Proof. There can be no model of class $\boldsymbol{P}_1$ since $\mathcal{F}$ has no such model. If there were a model of class $\boldsymbol{Q}_1$, then $A', B', \ldots, G'$ would be complements of recursively enumerable sets and relations and (since $n \in A \equiv n \notin A', \ldots, G(p,q,r) \equiv \text{non}\, G'(p,q,r)$) the sets and relations $A, B, \ldots, G$ would be recursive. Theorem 4 follows thus from theorem 3.

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Generalized dissimilarity of ordered sets *

by

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1. Introduction. The present paper arose from an attempt to solve the following problem: does there exist a (simply) ordered set E of more than one element, such that, for every pair of distinct elements a and b of E, the sets $E-\{a\}$ and $E-\{b\}$ are dissimilar (*i. e.*, there is no one-one order-preserving correspondence between the two sets)? An easy argument shows that there is no such set E of power $\aleph_0$; we shall prove, however, that there does exist a subset of the continuum, of power $\mathfrak{c}=2^{\aleph_0}$, possessing the property in question. Generalizations in various directions will also be obtained. In order to motivate these generalizations as they appear in the formal statements of the theorems in section 6 below, we shall give here a rough indication of their underlying ideas.

First of all, it is possible to find a subset E of the continuum such that not only is there no similarity transformation between $E-\{a\}$ and $E-\{b\}$, but there is not even a non-trivial "pseudo-similarity" transformation of $E-\{a\}$ onto $E-\{b\}$ (cf. Corollary 6.2 (d)), where we define (cf. 4.8) a pseudo-similarity transformation of an ordered set M to be a single-valued function (not necessarily one-one) defined on M that is, with respect to some decomposition of some dense subset of M into mutually exclusive subintervals of M, a similarity or anti-similarity on the interior of each of these subintervals. (This is clearly a more general kind of transformation than the "semi-similarity" introduced by Aronszajn [1]. In fact, there are $2^{\mathfrak{c}}$ pseudo-similarity transformations of the continuum into itself.) Then, it is not necessary that only single elements, a and b, be removed from E in order to obtain, say, dissimilar subsets of E; these single elements may be replaced by arbitrary distinct subsets of E of power less than $\mathfrak{c}$ (cf. Corollary 6.2 (d)). Another generalization is concerned with replacing the continuum by any ordered set M of power $\mathfrak{c}$ containing a subset of power $\mathfrak{c}$ that can be imbedded in the continuum; for any such M, we obtain a decomposition into $\mathfrak{c}$ mutually exclusive subsets (the sets E^{σ} in Theorem 6.1), each of which has the

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aforementioned properties involving dissimilarity, and is such that every one of its (non-empty) intervals is of power $\mathfrak{c}$. Theorems 6.5 and 6.7 deal with other decompositions of this sort. (For additional decomposition theorems of a related nature, cf. Ginsburg [3].)

Corollaries 6.2, 6.6 and 6.8 enumerate some of the special cases of Theorems 6.1, 6.5 and 6.7 that apply to decompositions of the continuum itself.

Theorems 6.1, 6.5 and 6.7 themselves are derived from three other theorems concerning decompositions of arbitrary sets (not necessarily ordered), namely, Theorems 3.2, 3.4 and 3.6. The transition from arbitrary sets to ordered sets is accomplished by means of Lemma 5.2 (or Corollary 5.3).

Finally, the remark following Corollary 6.2 is concerned with replacing $2^{\aleph_0}$ by higher powers.

This synopsis does not take into account all the results of this paper, but is intended merely to indicate some of the ramifications of the original problem that are considered herein.

The main tool used is a modification of an idea employed by Dushnik and Miller [2] to obtain a subset E of the continuum that is not similar to any proper subset of E (cf. also Sierpiński [7] and Ginsburg [3]).

2. Some definitions and notation. In what follows, C denotes the linear continuum; $\omega_{\mathfrak{c}}$ is the least ordinal number of power $\mathfrak{c}=2^{\aleph_0}$ (we make use of the well-ordering theorem); $|M|$ stands for the cardinal number of M; and the symbol $\subset$ signifies set inclusion, not necessarily proper.

2.1. *Function* or *mapping* means *single-valued* function. The symbol $g|A$ denotes g restricted to A, g being a function and A a subset of its domain of definition.

2.2. Definition. An $\aleph_\alpha$-*decomposition* of a set M is a family of $\aleph_\alpha$ mutually disjoint sets, each of power $\aleph_\alpha$, whose union is M.

2.3. The following will be convenient for dealing with decompositions of sets. Let α be any ordinal number. Put $W(\omega_\alpha)=$ set of all ordinals $<\omega_\alpha$, $P=P(\omega_\alpha)=$ lexicographically ordered set of all pairs (σ,τ) for which $\tau\leqslant\sigma<\omega_\alpha$. Clearly, P is well-ordered, of type ω_α; put $p=p_\alpha$ $=$(unique) similarity mapping of $W(\omega_\alpha)$ onto P.

For every $\tau<\omega_\alpha$, define a subset P_τ of P according to

(2.3a) $$P_\tau=\{(\sigma,\tau):\ \tau\leqslant\sigma<\omega_\alpha\};$$

it is obvious that the family of sets $\{P_\tau\}_{\tau<\omega_\alpha}$ form an $\aleph_\alpha$-decomposition of P. Let $q=q_\alpha$ be any function from $W(\omega_\alpha)$ onto itself, and for each pair (ξ,τ) $(\xi<\omega_\alpha,\ \tau<\omega_\alpha)$, put

(2.3b) $$Q_{\xi,\tau}=\{\delta:\ q(\delta)=\xi,\ p(\delta)\in P_\tau\};$$

it is an elementary matter to define q in such a way that for all such (ξ,τ), we have

(2.3c) $$|Q_{\xi,\tau}|=\aleph_\alpha,$$

and we shall suppose throughout the sequel that q has been so constructed.

3. Some decompositions of arbitrary sets. In this section we shall prove three theorems on decompositions of *arbitrary* sets (not necessarily ordered). These theorems will be applied later on, however, to ordered sets.

3.1. Definition. A function f on a set M (into any set N) is *non-trivial* ($\aleph_\alpha$), if there is a subset M' of M, of power $\aleph_\alpha$, free of fixed points of f, and on which f is one-one; *i. e.*, for all $x\in M'$, $y\in M'$, $x\neq y$, we have $x\neq f(x)\neq f(y)$.

