E would be an ideal containing M properly. This is impossible, for M is maximal.

THEOREM 3. *If M is a maximal ideal, then for all $x, y \in \mathfrak{A}$*

$$xy \in M \quad \text{is equivalent to} \quad x \in M \quad \text{or} \quad y \in M.$$

Proof. Implication to the left is evident. Let us suppose that implication to the right is not true, *i. e.* that there are an ideal M and two elements x and y with $xy \in M$, $x \notin M$, and $y \notin M$. By the lemma there are elements $m_0 \in M$ and $z_0 \in \mathfrak{A}$ such that $1 = m_0 + z_0 x$. We have $y = m_0 y + z_0 xy$, and as the left side is a sum of two elements of M, it follows that $y \in M$. This contradicts our supposition.

THEOREM 4. *A semi-ring $\mathfrak{A}$ is positive if and only if for all $M \in \mathfrak{M}$ and $x, y \in \mathfrak{A}$*

(1) $$x + y \in M \quad \text{implies} \quad x \in M \ \text{and}\ y \in M.$$

Proof. a) Suppose that $\mathfrak{A}$ is positive and that there are elements $x, y \in \mathfrak{A}$ and a maximal ideal M, with

$$x + y \in M \quad \text{and} \quad x \notin M.$$

By the lemma there are elements $m_0 \in M$ and $z_0 \in \mathfrak{A}$ such that $1 = m_0 + z_0 x$. The supposition that $\mathfrak{A}$ is positive implies that $1 + z_0 y$ has an inverse. This is impossible for $1 + z_0 y = m_0 + z_0 x + z_0 y = m_0 + z_0(x + y) \in M$.

Therefore if $\mathfrak{A}$ is positive condition (1) is satisfied.

b) Now let the semi-ring $\mathfrak{A}$ satisfy condition (1). It may be formulated in an equivalent form:

$$x \notin M \quad \text{or} \quad y \notin M \quad \text{implies} \quad (x + y) \notin M.$$

Assuming that $y = 1$ we have $1 + x \notin M$ for all $M \in \mathfrak{M}$ and all $x \in \mathfrak{A}$. By corollary 1 $\mathfrak{A}$ is positive.

COROLLARY 2. *If $\mathfrak{A}$ is a positive semi-ring, then for $M \in \mathfrak{M}$ and $x, y \in \mathfrak{A}$*

$$x + y \in M \quad \text{is equivalent to} \quad x \in M \ \text{and}\ y \in M.$$

COROLLARY 3. *In a positive semi-ring $\mathfrak{A}$ element zero belongs to all the maximal ideals.*

For a given maximal ideal M there exists at least one element $x \in M$. Then $x + 0 \in M$, $0(x + 0) = 0x + 0 \in M$ and by corollary 2 $0 \in M$.

COROLLARY 4. *In a positive semi-ring $\mathfrak{A}$, for all $x, y \in \mathfrak{A}$ the condition $x \in \Omega$ implies $x + y \in \Omega$.*

Definition 5. The intersection of all maximal ideals of a positive semi-ring $\mathfrak{A}$ we call its *radical*. We denote it by $\mathrm{Rad}\,\mathfrak{A}$.

Definition 6. A *semi-ring* $\mathfrak{A}$ is said to be *without radical* if $\mathrm{Rad}\,\mathfrak{A} = \{0\}$.

Examples. 1. The semi-ring of all real-valued non-negative functions on an abstract set T is without radical.

2. The semi-ring of all open sets of a topological T_1-space under usual operations is without radical.

3. In the positive semi-ring of all open sets of a T_0-space the radical may be different from 0. An example is the set of all real numbers with $U_a = \underset{x}{E}\,(a < x)$ as open sets.

THEOREM 5. *Let $\mathfrak{A}$ be a positive semi-ring. A necessary and sufficient condition for* $\mathrm{Rad}\,\mathfrak{A} = \{0\}$ *is that*

(2) *for every $x \neq 0$ there exist an $y \notin \Omega$ such that $x + y \in \Omega$.*

Proof. a) Let $\mathfrak{A}$ satisfy (2) and suppose that there is an element $x \neq 0$, $x \in \mathrm{Rad}\,\mathfrak{A}$. By (2) there is an element $y \notin \Omega$ such that $x + y \in \Omega$. By theorems 1 and 2 y belongs to a certain maximal ideal M. At the same time $x \notin M$, because otherwise $x + y \in M$, and this is impossible.

b) Suppose now that the radical contains only element 0. If $x \in \Omega$, then the element $y = 0$ has the desired properties: $y \notin \Omega$ and $x + y \in \Omega$. If $x \notin \Omega$ and $x \neq 0$, then there is a maximal ideal M with $x \in M$. By the lemma there are $m_0 \in M$ and $z_0 \in \mathfrak{A}$ such that $1 = m_0 + z_0 x$. If $x + m_0 \in \Omega$, there would exist a maximal ideal N with $x + m_0 \in N$. By theorem 3 we should have $x \in N$ and $m_0 \in N$, which together with $z_0 x \in N$ would give $z_0 x + m_0 \in N$. It follows that $x + m_0 \in \Omega$ and that element m_0 satisfies condition (2).

