# On the spaces of functions satisfying Hölder's condition

by

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We shall always suppose that functions, which we shall denote by $f(x)$, are continuous, periodic with period $l$, finite and defined for every value of the real variable $x$.

In this paper we shall denote by $\omega(h)$, $\omega_1(h)$ functions defined and differing from zero for $h>0$, monotonic, non-decreasing and tending to zero for $h\to 0$ [1]).

By $H_\omega$ we shall understand the space of all functions $f(x)$ satisfying Hölder's generalized condition, *i. e.* the inequality

$$(1)\qquad |f(x+h)-f(x)|\leqslant\omega(h)$$

for every $x$ and $h$. We shall always suppose here that

$$\overline{\lim_{h\to+0}}\frac{h}{\omega(h)}<\infty\ ^{2)}.$$

The distance $d$ between two elements of this space we define by

$$(2)\qquad d(f_1,f_2)=\max_{0\leqslant x<l}|f_1(x)-f_2(x)|.$$

Space $H_\omega$ is a complete space.

Let $R$ denote a set of functions $f(x)$ belonging to space $H_\omega$ and satisfying for every $x$ the condition

$$(3)\qquad \overline{\lim_{h\to 0}}\frac{|f(x+h)-f(x)|}{\omega_1(|h|)}=\infty.$$

Under the assumption

$$\lim_{h\to+0}\frac{h}{\omega(h)}=0$$

we shall show that set $R$ is either empty or residual in space $H_\omega$, in

[1]) We note that the theorems in this paper also be true if we suppose that $\omega(h) = \text{const}>0$. (Concerning functions $f(x)$, we shall also in this case always suppose that they are continuous and periodic.)

[2]) Otherwise only constant functions would belong to space $H_\omega$.

other words, that its complementary set in this space is a set of the first category.[3])

The theorem remains valid also in the case where $H_\omega$ is substituted by space $C$ of all functions $f(x)$ continuous and periodic with period $l$[4]).

LEMMA 1. *Let $\varphi(u)$ be a continuous function, periodic with period $l$ but non-constant, defined for every $u$.*

*For a function $\varphi(u)$ thus defined we can always determine two numbers $r, s$ satisfying the inequality $0 < r \leqslant s \leqslant l$ with the following property:*

*For every value of $u$ there exists a number $h_u$, satisfying the inequality $r \leqslant |h_u| \leqslant s$, such that*

$$|\varphi(u+h_u)-\varphi(u)| \geqslant D/2,$$

*where $D$ is the oscillation of $\varphi(u)$ in the interval $0 \leqslant u < l$*[5]).

LEMMA 2. *Given a function $f(x)$ belonging to space $H_\omega$, we can find a sequence $\{f_n(x)\}$ of functions also belonging to space $H_\omega$, satisfying Lipschitz's condition and tending uniformly to $f(x)$.*

Proof. For the proof it is sufficient to take

$$f_n(x) = n\int_x^{x+1/n} f(u)\,du. \tag{4}$$

Functions $f_n(x)$ are periodic with the same period as $f(x)$ and their sequence tends uniformly to $f(x)$. Since, in addition, we have the inequalities

$$|f_n'(x)| = n|f(x+1/n)-f(x)| \leqslant A,$$

where $A$ is constant for a fixed $n$, and

$$|f_n(x+h)-f_n(x)| \leqslant n\int_0^{1/n} |f(x+h+u)-f(x+u)|\,du \leqslant \omega(|h|),$$

therefore the functions $f_n(x)$ satisfy Lipschitz's condition and also belong to $H_\omega$.

We shall consider the functional space $H_\omega$, previously defined, which, as can easily be shown, is a complete space, and the set $R$ of functions $f(x)$ belonging to space $H_\omega$ and satisfying condition (3); as regards $\omega_1(h)$ we shall temporarily suppose that

$$\overline{\lim_{h\to+0}} \frac{h}{\omega_1(h)} < \infty. \tag{5}$$

[3]) The set $R$ is empty if $\omega_1(h)$ does not satisfy condition (6). This results from a theorem of W. Orlicz [3].