3.2. Theorem. *Let α be any ordinal, R any set of power $\aleph_\alpha$, $\{f_\xi\}_{\xi<\omega_\alpha}$ a family of non-trivial ($\aleph_\alpha$) functions from subsets of R into R. Put D_ξ $=$domain of f_ξ $(\xi<\omega_\alpha)$, $D=\bigcup_{\xi<\omega_\alpha}D_\xi$. Then there exists a subset F of R, having an $\aleph_\alpha$-decomposition $\{F_\tau\}_{\tau<\omega_\alpha}$, and with the following properties. Writing $V=R-F$:*

(3.2a) $$F\subset D;$$

(3.2b) $$\text{for all}\quad \xi<\omega_\alpha\quad\text{and}\quad\tau<\omega_\alpha,\qquad |D_\xi\cap F_\tau|=\aleph_\alpha;$$

(3.2c) $$|D\cap V|=\aleph_\alpha;$$

(3.2d) $$\text{for all}\quad \xi<\omega_\alpha\quad\text{and}\quad\tau<\omega_\alpha,\qquad |V\cap f_\xi(D_\xi\cap F_\tau)|=\aleph_\alpha.$$

Proof. We define sequences $F=\{x_\xi\}$, $G=\{y_\xi\}$, $H=\{z_\xi\}$ $(\xi<\omega_\alpha)$, as follows. Let $\delta<\omega_\alpha$, and suppose that elements x_ξ, y_ξ and z_ξ of R have been defined for all $\xi<\delta$. Since $f_{q(\delta)}$ (see 2.3) is non-trivial ($\aleph_\alpha$), there exists an x_δ such that

$$x_\delta\in D_{q(\delta)}-\{x_\xi\}_{\xi<\delta}-\{y_\xi\}_{\xi<\delta}-\{z_\xi\}_{\xi<\delta}$$

and

$$f_{q(\delta)}(x_\delta)\notin\{x_\xi\}_{\xi\leqslant\delta}\cup\{y_\xi\}_{\xi<\delta}.$$

Choose any such x_δ, put $y_\delta=f_{q(\delta)}(x_\delta)$, and then define z_δ as any element of

$$D-\{x_\xi\}_{\xi\leqslant\delta}-\{z_\xi\}_{\xi<\delta}.$$

This completes the definitions of F, G and H.

Clearly, we have (3.2a). Since $H \subset D \cap V$ and $|H| = \aleph_\alpha$, we get (3.2c). Putting

$$F_\tau = \{x_\delta \colon p(\delta) \in P_\tau\} \qquad (\tau < \omega_\alpha),$$

we see from (2.3a) ff. that the family $\{F_\tau\}$ is an $\aleph_\alpha$-decomposition of F.

Using (2.3b, c), we see at once that $\delta \in Q_{\xi,\tau}$ implies $x_\delta \in D_\xi \cap F_\tau$, whence $|D_\xi \cap F_\tau| = \aleph_\alpha$, *i. e.*, (3.2b). Putting

$$G_{\xi,\tau} = \{y_\delta \colon \delta \in Q_{\xi,\tau}\},$$

we find that $|G_{\xi,\tau}| = \aleph_\alpha$; since

$$G_{\xi,\tau} \subset f_\xi(D_\xi \cap F_\tau) \cap G,$$

and $G \subset V$, this gives us (3.2d). Our proof is now complete.

3.3. Definition. A family Φ of $\aleph_\alpha$ functions $\{\varphi_\tau\}$ are *completely distinct* ($\aleph_\alpha$) from M into N if every $\varphi_\tau \in \Phi$ is one-one from M into N, and if no two coincide at any point; *i. e.*, for all $x \in M$ and $x' \in M$, with $x \neq x'$, and for all σ, τ with $\sigma \neq \tau$, we have

$$\varphi_\sigma(x) \neq \varphi_\sigma(x') \neq \varphi_\tau(x').$$

3.4. Theorem. *Let α be any ordinal, R any set of power $\aleph_\alpha$, $\{f_\xi\}_{\xi<\omega_\alpha}$ a family of non-trivial ($\aleph_\alpha$) functions from subsets of R into R. Put D_ξ = domain of f_ξ ($\xi < \omega_\alpha$), $D = \bigcup_{\xi<\omega_\alpha} D_\xi$. Further, let $\{\varphi_\tau\}_{\tau<\omega_\alpha}$ be a family of completely distinct ($\aleph_\alpha$) functions on D into D, with φ_0 the identity. Then there exists a subset F of R, having an $\aleph_\alpha$-decomposition $\{F_\tau\}_{\tau<\omega_\alpha}$, and with the following properties. Writing $V = R - F_0$:*

(3.4a) $$F \subset D;$$

(3.4b) $$|R - F| = \aleph_\alpha;$$

(3.4c) *to every $\tau < \omega_\alpha$, there corresponds a $\gamma_\tau < \omega_\alpha$ such that* $\varphi_{\gamma_\tau}(F_0) = F_\tau$;

(3.4d) $$|D \cap V| = \aleph_\alpha;$$

(3.4e) *for every* $\xi < \omega_\alpha$, $|V \cap f_\xi(D_\xi \cap F_0)| = \aleph_\alpha$.

Proof. We define sequences $F = \{x_{\sigma,\tau}\}$, $G = \{y_\sigma\}$, $H = \{z_\sigma\}$, $\Gamma = \{\gamma_\tau\}$ ($\sigma < \omega_\alpha$, $\tau < \omega_\alpha$), as follows. Let $x_{0,0}$ be any element of $D_{q(0)}$ (see 2.3) for which

$$f_{q(0)}(x_{0,0}) \neq x_{0,0}$$

(such an element exists, since $f_{q(0)}$ is non-trivial ($\aleph_\alpha$)), and define $y_0 = f_{q(0)}(x_{0,0})$. Take for z_0 any element of R distinct from $x_{0,0}$. Put $\gamma_0 = 0$.

Let $0 < \delta < \omega_\alpha$, and suppose that elements $x_{\sigma,\tau}$, y_σ and z_σ of R, and ordinals $\gamma_\tau < \omega_\alpha$, have been defined for all $\sigma < \delta$ and $\tau < \delta$, such that

$$\varphi_{\gamma_\tau}(x_{\sigma,0}) = x_{\sigma,\tau} \qquad (\sigma < \delta, \ \tau < \delta).$$

First we shall define $x_{\delta,0}$ and y_δ, then $x_{\delta,\tau}$ ($\tau < \delta$), then γ_δ, then $x_{\sigma,\delta}$ ($\sigma \leqslant \delta$), and, finally, z_δ.