COROLLARY 5. *In a positive semi-ring $\mathfrak{A}$ without radical $0x = 0$ for all $x \in \mathfrak{A}$.*

Indeed, from definition 5 and corollary 3 element $0x$ belongs to the radical.

Definition 7. In a positive semi-ring $\mathfrak{A}$ a set of elements $\{x_1,x_2,\dots,x_k\}\subset\mathfrak{A}$ is a *dual finite covering* if $x_1x_2\dots x_k\in\operatorname{Rad}\mathfrak{A}$.

A set of elements $\{x_1,x_2,\dots,x_k\}\subset\mathfrak{A}$ is a *finite covering* if $x_1+x_2+\dots+x_k\in\Omega$.

Examples. 1. In the case of the semi-ring $\mathfrak{A}$ of all open sets of a topological space T, a finite class of sets $\{E_1,\dots,E_k\}\subset\mathfrak{A}$ is a dual finite covering if

$$T\subset(T-E_1)+(T-E_2)+\dots+(T-E_k).$$

The class $\{E_1,\dots,E_k\}$ is a finite covering if

$$T\subset E_1+E_2+\dots+E_k.$$

2. In the case of a semi-ring of all real-valued, non-negative functions on a set T, the set of functions $\{x_1,x_2,\dots,x_k\}$ is a dual finite covering if

$$T\subset F_1+F_2+\dots+F_k,$$

with $F_i=\underset{t}{E}\big(t\in T,\ x_i(t)=0\big)$.

The set of functions $\{x_1,x_2,\dots,x_k\}$ is a finite covering if

$$T\subset E_1+E_2+\dots+E_k,$$

where $E_i=\underset{t}{E}\big(t\in T,\ x_i(t)>0\big)$.

Theorem 6. *In a positive semi-ring $\mathfrak{A}$ a set of elements $\{x_1,x_2,\dots,x_k\}$ is*

(a) *a dual finite covering if and only if for every maximal ideal M at least one of the elements $x_1,x_2,\dots,x_k$ belongs to M;*

(b) *a finite covering if and only if there is no maximal ideal M such that $x_i\in M$ for $i=1,2,\dots,k$.*

Proof. (a) By corollary 3 the condition

$$x_1x_2\dots x_k\in\operatorname{Rad}\mathfrak{A}$$

ensures that for every $M\in\mathfrak{M}$ we have $x_1x_2\dots x_k\in M$, which by theorem 3 is equivalent to the fact that for every M at least one of the elements x_i $(i=1,2,\dots,k)$ belongs to M.

(b) By corollary 2 the condition

$$x_1+x_2+\dots+x_k\in\Omega$$

is equivalent to the fact that for every $M\in\mathfrak{M}$ $x_1+x_2+\dots+x_k\notin M$. By corollary 2 there exists i_0, $1\leqslant i_0\leqslant k$, with $x_{i_0}\notin M$.

Definition 8. A set of elements $\{x_\lambda\}\subset\mathfrak{A}$, λ belonging to a certain set of indices Λ, is a *covering* in a positive semi-ring $\mathfrak{A}$ if for every maximal ideal $M\in\mathfrak{M}$ there exists $\lambda_0\in\Lambda$ with $x_{\lambda_0}\notin M$.

By theorem 6 for finite sets Λ this definition agrees with definition 7 of finite covering.

Theorem 7. *In a positive semi-ring every covering contains a finite covering.*

Proof. Let $\{x_\lambda\}$, $\lambda\in\Lambda$, be a covering in $\mathfrak{A}$. The set of elements of the form

$$z=y_1x_{\lambda_1}+y_2x_{\lambda_2}+\dots+y_kx_{\lambda_k}$$

with $x_{\lambda_i}\in\{x_\lambda\}$, $y_i\in\mathfrak{A}$, $i=1,2,\dots,k$, is identical with the whole $\mathfrak{A}$. Otherwise, as an ideal, it would be contained in a certain maximal ideal M_0. This is impossible because the ideal M_0 would contain all x_λ, $\lambda\in\Lambda$, and the set $\{x_\lambda\}$ would not be a covering. Therefore there exist elements $a_1,a_2,\dots,a_k\in\mathfrak{A}$ and $x_{\lambda_1},x_{\lambda_2},\dots,x_{\lambda_k}\in\{x_\lambda\}$ such that

$$a_1x_{\lambda_1}+a_2x_{\lambda_2}+\dots+a_kx_{\lambda_k}=1.$$

By theorem 3 and corollary 3

$$x_{\lambda_1}+x_{\lambda_2}+\dots+x_{\lambda_k}\in\Omega,$$

and thus $\{x_{\lambda_1},x_{\lambda_2},\dots,x_{\lambda_k}\}$ is a finite covering.