[4]) This is Theorem 1 of H. Aeurbach and S. Banach [1].

[5]) The proof of the Lemma can be found in my paper [4].



THEOREM 1. *If, under assumption (5), $\omega_1(h)$ satisfies the condition*

$$\lim_{h\to+0} \frac{\omega_1(h)}{h} A(h) = 0, \tag{6}$$

*where*

$$A(h) = \sup_{0<t\leqslant h} \frac{t}{\omega(t)},$$

*then $R$ is a set residual in space $H_\omega$.*

Proof. Let us denote by $Z_n$ a set of functions $f(x)$ of space $H_\omega$ satisfying the inequality

$$|f(x+h)-f(x)| \leqslant n\omega_1(|h|) \tag{7}$$

for a certain $x$ with every $h$ ($n$ natural). The complementary set $R_n$ of set $Z_n$ in space $H_\omega$ is a set of functions $f(x)$ belonging to $H_\omega$ and satisfying the inequality

$$|f(x+h)-f(x)| > n\,\omega_1(|h|)$$

for every $x$ with a certain value of $h$.

In view of the fact that $\omega_1(h)$ is non-decreasing and $f(x)$ is bounded, we have

$$R = \prod_{n=1}^{\infty} R_n.$$

In order to prove the theorem it is therefore sufficient to prove that each of the sets $Z_n$ is non-dense in space $H_\omega$.

Let us suppose, for the proof, that $Z_{n_0}$ is not non-dense in space $H_\omega$. Since each of the sets $Z_n$ is obviously closed in space $H_\omega$, therefore there would exist in space $H_\omega$ a sphere $K_{\varrho_0}(f_0)$ of centre $f_0(x)$ and radius $\varrho_0$ belonging entirely to $Z_{n_0}$.

On the basis of Lemma 2, there exists in space $H_\omega$ a sequence of functions $\{f_n(x)\}$ tending uniformly to $f_0(x)$, whose terms satisfy Lipschitz's condition. We can therefore find in space $H_\omega$ a sphere $K_{\varrho_1}(y_1)$, of centre $y_1(x)$ and radius $\varrho_1$ such that $K_{\varrho_1}(y_1) \subset K_{\varrho_0}(f_0) \subset Z_{n_0}$, with $y_1(x) = \theta f_N(x)$, where $0<\theta<1$, and $f_N(x)$ is a sufficiently distant term on the sequence $\{f_n(x)\}$. Function $y_1(x)$ therefore satisfies Lipschitz's condition and belongs to $H_\omega$.

From (5) it follows that, with a certain $h_0$ and a certain constant $A$, the inequality $h/\omega_1(h) \leqslant A$ will be true for $0<h<h_0$. Hence, for every $x$ and every $h$ satisfying $0<|h|<h_0$ we shall have

$$|y_1(x+h)-y_1(x)| / \omega_1(|h|) \leqslant C, \tag{8}$$

where $C = k_1 A$ and $k_1$ is Lipschitz's constant of the function $y_1(x)$.

Let $\varphi(x)$ be a non-constant function, periodic with period $l$, satisfying Lipschitz's condition with Lipschitz's constant $k$. By $D$ we shall denote the oscillation of function $\varphi(x)$ in the interval $0 \leqslant x < l$.

Let number $a$ be so large as to satisfy the inequality

(9) $$aD/2 > C + n_0$$

and $h_0$ so small as to satisfy simultaneously, besides (8), the inequalities

(10) $$h_0 < l/2,$$

(11) $$kla \frac{\omega_1(h_0)}{h_0} \Lambda(h_0) < \frac{1}{2}(1-\theta),$$

(12) $$a\omega_1(h_0) \cdot \max_{0 \leqslant x < l} |\varphi(x)| < \varrho_1,$$

which, in view of (6), is possible.

