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Continuous mappings of a certain family

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1. We shall prove as a main result, using the axiom of choice ($\mathfrak{c}$ denotes the potency of the system of real numbers):

THEOREM I. *There exists a family of $2^{\mathfrak{c}}$ subsets of the plane such that no set of the family can be mapped continuously (degenerate maps excepted [1])) in (or on) another set of the family.*

This theorem is known if the "continuous maps in" are replaced by "topological maps in" or "continuous maps on" or even "continuous maps in, with $\mathfrak{c}$ image-points" [2]). While these theorems are true and have been proved in any complete separable space, this is clearly not the case with theorem I. In the discontinuum of Cantor D for example any subset of D having dimension 0 may be mapped continuously and non-degenerately in any subset of D, consisting of more than one point. Moreover it is clear that the family of theorem I must in any case consist of *connected* sets *containing no arcs*. On the other hand the other theorems are true both for (suitably selected) families of connected or not connected sets and (except in one case) for families containing arcs or no arcs.

From these considerations one might expect that a proof of theorem I might be rather difficult. This is, however, not the case if we add some topological features to the proof of a general set-theoretic theorem of Kuratowski [3]). Our proof parallels therefore to a great extent the proof of Kuratowski, which enables him to establish the mentioned theorem in the case of "continuous maps in with $\mathfrak{c}$ image-points". On the other hand this proof is only an existence-proof using the axiom of choice. It is easy *e. g.* to define effectively two sets neither of which can be map-

[1]) A map is called *degenerate* if the image consists of one point. Henceforth all maps considered will be non-degenerate.

[2]) Comp. C. Kuratowski, *Topologie I*, Warszawa 1952, p. 330-341. Associated with the theorems stated are the names of Kuratowski, Sierpiński, Lindenbaum, Waraszkiewicz.

[3]) O. c., p. 332, Théorème auxiliaire.

ped continuously on the other, whereas it is more difficult to find an effective example of two sets of this kind considering "continuous maps in". The sets of the family we shall find may be taken as totally imperfect; therefore the construction cannot be made effective (at the time being).

Let us mention finally two problems:

Does there exist a connected set which cannot be mapped continuously and non-degenerately on any proper subset?

Does there exist a 0-dimensional set A in a separable metric space S, which cannot be mapped continuously on any set A' with $A \subset \subset A' \subset S$?

2. LEMMA. *If $\Phi=\{f\}$ is a family with potency $\mathfrak{c}$ of transformations of subsets of n-dimensional Euclidean space R^n $(n>1)$ on subsets of R^n with potency $\mathfrak{c}$, and if in Φ are included transformations of all discontinua of Cantor D contained in R^n, there is a family $F=\{X, Y, \ldots\}$ of $2^{\mathfrak{c}}$ connected, totally imperfect subsets $X, Y, \ldots$ in R^n such that for each pair of different sets X, Y and every $f \in \Phi$ the relation*

$$\overline{\overline{f(X)-Y}}=\mathfrak{c}$$

holds true.

Proof. Put $\mathfrak{c}=\aleph_\delta$ and consider a well-ordering of the elements of Φ

$$f_0, f_1, \ldots, f_\omega, \ldots, f_\alpha, \ldots \qquad (\alpha<\omega_\delta) \tag{1}$$

in which every element f of Φ occurs $\mathfrak{c}$ times (this is possible since $\mathfrak{c}^2=\mathfrak{c}$). Consider a fixed f_α; introduce for each function-value y of f_α one value of the argument x such that $f_\alpha(x)=y$. A_α will be the union of all chosen arguments x. f_α is therefore one to one on A_α. Let $g_\alpha=f_\alpha|A_\alpha$. The potency of A_α is obviously $\mathfrak{c}$. By transfinite induction one proves the existence of a well ordered sequence:

$$p_0, p_1, \ldots, p_\omega, \ldots, p_\alpha, \ldots \qquad (\alpha<\omega_\delta) \tag{2}$$

such that

1° $p_\alpha \in A_\alpha$.
2° $p_\alpha \neq p_\xi$, $\quad (\xi<\alpha)$.
3° $p_\alpha \neq g_\eta(p_\xi)$, $\quad (\xi<\alpha,\ \eta<\alpha)$.
4° $g_\eta(p_\alpha) \neq p_\xi$, $\quad (\xi<\alpha,\ \eta<\alpha)$.