Theorem 8. *In a positive semi-ring $\mathfrak{A}$ the following three conditions are equivalent:*

(a) *for every two maximal ideals M and N there exists a dual finite covering $\{x,y\}$ with $x\notin M$ and $y\notin N$;*

(b) *for every maximal ideal M and every element $x\in M$, there exists a dual finite covering $\{a,b\}$, such that $\{x,a\}$ is a covering and $b\notin M$;*

(c) *for every covering $\{x,y\}$ there exists a dual finite covering $\{u,v\}$ such that $\{x,u\}$ and $\{y,v\}$ are coverings.*

If $\mathfrak{A}$ is a lattice of all open sets for a certain topological space T, this theorem has a simple topological interpretation. As we shall see later, the maximal ideals in $\mathfrak{A}$ may be taken, in this case, as elements of the space T. The theorem states here that for bicompact topological spaces T the following conditions are equivalent:

(a) *T is a Hausdorff space.*

(b) *T is a regular space.*

(c) *T is a normal space.*

Proof. We are going to prove three implications:

I. (a) implies (b). Let M_0 be a maximal ideal, and $x_0\notin M_0$. If $x_0\notin N$ for all $N\in\mathfrak{M}$ we put $a=0$, $b=1$ and (b) is true. If $x_0\in N$ for some $N\in\mathfrak{M}$, then $N\neq M_0$. With every ideal $N\in\mathfrak{M}$, such that $x_0\in N$ we may associate a dual finite covering $\{x,y\}$ in such a way that $x\notin N$ and $y\notin M_0$. This is possible by (a). Let us take a set A containing x_0 and all elements x which are taken from dual finite coverings corresponding to maximal ideals N. The set A is a covering. Indeed, for every maximal ideal M, either $x_0 \notin M$ or there exists a dual finite covering $\{x,y\}$ corresponding to M, such that $x \notin M$. By theorem 7 A contains a finite covering, say A_0. By theorem 3 and because of $xy \in \mathrm{Rad}\,\mathfrak{A} \subset M_0$ and $y \notin M_0$ we deduce that $x \in M_0$ for all $x \neq x_0$ in A. Therefore the covering A_0 must contain element x_0, and can be expressed as

$$A_0 = \{x_0, x_1, \ldots, x_k\}.$$

We put

$$a = x_1 + x_2 + \ldots + x_k, \qquad b = y_1 y_2 \ldots y_k$$

where $\{x_i y_i\}$, $i = 1, 2, \ldots, k$, is a dual finite covering corresponding to maximal ideal N_i. By theorem 3 and condition (a) $y_1 y_2 \ldots y_k \notin M_0$. Since $x_i y_i \in \mathrm{Rad}\,\mathfrak{A}$ then $ab \in \mathrm{Rad}\,\mathfrak{A}$. The dual finite covering $\{a,b\}$ satisfies condition (b).

II. (b) implies (c). Let us take the covering $\{x,y\}$. If x or y has an inverse, condition (c) is satisfied by the elements 0 and 1. Suppose now that x and y have no inverses. Let N be a maximal ideal containing y; $x + y \in \Omega$ implies that $x \notin N$. Making use of condition (b) we associate with every ideal N containing y a dual finite covering $\{a,b\}$ in such a way that $x + a \in \Omega$ and $b \notin N$. A set A which consists of y and all elements b from dual finite coverings corresponding to maximal ideals N is a covering. By theorem 7 A contains a finite covering A_0.

As x has no inverse, there exists a maximal ideal M containing it. Since $x + a \in \Omega$, none of the elements a may belong to M. On the other hand $ab \in \mathrm{Rad}\,\mathfrak{A} \subset M$, and by theorem 3 every element b must belong to M. This implies that the covering A_0 must contain the element y, and can be expressed as

$$A_0 = \{y, b_1, b_2, \ldots, b_k\}.$$

We put

$$v = b_1 + b_2 + \ldots + b_k, \qquad u = a_1 a_2 \ldots a_k$$

where $\{a_i, b_i\}$, $i = 1, 2, \ldots, k$, is a dual finite covering corresponding to N_i.

We must show that $x + u \in \Omega$. If it were not so, there would exist a maximal ideal M' containing $x + u$. By corollary 2 $x \in M'$, $u \in M'$ and by theorem 3 for some i_0 $a_{i_0} \in M'$, and this implies that $x + a_{i_0} \in M'$, in spite of $x + a_{i_0} \in \Omega$.