Since $f_{q(\delta)}$ is non-trivial ($\aleph_\alpha$), and since each φ_τ is one-one, we can choose $x_{\delta,0}$ such that

$$x_{\delta,0} \in D_{q(\delta)} - \{x_{\sigma,\tau}\}_{\sigma<\delta,\tau<\delta} - \{y_\sigma\}_{\sigma<\delta} - \{z_\sigma\}_{\sigma<\delta},$$

$$f_{q(\delta)}(x_{\delta,0}) \notin \{x_{\delta,0}\}_{\sigma\leqslant\delta} \cup \{y_\sigma\}_{\sigma<\delta},$$

and

$$\text{for all} \quad \tau < \delta, \qquad \varphi_{\gamma_\tau}(x_{\delta,0}) \notin \{x_{\sigma,\varrho}\}_{\sigma<\delta,\varrho<\delta} \cup \{z_\sigma\}_{\sigma<\delta}.$$

We then put $y_\delta = f_{q(\delta)}(x_{\delta,0})$, and define

$$x_{\delta,\tau} = \varphi_{\gamma_\tau}(x_{\delta,0}) \qquad (\tau < \delta)$$

(the case $\tau = 0$ agrees with $\gamma_0 = 0$, φ_0 = identity). We have now defined $x_{\sigma,\tau}$ for all $\sigma \leqslant \delta$ and $\tau < \delta$.

Likewise, since the family $\{\varphi_\tau\}_{\tau<\omega_\alpha}$ are completely distinct ($\aleph_\alpha$), we can find a $\gamma_\delta < \omega_\alpha$ such that for all $\pi \leqslant \delta$,

$$\varphi_{\gamma_\delta}(x_{\pi,0}) \notin \{x_{\sigma,\tau}\}_{\sigma\leqslant\delta,\tau<\delta} \cup \{z_\sigma\}_{\sigma<\delta};$$

we then define

$$x_{\sigma,\delta} = \varphi_{\gamma_\delta}(x_{\sigma,0}) \qquad (\sigma \leqslant \delta).$$

We have now defined $x_{\sigma,\tau}$ for all $\sigma \leqslant \delta$, $\tau \leqslant \delta$, and, in fact, we have

(3.4f) $$x_{\sigma,\tau} = \varphi_{\gamma_\tau}(x_{\sigma,0}) \qquad (\sigma \leqslant \delta, \ \tau \leqslant \delta).$$

Finally, we take for z_δ any element of

$$R - \{x_{\sigma,\tau}\}_{\sigma\leqslant\delta,\tau\leqslant\delta} - \{z_\sigma\}_{\sigma<\delta}.$$

This completes the definitions of the sets F, G, H and Γ.

Clearly, $H \subset R - F$, $F \subset D$, and $|H| = \aleph_\alpha$. In particular, we have (3.4a). Since $|H| = \aleph_\alpha$ and $H \subset R - F$, we get (3.4b). Putting

$$F_\tau = \{x_{\sigma,\tau}\}_{\sigma<\omega_\alpha} \qquad (\tau < \omega_\alpha),$$

we see that $\{F_\tau\}$ is an $\aleph_\alpha$-decomposition of F. Conclusion (3.4c) now follows from (3.4f). Since $|F_1| = \aleph_\alpha$ and $F_1 \subset D \cap V$, we get (3.4d).

Finally, (2.3b, c) show that for every $\xi < \omega_\alpha$,

$$|\{\delta \colon \delta < \omega_\alpha, \ q(\delta) = \xi\}| = \aleph_\alpha;$$

hence, putting

$$G_\xi = \{y_\delta \colon q(\delta) = \xi\},$$

we see that $|G_\xi| = \aleph_\alpha$. Since

$$G_\xi \subset f_\xi(D_\xi \cap F_0) \cap G,$$

and $G \subset V$, we get (3.4e). This completes the proof.

3.5. Definition. A family Φ of functions $\{\varphi_\tau\}$ are *pseudo-distinct* $(\aleph_\alpha)$ from M into N if every φ_τ is one-one from M into N, and if, for every $x \in M$ and every $y \in N$, there are $\aleph_\alpha$ functions $\varphi' \in \Phi$ for which $\varphi'(x)=y$ and such that the family of all these $\varphi'|(M-\{x\})$ are completely distinct $(\aleph_\alpha)$ from $M-\{x\}$ into N.

3.6. Theorem. *Let α be any ordinal, R any set of power $\aleph_\alpha$, and $\{f_\xi\}_{\xi<\omega_\alpha}$ a family of non-trivial $(\aleph_\alpha)$ functions from subsets of R into R. Put $D_\xi=$ domain of f_ξ $(\xi<\omega_\alpha)$, $D=\bigcup_{\xi<\omega_\alpha} D_\xi$. Finally, let $\{\varphi_\tau\}_{\tau<\omega_\alpha}$ be a family of pseudo-distinct $(\aleph_\alpha)$ functions from D into R, with φ_0 the identity. Then there exists an $\aleph_\alpha$-decomposition $\{F^\tau\}_{\tau<\omega_\alpha}$ of R such that*

(3.6a) $$F^0 \subset D;$$

(3.6b) *for every* $\tau<\omega_\alpha$, *there is a* $\gamma_\tau<\omega_\alpha$ *for which* $\varphi_{\gamma_\tau}(F^0)=F^\tau$;

(3.6c) *for every* $\xi<\omega_\alpha$, $|(R-F^0)\cap f_\xi(D_\xi\cap F^0)|=\aleph_\alpha$.

Proof. We define sequences $F=\{x_{\sigma,\tau}\}$, $G=\{y_\sigma\}$, and $\Gamma=\{\gamma_\tau\}$ $(\sigma<\omega_\alpha,$ $\tau<\omega_\alpha)$, as follows. Let [1])

(3.6d) $$\{r_\xi\}_{\xi<\omega_\alpha}$$

be an enumeration (without repetitions) of the elements of R. Let $x_{0,0}$ be any element of R for which

(3.6e) $$x_{0,0}\in D_{q(0)},\qquad f_{q(0)}(x_{0,0})\neq x_{0,0}$$

(such an element exists, since $f_{q(0)}$ (see 2.3) is non-trivial $(\aleph_\alpha)$). Put

$$y_0=f_{q(0)}(x_{0,0}),\qquad \gamma_0=0.$$

Let $0<\delta<\omega_\alpha$, and suppose that elements $x_{\sigma,\tau}$ and y_σ of R, and ordinals $\gamma_\tau<\omega_\alpha$, have been defined for all $\sigma<\delta$ and all $\tau<\delta$, such that

$$\varphi_{\gamma_\tau}(x_{\sigma,0})=x_{\sigma,\tau}\qquad(\sigma<\delta,\ \tau<\delta).$$

We shall define $x_{\delta,0}$ and y_δ, then $x_{\delta,\tau}$ $(\tau<\delta)$, then $x_{0,\delta}$ and γ_δ, and, finally, $x_{\sigma,\delta}$ $(0<\sigma\leqslant\delta)$.