Put

$$\bigcup_{\alpha<\omega_\delta} p_\alpha = P.$$

P has potency $\mathfrak{c}$ and even each set $P \cap A_\alpha = B_\alpha$ has potency $\mathfrak{c}$ in view of 1° (each f occuring $\mathfrak{c}$ times in (1), from which it follows that each A_α is repeated $\mathfrak{c}$ times though with different indices in the sequence $\{A_\alpha\}$, $\alpha<\omega_\delta$).

According to a known theorem (Kuratowski, o. c., p. 330) there is a family $F=\{X, Y, \ldots\}$ of $2^{\mathfrak{c}}$ subsets of P such that for two different sets the relation

$$\overline{\overline{B_\alpha \cap X - Y}}=\mathfrak{c} \tag{3}$$

holds for any $\alpha<\omega_\delta$.

We prove that each $X \in F$ is connected. Consider an arbitrary discontinuum $D \subset R^n$. There is an f_α which maps D. From the construction of A_α it follows that there is an A_α and B_α with $B_\alpha \subset A_\alpha \subset D$. In view of (3) we see that $D \cap X \neq 0$. Therefore any $X \in F$ contains points of every discontinuum $D \subset R^n$. This shows that $R^n - X$ is totally imperfect. On the other hand the complement in R^n of a totally imperfect set is connected according to a theorem of Sierpiński. This means: X itself is connected (the previously defined set P is also connected by the same argument). Moreover by analysing the proof (not mentioned here) which gives the family F it is easy to see that X itself can be chosen as a totally imperfect set.

From the relation (3) it follows easily that

$$\overline{\overline{f_\alpha(X)-Y}}=\mathfrak{c} \qquad (\alpha<\omega_\delta).$$

The proof is omitted here being a repetition of Kuratowski's proof (o. c., p. 333).

3. Proof of Theorem I. Substitute for Φ in the preceding lemma the family of all continuous maps of all G_δ-subsets of the plane into subsets with potency $\mathfrak{c}$. The family $F=\{X, Y, \ldots\}$ satisfies the propositions of theorem I. Indeed any non-degenerate continuous map f of a connected set $X \in F$ contains $\mathfrak{c}$ image-points. The continuous extension f^* of f to a G_δ-set $X^* \supset X$ therefore contains $\mathfrak{c}$ image-points. Hence $f^* \in \Phi$. From $f(X)=f^*(X)$ it follows

$$\overline{\overline{f(X)-Y}}=\overline{\overline{f^*(X)-Y}}=\mathfrak{c}.$$

This proves that $f(X)$ is not contained in any Y for any f, q. e. d.

4. Kuratowski (o. c., p. 333) proves theorem I in the case of continuous maps with $\mathfrak{c}$ image-points. It is easy to strengthen this result somewhat.

THEOREM II. *In each complete separable metric space S there exists a system of $2^{\mathfrak{c}}$ subsets $\{X\}$, each X of potency $\mathfrak{c}$, such that no X can be mapped continuously in another X, countable images excepted.*

Proof. Substitute in the "théorème auxiliaire" (Kuratowski o. c., p. 332; this theorem corresponds with our lemma) the family Φ of section 3, the plane being replaced by S. $F=\{X, Y, \ldots\}$ is the family which satisfies the conditions. Indeed, from $\overline{\overline{f(X)-Y}}=\mathfrak{c}$, $f\in\Phi$, $X\neq Y$, it follows that $\overline{\overline{X}}=\mathfrak{c}$ for any $X\in F$. Furthermore, if $g(X)$ is a continuous map of X with uncountable image, there is a continuous extension g^* on a G_δ-set X^*, containing X. X^* contains a discontinuum D such that $g^*(D)$ is a topological map. This follows from a well-known theorem (Kuratowski, o. c., p. 351), $g(X^*)$ being uncountable and our G_δ-set being topologically complete. Therefore $\overline{\overline{g^*(X^*)}}=\mathfrak{c}$. This implies $g^*\in\Phi$. The relation

$$\overline{\overline{g^*(X)-Y}}=\overline{\overline{g(X)-Y}}=\mathfrak{c}\qquad (X\neq Y)$$

proves the theorem.

In particular the potency of $g(X)$ for any $X\in F$ and any continuous map of X with uncountable image equals $\mathfrak{c}$, which gives

COROLLARY. *There exists a system of* $2^{\mathfrak{c}}$ *sets of dimension* 0 *and potency* $\mathfrak{c}$, *such that every continuous image of a set of the system either is countable or has* $\mathfrak{c}$ *many points.*

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