III. (c) implies (a). Let us take two maximal ideals $M \neq N$. The set

$$\mathop{E}_{z}(z = x + y,\ x \in M,\ y \in N)$$

is a whole $\mathfrak{A}$. Therefore there exist elements $x_0 \in M$ and $y_0 \in N$ with $x_0 + y_0 = 1$. By condition (c) there exist u and v such that

$$x_0 + u \in \Omega, \qquad y_0 + v \in \Omega \qquad \text{and} \qquad uv \in \mathrm{Rad}\,\mathfrak{A}.$$

Since $x_0 \in M$ and $y_0 \in N$, we have $u \notin M$ and $v \notin N$. Therefore the dual finite covering $\{u,v\}$ satisfies condition (a).

Definition 9. A positive semi-ring which satisfies condition (c) of theorem 8 we call a *normal semi-ring*.

Definition 10. (a). A *dual finite covering* K is *conjugate* to a covering A if for every element x in K there exists an element y in A such that $x + y \in \Omega$.

(b) *The order of a dual finite covering* K is the largest integer n for which there exists a subset $\{x_1, x_2, \ldots, x_k\} \subset K$ which is not a covering.

(c) *The order of a covering* A is the largest integer n for which there exists a subset $\{a_1, a_2, \ldots, a_n\} \subset A$, which is not a dual finite covering.

Definition 11. The *algebraic dimension of a positive semi-ring* $\mathfrak{A}$ is the least number n for which to every covering there corresponds a conjugate dual finite covering with the order $\leqslant n+1$. We denote it by $\dim \mathfrak{A} = n$.

Let $\mathfrak{A}$ be a semi-ring. We define a certain relation of equivalence in $\mathfrak{A}$. We write

$$x \equiv y,$$

if and only if for every maximal ideal $M \in \mathfrak{M}$

$$x \in M \quad \text{is equivalent to} \quad y \in M.$$

It is easy to see that relation $\equiv$ is reflexive, symmetric and transitive. The abstraction classes are denoted by $[x], [y]$, $x, y \in \mathfrak{A}$, and the set of these classes by $[\mathfrak{A}]$. If we define, in a natural way, operations $+$ and $\cdot$ in $[\mathfrak{A}]$ as

$$[x] + [y] = [x + y], \qquad [x][y] = [xy],$$

then $[\mathfrak{A}]$ is a semi-ring. Addition and multiplication in $[\mathfrak{A}]$ are idempotent, *i. e.*

$$[x] + [x] = [x] \qquad \text{and} \qquad [x][x] = [x].$$

If $\mathfrak{A}$ is positive, then so is $[\mathfrak{A}]$.

Example. Let $\mathfrak{A}$ be the semi-ring of real-valued, non-negative, continuous functions $x(t)$ on a certain bicompact Hausdorff space T. It is easy to verify that in this case all maximal ideals M in $\mathfrak{A}$ have the form

$$M = \mathop{E}_{x}(x \in \mathfrak{A},\ x(t_0) = 0), \qquad t_0 \in T.$$

In this case $[\mathfrak{A}]$ may be identified with a lattice of all open sets in T by a one-to-one correspondence

$$[x] \to \underset{t}{E}\left(t \in T,\ x(t) > 0\right).$$

By corollary 2 and definition 5 we see that

$$[1] = \Omega, \qquad [0] = \operatorname{Rad} \mathfrak{A} \qquad \text{(for positive } \mathfrak{A}).$$

Remark. If $\mathfrak{A}$ is positive, then every maximal ideal in $[\mathfrak{A}]$ may be expressed as $[M]$ where $M \in \mathfrak{M}$ is uniquely determined, so that we can identify the set $\mathfrak{M}([\mathfrak{A}])$ with the set $\mathfrak{M}(\mathfrak{A})$ of maximal ideals in $\mathfrak{A}$.

THEOREM 9. *If $\mathfrak{A}$ is a positive semi-ring, then $[\mathfrak{A}]$ is without radical.*

Proof. This is a trivial conclusion from definition 6.

THEOREM 10. *In a positive semi-ring $\mathfrak{A}$ $x \equiv y$ is equivalent to $xy \equiv x+y$.*

Proof. By theorem 3 and corollary 4 $x \in M$, $y \notin M$ is equivalent to $x+y \notin M$, $xy \in M$ for all $M \in \mathfrak{M}$.

THEOREM 11. *$x \equiv y$ is in a positive semi-ring $\mathfrak{A}$ equivalent to:*

$$[z+x+y] = [1] \quad \textit{implies} \quad [z+xy] = [1] \quad \textit{for every} \quad z \in \mathfrak{A}.$$

Proof. (a) Let $M \in \mathfrak{M}$, $x \in M$ and $y \notin M$. This implies that $x+y \notin M$, and by the lemma there exist elements $a \in \mathfrak{A}$ and $m \in M$, such that $a(x+y)+m = 1$. This equality implies that $x+y+m \in \Omega$, which together with $xy \in M$ and $m \in M$ gives $m+xy \in M$ and consequently $m+xy \notin \Omega$. This completes the proof of the necessity of the condition.