Since the function $f_{q(\delta)}$ is non-trivial $(\aleph_\alpha)$, and since every φ_τ is one-one, we can choose $x_{\delta,0}$ so that

(3.6f) $$\begin{aligned}&x_{\delta,0}\in D_{q(\delta)}-\{x_{\sigma,\tau}\}_{\sigma<\delta,\tau<\delta}-\{y_\sigma\}_{\sigma<\delta},\\ &f_{q(\delta)}(x_{\delta,0})\notin\{x_{\sigma,0}\}_{\sigma\leqslant\delta}\cup\{y_\sigma\}_{\sigma<\delta},\\ &\text{for all}\quad \tau<\delta,\quad \varphi_{\gamma_\tau}(x_{\delta,0})\notin\{x_{\sigma,\varrho}\}_{\sigma<\delta,\varrho<\delta}.\end{aligned}$$

[1]) The authors are indebted to P. Erdös for the suggestion of well-ordering R, and then defining $x_{0,\delta}$ as is done below, in order to obtain the mutual similarity of the sets F^τ in Theorem 6.7.

We then put $y_\delta=f_{q(\delta)}(x_{\delta,0})$, and define

$$x_{\delta,\tau}=\varphi_{\gamma_\tau}(x_{\delta,0})\qquad(\tau<\delta)$$

(the case $\tau=0$ agrees with the earlier definitions $\gamma_0=0$, $\varphi_0=$identity). We have so far defined $x_{\sigma,\tau}$ for all $\sigma\leqslant\delta$ and $\tau<\delta$.

Now select as $x_{0,\delta}$ the first element r_ξ in the sequence (3.6d) for which

$$r_\xi\notin\{x_{\sigma,\tau}\}_{\sigma\leqslant\delta,\tau<\delta}.$$

Since the family $\{\varphi_\tau\}$ are pseudo-distinct $(\aleph_\alpha)$ from D into R, there is a function φ_{γ_δ} such that

$$\varphi_{\gamma_\delta}(x_{0,0})=x_{0,\delta}$$

and

$$\text{for all}\quad \pi\leqslant\delta,\quad \varphi_{\gamma_\delta}(x_{\pi,0})\notin\{x_{\sigma,\tau}\}_{\sigma\leqslant\delta,\tau<\delta};$$

we then define

$$x_{\sigma,\delta}=\varphi_{\gamma_\delta}(x_{\sigma,0})\qquad(\sigma\leqslant\delta)$$

(this agrees with the case $\sigma=0$ already defined). We now have

(3.6g) $$x_{\sigma,\tau}=\varphi_{\gamma_\tau}(x_{\sigma,0})\qquad(\sigma\leqslant\delta,\ \tau\leqslant\delta).$$

This completes the inductive definitions of the sets F, G and Γ. Clearly $F=R$.

Put

$$F^\tau=\{x_{\sigma,\tau}\}_{\sigma<\omega_\alpha}\quad(\tau<\omega_\alpha);$$

then $\{F^\tau\}$ is an $\aleph_\alpha$-decomposition of R, and $G\subset R-F^0$. Now (3.6a) follows from (3.6e, f), and (3.6b) follows from (3.6g). Finally, (2.3b, c) show that, for every $\xi<\omega_\alpha$, we have

$$|\{\delta\colon\ \delta<\omega_\alpha,\ q(\delta)=\xi\}|=\aleph_\alpha;$$

hence, putting

$$G_\xi=\{y_\delta\colon\ q(\delta)=\xi\},$$

we see that $|G_\xi|=\aleph_\alpha$. Since

$$G_\xi\subset f_\xi(D_\xi\cap F^0)\cap(R-F^0),$$

we get (3.6c). This completes the proof.

We now turn to ordered sets.

4. Pseudo-similarity transformations. In the present section, we introduce the notion of a pseudo-similarity transformation of an ordered set, and establish some lemmas pertaining thereto.

4.1. Let M be an ordered set. An element of M having two neighbours (*i. e.*, having both an immediate predecessor and an immediate successor) is called an *isolated* element of M; a border element of M is also called isolated, provided that it either has a neighbour or is the only element of M. A subset I of M is called an *interval* of M if both

(4.1a) I is empty, or I consists of one isolated element of M, or I contains at least two elements;

and

(4.1b) with every x, y in I, every element of M between x and y is also in I.

For any $a \in M$, $M^{(a)}$ resp. $M_{(a)}$ denotes the interval of M consisting of all $x \in M$ resp. $y \in M$ for which $x<a$ resp. $y>a$.

4.2. Let M, K be ordered sets, with $K \subset M$. Then K is *dense in* M if for every $m_1<m_2$ $(m_j \in M)$, there exist k_1, k_2 $(k_j \in K)$ such that $m_1 \leqslant k_1 < k_2 \leqslant m_2$. Whenever we say "$K$ is dense in M", it will be understood that $K \subset M$.

This is essentially the definition given in [4], p. 89, and appears to be the natural one for us to use [2]). It does not coincide with "dense" in the topological sense (cf. [4], p. 249), with respect to the interval topology [3]) on M. *E. g.*, if M is the set of reals in $(0,1]+[2,3)$, and $K=(0,1)+(2,3)$, then K is dense in M in the latter sense, but not in our present sense. On the other hand, the two definitions coincide if, *e. g.*, M is a dense set (*i. e.*, with at least two elements but without neighbouring elements) or an isolated set (*i. e.*, every element is isolated) of more than one element.

Our occasional use of topological terms, such as "open" and "closed", will always refer to the interval topology.

4.3. We shall also extend the symbols $M^{(a)}$ and $M_{(a)}$ to include the case in which a is an arbitrary element of an ordered set H that contains M. We define: $M^{(a)}=M \cap H^{(a)}$, $M_{(a)}=M \cap H_{(a)}$.

In case $a \in H-M$, these sets need not be intervals of M. For example, if H is the subset $\{0\}+[1,2)$ of the reals, and M the set $\{0\}+(1,2)$, then $M^{(1)}=\{0\}$, which is not an interval of M. On the other hand, if M is unbordered, or if M is dense in H, then the sets in question will always be intervals of M.

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[2]) According to this definition, the null set is dense in any one-element set. This does not disturb us.

[3]) The interval topology on an ordered set M of more than one element is formed by taking as a base of open neighbourhoods all intervals of the form $M_{(a)}$, $M^{(b)}$, and $M_{(a)} \cap M^{(b)}$ $(a \in M,\ b \in M)$; cf. [4], p. 214. Note that the relative topology on a subset need not coincide with the intrinsic interval topology thereon. *E. g.*, if K is the subset of C (the reals) consisting of all $x \leqslant 0$ and all $x>1$, then the interval $K_{(1)}$ (cf. 4.3) is a closed set under the topology induced from C, but it is not a closed set under the interval topology on K.

4.4. Definition. Let H, N be ordered sets, with $N \subset H$. Then N is *essentially nowhere dense* in H if for every interval I of H that is a dense set, there is a non-empty interval J of I that is free of elements of N. (If H is a dense set, we may drop the word "essentially").