Moreover, let us write

(13) $$b = [l/h_0] + 1$$

and

$$y_2(x) = a\omega_1(l/b)\varphi(bx).$$

We shall prove that

$$f^*(x) = y_1(x) + y_2(x)$$

belongs to $Z_{n_0}$.

In view of (13) and (10) we shall have $l/b < h_0 < 2l/b$, and hence inequality (11) becomes

$$kla \frac{\omega_1(lb^{-1})}{lb^{-1}} \Lambda(lb^{-1}) < 1 - \theta.$$

From the latter we obtain the inequality

(14) $$\frac{|y_2(x+h) - y_2(x)|}{\omega(|h|)} = a\omega_1\left(\frac{l}{b}\right) \frac{|\varphi(bx+bh) - \varphi(bx)|}{bh} \cdot \frac{bh}{\omega(|h|)} < 1 - \theta$$

valid for $0 < bh \leqslant l$ and for every $x$. Inequality (14) remains valid for the case $bh > l$, in view of the periodicy of $\varphi(x)$ and monotony of $\omega(h)$. It is also valid for $h < 0$, which we verify substituting in it $x-h$ for $x$.

From (14), in view of $y_1(x) = \theta f_N(x)$ and $f_N(x) \in H_\omega$, it follows that $f^*(x) \in H_\omega$. Taking into account (12) we have further

$$d(f^*, y_1) = \max_{0 \leqslant x < l} |y_2(x)| < \varrho_1,$$

from which it follows that $f^*(x) \in Z_{n_0}$.



Thus, from the supposition that set $Z_{n_0}$ is non-dense in the space $H_\omega$, it would follow that $f^*(x) \in Z_{n_0}$. We shall show that it is not possible.

Let us apply to $\varphi(x)$ Lemma 1, in which we put $u = bx$ and $hu = bh_x$. In view of (9), for every $x$ we can choose such an $h_x$ satisfying

$$0 < r/b \leqslant |h_x| \leqslant s/b \leqslant l/b,$$

that we shall have

$$|y_2(x+h_x) - y_2(x)| / \omega_1(|h_x|) \geqslant aD/2 > C + n_0.$$

Taking into account (8), we should obtain for every $x$ and a certain $h_x$

$$\frac{|f^*(x+h_x) - f^*(x)|}{\omega_1(|h_x|)} \geqslant \frac{|y_2(x+h_x) - y_2(x)|}{\omega_1(|h_x|)} - \frac{|y_1(x+h_x) - y_1(x)|}{\omega_1(|h_x|)} > n_0,$$

which would be contradictory to the fact that $f^*(x) \in Z_{n_0}$.

It follows hence that each of the sets $Z_n$ is non-dense in $H_\omega$ and therefore set $R$ is a set residual in $H_\omega$. Thus Theorem 1 is proved.

Before examining the case where (5) is not satisfied, we shall state, for the space $H_\omega$ previously defined, the following

THEOREM 2. *Let $\{h_i\}$ denote a sequence of positive numbers tending to zero for $i \to \infty$. Then set $R^*$ of functions $f(x)$ belonging to the space $H_\omega$ and satisfying for every $x$ the condition*

(15) $$\overline{\lim_{i\to\infty}} \left| \frac{f(x+h_i) - f(x)}{h_i} \right| = \infty$$

*is empty in the case of $\lim_{h\to+0} \Lambda(h) > 0$, and residual in space $H_\omega$ in the case of $\lim_{h\to+0} \Lambda(h) = 0$.*

Proof. Let us take $\lim_{h\to+0} \Lambda(h) > 0$. In this case space $H_\omega$ is a set of functions, all of which satisfy Lipschitz's condition [6]), and therefore do not satisfy (15). In this case set $R^*$ is empty.