(b) Now let $m \in \mathfrak{A}$ be such that $m+x+y \in \Omega$ and $m+xy \notin \Omega$. Then there exists a maximal ideal M such that $m+xy \in M$. Consequently $m \in M$ and $xy \in M$, or in other words, $x \in M$ or $y \in M$. On the other hand $m+x+y \in \Omega$ and therefore x and y cannot at the same time belong to M. Thus $x \not\equiv y$ and this proves the sufficiency of the condition.

Definition 12. A *semi-ring* is *reduced* if it is positive and if $x \equiv y$ is equivalent to $x = y$.

COROLLARY 6. *A positive semi-ring $\mathfrak{A}$ is reduced, if and only if, for every two elements $x, y \in \mathfrak{A}$ the relation $x = y$ is equivalent to:*

$$z+x+y = 1 \quad \textit{implies} \quad z+xy = 1 \quad \textit{for all} \quad z \in \mathfrak{A}.$$

The above considerations may be formulated in the following

THEOREM 12. *Every reduced semi-ring $\mathfrak{A}$ is a distributive lattice, with 0 and 1, such that for every $x, y \in \mathfrak{A}$ $x = y$ if and only if*

$$z+x+y = 1 \quad \textit{implies} \quad z+xy = 1 \quad \textit{for every} \quad z \in \mathfrak{A}.$$

Every distributive lattice with 0 and 1 which satisfies this condition is a reduced semi-ring.

§ 2. Topology in the space $\mathfrak{M}$ of maximal ideals

Let $\mathfrak{A}$ be a positive semi-ring. We introduce topology in the set $\mathfrak{M}$ of maximal ideals of $\mathfrak{A}$ in the following way. As a basis of open sets in $\mathfrak{M}$ we take all the sets of the form

$$\Gamma_x = \underset{M \in \mathfrak{M}}{E} (x \notin M).$$

Dually the sets

$$\Delta_x = \underset{M \in \mathfrak{M}}{E} (x \in M)$$

form a basis of closed sets in the same topology.

It is easy to see that $\mathfrak{M}$ is a topological T_1-space and satisfies four axioms of Kuratowski. By theorem 7, $\mathfrak{M}$ is bicompact.

We have seen that it is possible to identify the set $\mathfrak{M}$ of all maximal ideals in $\mathfrak{A}$ and such a set for $[\mathfrak{A}]$ by a one-to-one correspondence

$$M \leftrightarrow [M].$$

It is a direct consequence of the definition of neighbourhoods that $[\mathfrak{A}]$ yields in $\mathfrak{M}$ the same topology as $\mathfrak{A}$.

The Γ-sets have the following properties:

(a) $\Gamma_{xy} = \Gamma_x \Gamma_y$,

(b) $\Gamma_{x+y} = \Gamma_x + \Gamma_y$,

(c) $\Gamma_x = \mathfrak{M}$ *if and only if* $x \in \Omega$,

(d) $\Gamma_x = 0$ *if and only if* $x \in \operatorname{Rad} \mathfrak{A}$.

The first three properties can be derived from theorem 3, the last from theorem 5.

Dually, for Δ-sets, we have

(a′) $\Delta_{xy} = \Delta_x + \Delta_y$,

(b′) $\Delta_{x+y} = \Delta_x \Delta_y$,

(c′) $\Delta_x = 0$ *if and only if* $x \in \Omega$,

(d′) $\Delta_x = \mathfrak{M}$ *if and only if* $x \in \operatorname{Rad} \mathfrak{A}$.

THEOREM 13. *In the sense of topology defined, as above, in $\mathfrak{M}$, for every maximal ideal M and each of the sets $E \subset \mathfrak{M}$ we have*

$$M \in \bar{E} \quad \textit{if and only if} \quad \prod_{N \in E} N \subset M.$$

Proof. $M \in \bar{E}$ is equivalent to: $M \in \Gamma_x$ implies $\Gamma_x E = 0$. The latter statement is equivalent to: for every $x \notin M$ there exists a maximal ideal $N \in E$ such that $x \notin N$. This is further equivalent to:

$$x \in \prod_{N \in E} N \quad \text{implies} \quad x \in M.$$

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THEOREM 14 [4]). *A reduced semi-ring $\mathfrak{A}$ is isomorphic with a semi-ring of sets which form an open basis for a bicompact topological T_1-space, namely the space $\mathfrak{M}$ of its maximal ideals.*

Proof. To every element $x \in \mathfrak{A}$ corresponds a set $\Gamma_x \subset \mathfrak{M}$. The correspondence is one-to-one because $\mathfrak{A}$ is reduced. Moreover, it preserves algebraical operations.