For example, every finite subset of H is essentially nowhere dense in H.

4.5. For any ordered set H, $i(H)$ denotes the set obtained from H by deleting its border elements (if there are any).

Thus if J is any interval of H, then $i(J)$ is an open interval of H (contained in, but not necessarily coinciding with, the interior of J). Note also that if J contains at most two elements, then $i(J)$ is empty.

We now proceed to generalize the notion of a similarity transformation of an ordered set.

4.6. Definition. Let H be an ordered set. A set $\{H_t\}_{t \in T}$ of non-overlapping intervals of H whose union is dense in H will be called an *essential partition* — or, for brevity, simply a *partition* — of H.

Note that the corresponding intervals $i(H_t)$ are mutually exclusive.

The proof of the following lemma is obvious.

4.7. Lemma. *Let $\{E_t\}_{t \in T}$ be a partition of an ordered set E, and let $E'=\bigcup_{t \in T} i(E_t)$. Then*

(4.7a) *$E-E'$ is essentially nowhere dense in E;*

(4.7b) *if E is a dense set, then E' is dense in E.*

4.8. Definition. Let H, M be ordered sets. We shall say that H is *pseudo-similar* to M if there exists a function f defined on H, with $f(H)=M$, and a partition $\mathfrak{H}=\{H_t\}_{t \in T}$ of H, such that, for every $t \in T$, the restriction $f|i(H_t)$ is either a similarity or an anti-similarity. The values of f elsewhere on H are arbitrary elements of M.

Such a function f will be called a *pseudo-similarity (transformation)* of H onto M. The partition $\mathfrak{H}$ will be called a partition *belonging to* f, and f will be said to be *based upon* this partition.

Note that f is not assumed to be one-one. Note also that a given f may be based upon several different partitions.

Furthermore, it may be seen without difficulty that if f is a pseudo-similarity of H onto M, and if J is dense in some *interval* of H, then the restriction $f|J$ is a pseudo-similarity of J into M (*i. e.*, onto some subset of M). The example H=lexicographically ordered plane, $J=x$-axis, shows that this need not be the case for an arbitrary subset J of H.

4.9. Definition. A pseudo-similarity transformation f of H onto M is *essentially the identity* if there is a partition $\{H_t\}_{t \in T}$ belonging to f

such that $f|i(H_t)$ is the identity for every $t \in T$; more particularly, f is essentially the identity *with respect to this partition* [4]).

We also speak of f as an *essentially identical mapping* of H onto M, and write $H='M$; if no such mapping exists, we write $H\neq'M$.

Observe, in this connection, that the relation $='$ is not symmetric. For example, if H denotes the irrationals, then obviously C (=reals)$\neq'H$ (since every interval of C contains a rational). On the other hand, we do have $H='C$: express H as the union of infinitely many non-overlapping intervals H_t (of H) each with irrational endpoints, map each $i(H_t)$ onto itself identically, and map the set of all the endpoints onto the set of all rationals in an arbitrary fashion (one-one or many-one).

4.10. Lemma. *Let H, N be ordered sets, with $N\subset H$, and suppose that N is closed and essentially nowhere dense* (cf. 4.4) *in H. Express $H-N$ as the union of mutually exclusive open intervals of H. Then the collection of all these intervals, along with all the two-element intervals of H that are not contained in $H-N$, constitutes a partition of H. In particular, if $N=H$ (i. e., if H is essentially nowhere dense in itself), and if D is an arbitrary ordered set, then every pseudo-similarity transformation of H into D is essentially the identity.*

Proof. Let K denote the union of $H-N$ with all the two-element intervals of H that are not contained in $H-N$. It suffices to show that K is dense in H. Let a, b be any two elements of H, with $a<b$, and denote the interval $a\leqslant x\leqslant b$ of H by I. If I contains two neighbouring elements, then these elements are both in K (as follows from the definition of K). In the contrary case, I is a dense set. Then, since N is essentially nowhere dense, there is an interval J of I contained in $H-N$, and hence in K; and, of course, J contains at least two elements. This completes the proof that K is dense in H, and hence the proof of the lemma.

4.11. Definition. For any ordered sets K and L, we write

(4.11a) $\mathcal{G}(K,L)$=family of all non-identical similarity transformations and all anti-similarity transformations of K into L;

(4.11b) $\mathcal{F}(K,L)=\bigcup_J\mathcal{G}(J,L)$, J ranging over all (non-empty) *unbordered* intervals of K.

It follows that $\mathcal{F}(K,L)=\bigcup_J\mathcal{F}(J,L)$, J ranging over all unbordered intervals of K.

If K is finite, then, trivially, the family $\mathcal{F}(K,L)$ is empty. Our goal in this paper is, essentially, to find non-trivial examples for which $\mathcal{G}$ and $\mathcal{F}$ are empty (cf. section 5).

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[4]) Note that f need not be essentially the identity with respect to every partition belonging to it.

For the linear continuum C, as is well-known, $|\mathcal{G}(C,C)|=|\mathcal{F}(C,C)|=\mathfrak{c}$. On the other hand, the number of pseudo-similarity transformations from C into C — even the number of essentially identical transformations — is $2^\mathfrak{c}$. For let C' be any nowhere dense, perfect subset of C (*e. g.*, the Cantor ternary set), and consider the corresponding partition of C (Lemma 4.10). Any pseudo-similarity transformation based upon this partition may assume arbitrary values on C'; since $|C'|=\mathfrak{c}$, there are $2^\mathfrak{c}$ such transformations.

4.12. Lemma. *Let D, A_1 and A_0 be ordered sets, with $A_1\subset A_0$ and such that A_1 is dense in A_0 and contains the border elements of A_0 (if there are any). Let h be any similarity or anti-similarity mapping of A_0 into D, and suppose that $h(A_1)$ is dense in some interval H of D. Then h is determined by its values on A_1.*

Proof. Every element x of A_0-A_1 determines a gap of A_1. The element $h(x)$ of $h(A_0)$ determines the corresponding gap of $h(A_1)$. But $h(A_1)$ is dense in the interval H of D, so this gap can be filled by only one element of D.

4.13. Corollary. *Let D, H, A_1 and A_0 be ordered sets, with $A_1\subset A_0\subset H\subset D$, H an interval of D, and such that A_1 is dense in H and contains the border elements of A_0 (if there are any). Let h be any similarity mapping of A_0 into D such that the restriction $h|A_1$ is the identity. Then h itself is the identity.*

Proof. It is easily seen that $h(A_0)\subset H$. The result now follows from Lemma 4.12.

In this corollary, the condition that A_1 contain the border elements of A_0 is critical. For let D be the reals in $(0,1]+\{2\}$, $A_0=H=(0,1]$, $A_1=(0,1)$, and $h(A_0)=(0,1)+\{2\}$. Then A_1 is dense in H, but does not contain the last element of A_0; and h is not the identity, since $h(1)=2$.