Suppose that $\lim_{h\to+0} \Lambda(h) = 0$. With this supposition the first part of the proof runs analogically to the first part of the proof of Theorem 1 with the following adaptation of the notation and formulas used in it:

We must take $\omega_1(h) = h$. In this way conditions (5) and (6) are satisfied. Symbols $Z_n$, $R_n$, $R$ must be replaced respectively by $Z_n^*, R_n^*, R^*$. Set $Z_n^*$ we define as a set of functions $f(x)$ of space $H_\omega$ satisfying the inequality

$$|f(x+h_i) - f(x)| \leqslant nh_i$$

[6]) Compare the Remark to Theorem 6 in my paper [4].

for a certain $x$ with every value of $i$ $(i=1,2,\ldots)$. Set $R_n^*$ we define as a set of functions $f(x)$ of space $H_\omega$ satisfying the inequality

$$|f(x+h_i)-f(x)|>nh_i$$

for every $x$ with a certain value of $i$ $(i=1,2,\ldots)$.

Function $\varphi(x)$ we define by the formula

$$\varphi(x)=l/2-|x-l/2|$$

for $0\leqslant x\leqslant l$, taking $\varphi(x+l)=\varphi(x)$ for the remaining $x$. On the basis of this definition we have $k=1$ and $D=l/2$ [7]).

Since the conditions of Theorem 1 are all satisfied with the notation thus changed, therefore, supposing that set $Z_{n_0}^*$ is not a non-dense set we should prove that $f^*(x)$ belongs to $Z_{n_0}^*$.

In order to prove that this is not possible, (whence follows the proof of Theorem 2), we do not apply Lemma 1. Taking into account (8) and (9) we obtain for every $x$

$$|f_+^{*\prime}(x)|\geqslant|y_{2+}'(x)|-|y_{1+}'(x)|>al-C>n_0$$

in contradiction to the fact that $f^*(x)$ belongs to $Z_{n_0}^*$.

In this manner Theorem 2 has been completely proved. Since it is obviously valid for the sequence $\{h_i\}$ of negative terms tending to zero, hence, in particular the following theorem is valid:

THEOREM 2*. *The set of functions having the properties:*

1° *belong to the space* $H_\omega$,

2° *for every $x$ at least one of the right-side derivatives of $f(x)$ (or at least one of the left-side derivatives) is infinite,*

*is in space $H_\omega$ residual if* $\lim\limits_{h\to+0}\Lambda(h)=0$ *or empty if* $\lim\limits_{h\to+0}\Lambda(h)>0$ [8]).

We shall now prove that Theorem 1 is valid also in the case of a supposition contrary to (5). Suppose that

$$\overline{\lim_{h\to+0}}\frac{h}{\omega_1(h)}=\infty.$$

Thus there exists a sequence $\{h_i\}$ of positive terms and tending to zero, such that

$$\lim_{i\to\infty}\frac{h_i}{\omega_1(h_i)}=\infty.$$

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[7]) Instead of (9) we can take $2aD>C+n_0$ in this case.

[8]) Replacing in Theorem 2* space $H_\omega$ by space $C$ we obtain Theorem 1 of S. Banach's paper [2].



Let us suppose that $f(x)$ belonging to $H_\omega$ does not satisfy, for a certain $x$, condition (3). In this case that function does not satisfy for this $x$ simultaneously condition

$$\overline{\lim_{i\to\infty}}\frac{|f(x+h_i)-f(x)|}{\omega_1(h_1)}=\overline{\lim_{i\to\infty}}\frac{|f(x+h_i)-f(x)|}{h_i}\cdot\frac{h_i}{\omega_1(h_i)}=\infty$$

and condition

$$\overline{\lim_{i\to\infty}}\frac{|f(x+h_i)-f(x)|}{h_i}=\infty.$$

A set of functions of space $H_\omega$ which do not satisfy for a certain $x$ the latter condition is on the basis of Theorem 2, with the supposition $\lim\limits_{h\to+0}\Lambda(h)=0$, a set of the first category in this space. Therefore the same applies to the set of functions $f(x)$ of space $H_\omega$ not satisfying condition (3) for a certain $x$. Thus in the place of Theorem 1 we obtain the following

THEOREM 1*. *Under the assumption*

$$\lim_{h\to+0}\frac{h}{\omega(h)}=0,$$

*if $\omega_1(h)$ satisfies* (6), *then set $R$ is a set residual in space $H_\omega$.*