COROLLARY 7. *Every positive semi-ring may be homomorphically mapped on a semi-ring of sets, which form an open basis for some bicompact T_1-space.*

By a homomorphism we mean here a mapping preserving algebraical operations. In fact the mapping

$$x \to \Gamma_x$$

is something more than a homomorphism (see § 3). It is such that the element mapped on the whole space has an inverse.

We are now able to give a topological interpretation in the space $\mathfrak{M}$ of some notions introduced in § 1. We have the following correspondences:

(3) $$x_1 + x_2 + \ldots + x_k \in \Omega$$

if and only if

$$\mathfrak{M} \subset \Gamma_{x_1} + \Gamma_{x_2} + \ldots + \Gamma_{x_k},$$

i. e. $\{x_1, x_2, \ldots, x_k\}$ *is a covering in* $\mathfrak{A}$, *if and only if* $\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_k}$ *is an open covering of* $\mathfrak{M}$.

(4) $$x_1 x_2 \ldots x_k \in \operatorname{Rad} \mathfrak{A}$$

if and only if

$$\mathfrak{M} \subset \Delta_{x_1} + \Delta_{x_2} + \ldots + \Delta_{x_k},$$

i. e. $\{x_1, x_2, \ldots, x_k\}$ *is a dual finite covering in* $\mathfrak{A}$ *if and only if* $\Delta_{x_1}, \Delta_{x_2}, \ldots, \Delta_{x_k}$ *is a closed covering of* $\mathfrak{M}$.

(5) *The order of a covering* $\{x_1, x_2, \ldots, x_k\}$ *is* n *if and only if the order of the open covering*

$$\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_k}$$

is n.

(6) *The order of a dual finite covering* $\{x_1, x_2, \ldots, x_k\}$ *is* n *if and only if the order of the closed covering*

$$\Delta_{x_1}, \Delta_{x_2}, \ldots, \Delta_{x_k}$$

is n.

By passing from $\mathfrak{A}$ *to* $\mathfrak{M}$ *the relation* $x \in M$ *passes into* $M \notin \Gamma_x$ *and the relation* $x \notin M$ *passes into* $M \in \Gamma_x$.

[4]) By theorem 12 this theorem states that every distributive lattice with 0 and 1 such that for $x, y \in \mathfrak{A}$ $x \neq y$ implies that there exists $z \in \mathfrak{A}$ such that $xyz = 0$ and $(x+y)z = 0$ is isomorphic with the basis of closed sets for a certain bicompact T_1-space; it was obtained by H. Wallman in [6].

In terms of Δ-sets $x \in M$ *passes into* $M \in \Delta_x$, $x \notin M$ *passes into* $M \notin \Delta_x$.

With this in mind we see that theorems 3 and 4 as well as corollary 4 characterize trivial properties of union and intersection of sets in terms of positive semi-rings.

THEOREM 15. *The space* $\mathfrak{M}$ *is normal if and only if the semi-ring* $\mathfrak{A}$ *is normal.*

Proof. (a) Let $\mathfrak{M}$ be a normal space. For every two distinct points $M \in \mathfrak{M}$, $N \in \mathfrak{M}$, $M \neq N$ there exist disjoint neighbourhoods Γ_x and Γ_y such that $M \in \Gamma_x$ and $N \in \Gamma_x$. In terms of semi-rings it means that there are elements $x, y \in \mathfrak{A}$ such that $xy = 0$ and $x \notin M$, $y \notin N$. By theorem 8 $\mathfrak{A}$ is thus normal.

(b) If $\mathfrak{A}$ is normal, then, translating for example condition (a) of theorem 8 into the language of Γ-sets in $\mathfrak{M}$, we see that $\mathfrak{M}$ is a Hausdorff space. Being bicompact and a Hausdorff space, $\mathfrak{M}$ is a normal space.

THEOREM 16. *The space* $\mathfrak{M}$ *has the (Lebesgue) dimension* n *if and only if the corresponding positive semi-ring has the algebraic dimension* n.

Proof. It is a direct consequence of definitions 7 and 11.

As an example we deduce the following representation theorem of M. H. Stone:

Every Boolean algebra is isomorphic with a class of all open-closed sets of a certain topological bicompact space which is totally disconnected.

Proof. Every Boolean algebra $\mathfrak{B}$ is a reduced semi-ring. This follows at once from theorem 12.

Since for given x and y with $x+y=1$ the condition (c) of theorem 8 is satisfied by the elements $u=1-x$ and $v=1-x$, it follows that $\mathfrak{B}$ is a normal semi-ring. Therefore by theorem 15 $\mathfrak{M}$ is a Hausdorff space.