Again, it is critical that A_0 be dense in H. For let $D=H=$reals in $(0,1]+[2,3)$, $A_0=(0,1]+(2,3)$, $A_1=(0,1)+(2,3)$, and $h(A_0)=(0,1)+[2,3)$. Then A_0 is unbordered, but is not dense in H; and h is not the identity, since $h(1)=2$.

4.14. Corollary. *Let D, K and J be ordered sets, with $J\subset K\subset D$, K a dense set and an interval of D, and J dense in K. Let h be any similarity mapping of $E=i(J)$ into D that is essentially the identity. Then h is the identity.*

Proof. Let $\{E_t\}_{t\in T}$ be a partition of E with respect to which h is essentially the identity. Then $h|A_1$ is the identity, where $A_1=\bigcup_{t\in T}i(E_t)$. Since E is dense in the dense set $H=i(K)$, E is a dense set, whence, by Lemma 4.7(b), A_1 is dense in E. Therefore A_1 is dense in H. Since, clearly,

E is unbordered, the result is now immediate from Corollary 4.13 (with $A_0=E$).

The condition that K be a dense set is critical here. For let $D=K=J=E=T=$ set of all integers, and let $h(t)=t+1$ $(t \in E)$. The partition of E defined by $E_t=\{t\}$ $(t \in T)$ belongs to h, and for this partition, the set A_1 is empty. Hence h (as well as every other pseudo-similarity on E) is essentially the identity (cf. Lemma 4.10).

4.15. Lemma. *Let L be a continuous set, and let E and H be ordered sets, with $E \subset H$, and such that E is dense in H and contains the border elements of H (if there are any). Then any similarity resp. anti-similarity mapping h of E into L can be extended to a like mapping f of H into L. (And, obviously, if h is not the identity, then f is not the identity.)*

Proof. Every $x \in H-E$ determines a gap $(A|B)$ of E. Since L is continuous, there is a $y \in L$ such that $h(A)<y<h(B)$ resp. $h(A)>y>h(B)$; define $f(x)$ to be any such y.

Of course the continuity of L is critical here.

As a corollary to the lemma, any *pseudo-similarity* h on E has an extension to H (and for this, we do not require that E contain the border elements of H). For let the partition $\{E_t\}_{t \in T}$ of E belong to h; then $\{H_t\}_{t \in T}$ is a partition of H, where H_t denotes the smallest interval of H that contains E_t $(t \in T)$. By the lemma, each transformation $h_t=h|i(E_t)$ has an extension to a like transformation f_t of $i(H_t)$ into L, and the desired conclusion now follows at once.

4.16. Definition. A non-empty ordered set M is called *$\aleph_\alpha$-homogeneous* if every non-empty interval of M is of power $\aleph_\alpha$.

4.17. Lemma. *Let L be an ordered set, H an unbordered interval of L, and J an $\aleph_\alpha$-homogeneous subset of H dense in H. Then every function $h \in \mathcal{G}(J,L)$ is non-trivial* $(\aleph_\alpha)$ (cf. 4.11 and 3.1).

Proof. Since every such h is one-one, we need only find a subset of J of power $\aleph_\alpha$ that is free of fixed points of h. This is trivial if h reverses order. Let h preserve order, and denote by A_1 its set of fixed points. Since J is unbordered and h is not the identity, A_1 cannot be dense in H (Corollary 4.13, with $A_0=J$, $D=L$). Consequently, there is an entire interval of H, hence an entire interval of J, free of fixed points of h. The result now follows from the fact that J is $\aleph_\alpha$-homogeneous.

4.18. Corollary. *Let L be an unbordered ordered set, and let D be an $\aleph_\alpha$-homogeneous subset of L dense in L. Then every function $f \in \mathcal{F}(D,L)$ is non-trivial* $(\aleph_\alpha)$.

No proof is required. As a further corollary, it can easily be seen that every *pseudo-similarity* f of J into L, or of D into L, is also non-trivial $(\aleph_\alpha)$, provided that f is not essentially the identity.

5. The fundamental lemma. In this section, we establish the lemma that serves as the link between the decomposition theorems of section 3 (for arbitrary sets) and those of section 6 (for ordered sets). First we introduce an abbreviation that will be convenient in simplifying the statements of our results.

5.1. Definition. Let α be an ordinal, let A, B, D and I be ordered sets, with $I \subset D$ and $|I| \geqslant 2$, and denote by K the smallest interval of D that contains I. Then, by the proposition

$$\mathbf{P}(\aleph_\alpha;\ A,B,D,I),$$

we shall mean the following composite statement:

If

(**P**.a) $\quad |I-A|<\aleph_\alpha$,

then

(**P**.1) $\quad$ the sets A,B^* are dissimilar [5];

if

(**P**.b) $\quad |I-A|<\aleph_\alpha$, $A \cup B \subset D$, and $i(K) \cap A \neq i(K) \cap B$,

then

(**P**.2) $\quad$ the sets A,B are dissimilar;

if

(**P**.c) $\quad |I-A|<\aleph_\alpha$, $A \cup B \subset D$, $A-I$ is essentially nowhere dense in A, and $A \neq' B$,

then

(**P**.3) $\quad$ there is no pseudo-similarity transformation whatsoever of A onto B.

We may now state our lemma as follows.

5.2. Lemma. *Let α be an ordinal, and let A, B, D, I, K, L, U and V be ordered sets, such that*

(5.2a) $\quad$ *L is continuous;*

(5.2b) $\quad$ *$I \subset K \subset D \subset L$, with K the smallest interval of D that contains I, and I $\aleph_\alpha$-homogeneous and dense in K;*

(5.2c) $\quad D \cap V \subset U$;

(5.2d) $\quad |B-D|<\aleph_\alpha$;

(5.2e) $\quad |B \cap U|<\aleph_\alpha$;

(5.2f) $\quad$ *for all $f \in \mathcal{F}(i(K),L)$* (cf. 4.11), *$|V \cap f(D_f \cap I)| \geqslant \aleph_\alpha$, where D_f denotes the domain of f.*

Then the proposition $\mathbf{P}(\aleph_\alpha;\ A,B,D,I)$ (5.1) *holds.*

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[5] As is customary, the symbol "*" signifies inverse order.

Remark. For the proofs of (**P**.1,2), the hypothesis (5.2f) can be weakened by replacing $\mathcal{F}$ therein by $\mathcal{G}$. This detail is of little concern, however, as in the application to the theorems of the next section, the present hypotheses, and, in fact, the even stronger hypotheses of the corollary that follows, are just as easily derivable in full.