It is not difficult to verify that the proof of Theorems 1 and 2 and consequently 1* and 2* can be carried out analogically in the case of space $H_\omega$ being replaced by space $C$ of all functions $f(x)$ continuous and periodic with period $l$. In the last case the proofs are simplified. Formulas (6), (11), (14) are unnecessary, since it is obvious that $f^*(x)\in C$. Supposition (5) becomes superfluous and Theorem 1 takes in this case the following form:

THEOREM 3. *The set of functions $f(x)$ which satisfy* (3) *for every $x$ is in space $C$ of all functions continuous and periodic with period $l$ a residual set.*

Remark. To prove the superfluity of (5) for space $H_\omega$ there is no need to refer to Theorem 2. We define, for this purpose, the function $\omega_1^*(h)$ as follows:

Let $\omega_1^*(h)$ assume for every $h$ one, but not the smaller, of two values $h$ and $\omega_1(h)$. We note that $\omega_1^*(h)$ satisfies (5), (6) with the supposition

$$\lim_{h\to+0}\frac{h}{\omega(h)}=0$$

and that Theorem 1 is valid for it. For every $x$ we have

$$\overline{\lim_{h\to0}}\frac{|f(x+h)-f(x)|}{\omega_1^*(|h|)}\leqslant\overline{\lim_{h\to0}}\frac{|f(x+h)-f(x)|}{\omega_1(|h|)}.$$

Thus the set of functions $f(x)$ satisfying condition (3) for every $x$ is a residual set in space $H_\omega$ not only for $\omega_1^*(h)$, but also for $\omega_1(h)$.

The same applies to space $C$.

We note also that the supposition

$$\lim_{h\to+0}\frac{h}{\omega(h)}=0$$

is evidently satisfied in view of (6) in the case of Theorem 1, *i. e.* in the case of accepting supposition (5). In the case of the contrary supposition, *i. e.* $\lim_{h\to+0} \Lambda(h)>0$, space $H_\omega$ is a set of functions, all of which satisfy Lipschitz's condition [6]).

## References


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# A generalization of maximal ideals method of Stone and Gelfand

by

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It is well known that the ring $C(T)$ of all real-valued continuous functions defined on a bicompact Hausdorff space $T$ characterizes topologically the space $T$. More exactly, two bicompact Hausdorff spaces $T$ and $T'$ are homeomorphic if and only if the rings $C(T)$ and $C(T')$ are algebraically isomorphic. This theorem is usually referred to as Stone's theorem (see [4] and also [2]).

The analysis of the proof of Stone's theorem shows that instead of considering the whole ring $C(T)$ it suffices to consider some weaker classes of functions, *e. g.* the class $C^+(T)$ of all non-negative continuous functions on $T$. The class $C^+(T)$ can be considered as an abstract algebra with the usual operations of addition and multiplication. Two algebras $C^+(T)$ and $C^+(T')$ are algebraically isomorphic, if and only if the bicompact Hausdorff spaces $T$ and $T'$ are homeomorphic.

On the other hand, it is known that the bicompact Hausdorff spaces $T$ are also topologically characterized by the lattices off all open subsets of $T$ [1]).

The usual methods of proofs are similar in both characterizations: by real functions or by open subsets. This method may be called the method of maximal ideals. In both cases we consider certain abstract algebras with two operations, addition and multiplication, and we define the notion of maximal ideals. This notion is the algebraic analogue of the notion of a point in a space.

The purpose of this paper *) is to develop the common idea of both characterizations of bicompact spaces. We introduce a general notion of a semi-ring with two operations, addition and multiplication, characterized by a set of simple axioms. Algebras $C^+(T)$ and some lattices (in particular, Boolean algebras and lattices of open subsets which ap-

*) Presented to the Mathematical Institute of the Polish Academy of Sciences Group of Topology, in October 1952.

[1]) This is valid for more general spaces. See [5] and [6].