It may also be directly deduced from the fact that in this case a complement of a Γ-set is a Γ-set

$$\mathfrak{M} - \Gamma_x = \Gamma_{1-x}.$$

It suffices to demonstrate that every open-closed set may be expressed as a Γ-set. This follows from the fact that in bicompact Hausdorff spaces every open basis, closed under formation of finite unions, contains every open-closed set.

As another example we prove the following

THEOREM 17. *Every distributive lattice* $\mathfrak{A}$ *with* 0 *and* 1 *such that for every* $x, y \in \mathfrak{A}$ *with* $x+y=1$ *there exist* $u, v \in \mathfrak{A}$ *with* $uv=0$, *such that*

$$x+u=1 \quad \text{and} \quad y+v=1;$$

is isomorphic with a basis of open sets for a certain bicompact Hausdorff space.

Proof. This is an immediate consequence of definition 8 and theorems 12, 14 and 15.

§ 3. Γ-homomorphisms

Given two semi-rings $\mathfrak{A}$ and $\mathfrak{A}'$ we say that $\mathfrak{A}'$ is a *Γ-homomorphic image* of $\mathfrak{A}$ if and only if $\mathfrak{A}'$ is a homomorphic image of $\mathfrak{A}$ in the usual algebraical sense and if the inverse image of the set Ω' is identical with Ω. In other words, the mapping F of $\mathfrak{A}$ onto $\mathfrak{A}'$ is a *Γ-homomorphism* if and only if

(a) $F(x+y)=F(x)+F(y)$,

(b) $F(xy)=F(x)F(y)$,

(c) $F^{-1}(\Omega')\subset\Omega$.

If $\mathfrak{A}'$ is a Γ-homomorphic image of $\mathfrak{A}$, we write

$$\mathfrak{A}\xrightarrow{\Gamma}\mathfrak{A}'.$$

Conditions (a) and (b) imply that

$$F^{-1}(\Omega')\supset\Omega,$$

so that from (c) we have

$$F^{-1}(\Omega')=\Omega.$$

The importance of Γ-homomorphisms is due to the fact that the property of having an inverse is invariant under this type of mappings. The structure of the sets of elements which have no inverses of two Γ-homomorphic semi-rings is for topological purposes the same.

Examples. 1. For positive semi-rings the natural mapping $F(x)=[x]$ is a Γ-homomorphism of $\mathfrak{A}$ onto $[\mathfrak{A}]$. In particular:

2. Every positive semi-ring $\mathfrak{A}$ is Γ-homomorphic with a basis of open sets $\Gamma_x, x\in\mathfrak{A}$ in the space of its maximal ideals $F(x)=\Gamma_x$.

3. The semi-ring $C^+(T)$ of all real-valued, non-negative continuous functions on a certain topological bicompact Hausdorff space T with the usual operations of addition and multiplication is Γ-homomorphic with the semi-ring γ of all open sets in T if we put for $x\in C^+(T)$

$$F(x)=\mathop{E}_{t}\big(x(t)>0\big),\qquad t\in T.$$

4. The semi-ring $\mathfrak{R}$ (see example 4 to definition 1) generated by characteristic functions of open sets in T is Γ-homomorphic with the semi-ring of open sets in T if we put, as before,

$$F(x)=\mathop{E}_{t}\big(x(t)>0\big),\qquad t\in T.$$

THEOREM 18. *Let $\mathfrak{A}$ be a positive semi-ring, T a bicompact topological T_1-space, $\mathfrak{A}'$ a semi-ring of some open sets in T. If $\mathfrak{A}$ is Γ-homomorphic with $\mathfrak{A}'$, then every maximal ideal M in $\mathfrak{A}$ has the form*

$$M=\mathop{E}_{x\in\mathfrak{A}}\big(t\notin F(x)\big);\tag{7}$$

here F is a Γ-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$, and t a certain point of T, chosen for M.

Proof. If there is a maximal ideal M not having the form (7), then for every point $t\in T$ there exists an element $x\in M$ such that $t\notin F(x)$. It means that the class of open sets $F(M)$ is a covering of T.

Since T is bicompact, there are elements $x_1,x_2,\dots,x_k$ in $\mathfrak{A}$ such that

$$T=F(x_1)+F(x_2)+\dots+F(x_k)=F(x_1+x_2+\dots+x_k).$$

T as identity of $\mathfrak{A}'$ has an inverse, and thus

$$x_1+x_2+\dots+x_k\in\Omega,$$

in spite of $x_1+x_2+\dots+x_k\in M$.

If to different points $t\neq t'$ correspond different sets (7), then we have a one-to-one correspondence between the sets $\mathfrak{M}$ and T. It is so if $\mathfrak{A}'$ is an open basis in T.