Proof of Lemma 5.2. Let g be an assumed pseudo-similarity mapping of A onto B. Put

$$I'=i(I), \qquad K'=i(K),$$

$$A_1=I'\cap g^{-1}(D\cap B), \qquad A_0=K'\cap A \tag{5.2g}$$

($g^{-1}(X)$ denotes the complete inverse image of X). Note that $A_1\subset I'\cap A\subset A_0$.

We show first that $|I'-A_1|<\aleph_\alpha$. From (**P**.a,b,c), we have $|I'-A|<\aleph_\alpha$. If g is one-one, the result then follows easily from (5.2d,g). If g is not one-one, then we assume $B\subset D$ (**P**.c), which, with (5.2g), yields $A_1=I'\cap A$. Then $I'-A_1=I'-A$, and again we have our result.

Since I' is $\aleph_\alpha$-homogeneous and dense in K' (5.2b), it follows that I', K', A_1 and A_0 are all unbordered dense sets, with A_1 dense in each of the others, and A_0 dense in K'.

If g is a similarity or an anti-similarity, then so, of course, is the restriction $g|A_1$. If g is an arbitrary pseudo-similarity, then we assume $A\subset D$ (**P**.c), whence A_0 is an *interval* of A (since K' is an interval of D), and again $g|A_1$ is a pseudo-similarity (cf. 4.8).

Suppose, first, that $g|A_1$ is not essentially the identity. Then (see 4.9) there is a (non-empty) interval J of A_1 such that $g|i(J)$ is either a non-identical similarity or an anti-similarity. Since A_1 is a dense set, the interval $E=i(J)$ of A_1 is unbordered. Now $A_1\subset K'$; let H be the smallest interval of K' that contains E. Then H is unbordered; and, since A_1 is dense in K', E is dense in H. Furthermore, using (5.2g, b) we have

$$g(E)\subset g(A_1)\subset D\subset L.$$

Consequently, by Lemma 4.15 (with $h=g|E$), $g|E$ can be extended to a like mapping $f\in\mathcal{G}(H,L)$ (cf. 4.11). But H is an unbordered interval of K'; therefore $f\in\mathcal{F}(K',L)$. It follows from (5.2f) that

$$|V\cap f(H\cap I')|=|V\cap f(H\cap I)|\geqslant\aleph_\alpha.$$

Now $|(H\cap I')-A_1|<\aleph_\alpha$, since $|I'-A_1|<\aleph_\alpha$. Therefore, since f is single-valued, we have $|V\cap f(H\cap I'\cap A_1)|\geqslant\aleph_\alpha$. Next, we observe that $H\cap I'\cap A_1=H\cap A_1=E$, the first equality being a consequence of the relation $A_1\subset I'$, and the second resulting from the fact that E is an *interval* of A_1. Accordingly, $|V\cap f(E)|\geqslant\aleph_\alpha$. Now $f(E)=g(E)\subset g(A_1)\subset D\cap B$ (5.2g). Consequently, $|V\cap D\cap B|\geqslant\aleph_\alpha$. But $V\cap D\subset U$ (5.2c), so we have $|U\cap B|\geqslant\aleph_\alpha$. This, however, contradicts (5.2e).

Henceforth suppose, then, that $g|A_1$ is essentially the identity. Then g cannot be an anti-similarity (since A_1 is a dense set). We assume, accordingly, that $A\subset D$ and $B\subset D$ (**P**.b,c). As already noted, the former implies that A_0 is an *interval* of A, and the latter that $A_1=I'\cap A$.

If g is a similarity, then $g|A_1$ is also a similarity. Since $g(A_1)\subset D$, it follows from Corollary 4.14 (with $J=A_1$, $h=g|A_1$) that $g|A_1$ is the identity. Next, $g(A_0)\subset B\subset D$, and it is obvious that $g|A_1=(g|A_0)|A_1$. It therefore follows from Corollary 4.13 (with $H=K'$, $h=g|A_0$) that $g|A_0$ is the identity. Consequently, $g(A_0)=A_0\subset K'$, and $A_0=g(A_0)\subset B$, so that $A_0=K'\cap A\subset K'\cap B$.

Now let b be an arbitrary element of B, and put $a=g^{-1}(b)$. If $a<K'$, then, obviously, $a<A_0$, whence, for every $a'\in A_0$, $b=g(a)<g(a')=a'$ (since g is order-preserving and $g|A_0$ is the identity). Thus $a<K'$ implies $b<A_0$. But A_0 is coinitial with K'; therefore $a<K'$ implies $b<K'$. Likewise, $a>K'$ implies $b>K'$. Now $a\in D$, and K' is an *interval* of D. Hence either $a<K'$ or $a>K'$ or $a\in K'$. It therefore follows from the preceding that $b\in K'$ implies $a\in K'$, whence $a\in K'\cap A=A_0$. But then $b=g(a)=a$. Thus $K'\cap B\subset K'\cap A$. With the reverse inclusion established previously, this yields $K'\cap B=K'\cap A$, contradicting (**P**.b).

Finally, if g is an arbitrary pseudo-similarity, we assume (**P**.c) that $A-I$ is essentially nowhere dense in A (cf. 4.4). Therefore $A-I'$ is essentially nowhere dense in A. Then, since A_0 is an *interval* of A, the set $A_0\cap(A-I')$, $=A_0-A_1$, is (essentially) nowhere dense in A_0. Now let $\{J_t\}_{t\in T}$ be a partition of A_1 with respect to which $g|A_1$ is essentially the identity (cf. 4.9). We "refine" this partition, as follows. Put $A'=\bigcup_{t\in T}i(J_t)$. By Lemma 4.7(a), A_1-A' is nowhere dense in A_1. Therefore, since $A_1\subset A_0$, A_1-A' is nowhere dense in A_0. It follows that the set $(A_1-A')\cup(A_0-A_1)$, $=A_0-A'$, is nowhere dense in A_0. So, then, is N, the closure (in A_0) of A_0-A'. It therefore follows from Lemma 4.10 that there is a partition $\mathfrak{E}_0=\{E_s\}_{s\in S}$ of A_0 such that $\bigcup_{s\in S}E_s=A_0-N$. But $A_0-N\subset A'\subset A_1$, and $g|A'$ is the identity. Therefore $\mathfrak{E}_0$ is a partition of A_1, and $g|A_1$ is essentially the identity with respect to this partition. On the other hand, $\mathfrak{E}_0$ is also a partition of A_0; therefore $g|A_0$ itself is essentially the identity with respect to $\mathfrak{E}_0$.