It is true that if every maximal ideal in $\mathfrak{A}$ has the form (7), then the topology which is determined in T by taking $\mathfrak{A}'$ as an open basis is bicompact. We prove a somewhat more general

THEOREM 19. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two positive semi-rings, $\mathfrak{M}$ and $\mathfrak{M}'$ the corresponding spaces of maximal ideals. Then*

$$\mathfrak{A}\xrightarrow{\Gamma}\mathfrak{A}'\quad\textit{implies}\quad\mathfrak{M}\overset{\text{top}}{=\!=}\mathfrak{M}'$$

i. e. a Γ-homomorphism of positive semi-rings implies a homeomorphism of the corresponding spaces of maximal ideals.

Proof. Let us assume that F is a Γ-homomorphism such that $F(\mathfrak{A})=\mathfrak{A}'$.

If $M'\in\mathfrak{M}'$ then $F^{-1}(M')$ is a maximal ideal in $\mathfrak{A}$. We define the mapping f of $\mathfrak{M}'$ in $\mathfrak{M}$ as follows:

$$f(M')=F^{-1}(M').$$

We have to show that f maps $\mathfrak{M}'$ onto $\mathfrak{M}$, that it is one-to-one and bicontinuous.

a) Let $M=F^{-1}(M')$, i. e. $M=\mathop{E}_{x\in\mathfrak{A}}\big(F(x)\in M'\big)$. Suppose there is an $M\in\mathfrak{M}$ which is not an inverse image of any $M'\in\mathfrak{M}'$. Then, for every

$M' \in \mathfrak{M}'$, there exists $x \in M$ such that $F(x) \notin M'$. Therefore the set $F(M) \subset \mathfrak{A}'$ is a covering. By theorem 7 there are elements $x_1, x_2, \ldots, x_k \in M$ such that

$$F(x_1)+F(x_2)+\ldots+F(x_k)=F(x_1+x_2+\ldots+x_k).$$

Therefore $x_1+x_2+\ldots+x_k \in \Omega$ in spite of $x_1+x_2+\ldots+x_k \in M$.

(b) Suppose that $M' \neq N'$, $M' \in \mathfrak{M}'$, $N' \in \mathfrak{M}'$ and $F^{-1}(M')=F^{-1}(N')=M$. There are x', y', $x' \in M'$, $y' \notin M'$, $x' \notin N'$, $y' \in N'$. Since

$$x'+y' \in M' \quad \text{and} \quad x'+y' \notin N',$$

it follows that

$$F^{-1}(x'+y') \notin M,$$

in spite of

$$F^{-1}(x')+F^{-1}(y') \in M.$$

c) From the identities

$$\mathop{E}_{M}(x \notin M)=f\Big(\mathop{E}_{M'}\big(F(x) \notin M'\big)\Big),$$

$$\mathop{E}_{M'}\big(F(x) \in M'\big)=f^{-1}\Big(\mathop{E}_{M}(x \notin M)\Big)$$

we see that f maps the open basis of $\mathfrak{M}'$ onto the open basis of $\mathfrak{M}$ and therefore f is a homeomorphism.

COROLLARY 8. *Let T and T' be bicompact topological T_1-spaces, $\mathfrak{A}$ and $\mathfrak{A}'$ positive semi-rings, F and F' two Γ-homomorphisms defined on $\mathfrak{A}$ and $\mathfrak{A}'$, such that $F(\mathfrak{A})$ and $F'(\mathfrak{A}')$ are open bases in T and T'.*

Under these assumptions

$$\mathfrak{A} \xrightarrow{\Gamma} \mathfrak{A}' \quad \textit{implies} \quad T \overset{\text{top}}{=} T'.$$

Examples. 1. If T is a bicompact topological Hausdorff space and $\mathfrak{A}$ a semi-ring of all real-valued non-negative continuous functions on T, then the sets

$$\mathop{E}_{t \in T}\big(x(t)>0\big), \quad x \in \mathfrak{A},$$

form an open basis γ in T. γ is a semi-ring and the mapping

$$F(x)=\mathop{E}_{t \in T}\big(x(t)>0\big)$$

is a Γ-homomorphism between $\mathfrak{A}$ and γ.

From corollary 7 we obtain in this case the well known theorem of M. H. Stone:

The necessary and sufficient condition for two topological Hausdorff spaces to be homeomorphic is that the rings of real-valued continuous functions on these spaces be algebraically isomorphic.

It must only be noted that isomorphism of the semi-rings of all non-negative functions induces an isomorphism between rings of all continuous functions.

2. Assuming in corollary 7 that $\mathfrak{A}$ and $\mathfrak{A}'$ are open bases in T and T' we obtain another well known theorem:

A necessary and sufficient condition for two topological T_1-spaces to be homeomorphic is the existence of bases of open sets in these spaces which are algebraically isomorphic (considered as lattices).

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