Next, let H_1 resp. H_2 denote the set of all $x\in A$ resp. $y\in A$ for which $x<A_0$ resp. $y>A_0$. Then H_j $(j=1,2)$ is an *interval* of A, except in the case where it consists of only one element. In any event, the set $H_j\cap(A-I')$, $=H_j$, is essentially nowhere dense in itself. By Lemma 4.10, $g|H_j$ is essentially the identity with respect to some partition $\mathfrak{E}_j$ (in fact, the partition consisting of (the null set and) all the two-element intervals of H_j). Since A_0 is unbordered, it is evident that the family $\mathfrak{E}_1\cup\mathfrak{E}_0\cup\mathfrak{E}_2$

is a partition of A [6]), and that g is essentially the identity with respect to this partition. Since $g(A)=B$, we have, therefore, $A='B$, contradicting (**P**.c). This completes the proof of the lemma.

In our applications of this lemma, it will be more convenient to refer instead to the following corollary.

5.3. COROLLARY. *Let a be an ordinal, and let $A, B, D, \widetilde{F}, I, L, M, U$ and V be ordered sets, with*

$$V \cap \widetilde{F}=0 \quad \text{and} \quad B \subset M,$$

and such that

(5.3 a) *L is continuous;*

(5.3 b) *$I \subset \widetilde{F} \subset D \subset L$, with $\widetilde{F}$ $\aleph_a$-homogeneous and dense in D, and I a non-empty interval of $\widetilde{F}$;*

(5.3 c) $$D \cap V \subset U;$$

(5.3 d) $$U \subset V \cup (M-D) \quad \text{and} \quad |B-\widetilde{F}|<\aleph_a,$$

or

(5.3 e) $$B-D \subset U \quad \text{and} \quad |B \cap U|<\aleph_a;$$

(5.3 f) *for all $f \in \mathcal{F}(D,L)$ (cf. 4.11), $|V \cap f(D_f \cap \widetilde{F})| \geqslant \aleph_a$, where D_f denotes the domain of f.*

Then the proposition $\mathbf{P}(\aleph_a; A,B,D,I)$ (see 5.1) *holds.*

Proof. It suffices to verify the hypotheses of Lemma 5.2. First, (5.3 a, c) are merely restatements of (5.2 a, c), respectively. Secondly, (5.3 b) is easily seen to imply (5.2 b). Next, we show that (5.3 f) implies (5.2 f) (where K denotes the smallest interval of D that contains I). Clearly, $i(I)=i(K) \cap \widetilde{F}$. Let $f \in \mathcal{F}(i(K),L)$. Then $D_f \subset i(K)$, whence

$$D_f \cap \widetilde{F} \subset i(K) \cap \widetilde{F}=i(I).$$

Therefore $D_f \cap \widetilde{F} \subset D_f \cap I$. But $f \in \mathcal{F}(D,L)$ (since $i(K)$ is an unbordered interval of D), and our conclusion now follows at once.

Finally, since both (5.2 d, e) are immediate consequences of (5.3 e), it remains to be shown only that these two also follow from (5.3 d). Since $V \cap \widetilde{F}=0$ and $\widetilde{F} \subset D$, the second half of (5.3 d) implies $|B \cap V|<\aleph_a$ and $|B-D|<\aleph_a$, the latter being (5.2 d). Since $B \subset M$, the first half of (5.3 d) yields $B \cap U \subset (B \cap V) \cup (B-D)$. Combining this with the inequalities just established, we find that $|B \cap U|<\aleph_a$, which is (5.2 e).

We shall also make use, in the next section, of the following well-known result.

[6]) In order to be able to draw this conclusion in case A_0 were not unbordered, we would simply have included the first resp. last element of A_0 in the set H_1 resp. H_2.

5.4. LEMMA. *Every subset of C (= reals) of power $\mathfrak{c}$ contains an unbordered $\mathfrak{c}$-homogeneous subset.*

Proof. An unbordered $\mathfrak{c}$-homogeneous subset of $C_1 \subset C$, where $|C_1|=\mathfrak{c}$, is furnished by the set of all points $x \in C_1$ such that every neighbourhood of x contains $\mathfrak{c}$ points of C_1 to the right of x as well as $\mathfrak{c}$ points of C_1 to the left of x.

We conclude this section with the following definition.

5.5. Definition. An *$\aleph_a$-homogeneous decomposition* of an ordered set M is an $\aleph_a$-decomposition of M consisting of sets that are $\aleph_a$-homogeneous (cf. 2.2 and 4.16).

6. The main theorems.

6.1. THEOREM. *Let M be any ordered set of power $\mathfrak{c}$. Suppose that M contains a subset of power $\mathfrak{c}$ that can be imbedded in the continuum C — whence, by Lemma 5.4, M has an unbordered $\mathfrak{c}$-homogeneous subset D that can be imbedded in C. Then M has a $\mathfrak{c}$-homogeneous decomposition $\{E^\sigma\}_{\sigma<\omega_\mathfrak{c}}$* (cf. 5.5) *with the following properties:*

There is associated with every E^σ a certain "distinguished element" $u^\sigma \in E^\sigma$, such that the set $F^\sigma=E^\sigma-\{u^\sigma\}$ is a $\mathfrak{c}$-homogeneous subset of D dense in D.

Denote by U the set of all distinguished elements. Let A, B be any subsets of M such that

(6.1 a) $$|B \cap U|<\mathfrak{c};$$

let σ be any ordinal $<\omega_\mathfrak{c}$, and let I^σ be any non-empty interval of F^σ. Then the proposition $\mathbf{P}(\mathfrak{c}; A,B,D,I^\sigma)$ (see 5.1) *holds.*

Proof. Since D is an unbordered dense set imbeddable in C, we may (if necessary) add suitable elements to the set M so as to obtain a set $M' \supset M$ that has a subset L similar to C, and such that D is dense in L.

It is easily seen that the family of transformations $\mathcal{F}(D,L)$ (cf. 4.11) is at least, and hence exactly, of power $\mathfrak{c}$; denote this family by $\{f_\xi\}_{\xi<\omega_\mathfrak{c}}$. By Corollary 4.18, each of these transformations is non-trivial ($\mathfrak{c}$). Accordingly, the hypotheses of Theorem 3.2, with $\aleph_a=\mathfrak{c}$, $R=L$, and the sets D_ξ the non-empty unbordered intervals (with repetitions) of D, are satisfied. Let V, F and $\{F_\tau\}_{\tau<\omega_\mathfrak{c}}$ be as in the conclusion of Theorem 3.2. From (3.2 a, b), every F_τ is a $\mathfrak{c}$-homogeneous subset of D dense in D.

Put

(6.1 b) $$U=M-F.$$

Since $F \cap V=0$ and $D \cap V \subset D \subset M$, we have $D \cap V \subset M-F$, *i. e.*,

(6.1 c) $$D \cap V \subset U.$$

We shall later rearrange the sequence $\{F_\tau\}$ into a new sequence $\{F^\sigma\}$; for the moment, let us suppose that this has already been done. Let A, B and I^σ be as in (6.1a) ff. We now verify the hypotheses of Corollary 5.3, with $\aleph_\alpha$, A, B, D, L, M, U and V as above, $\widetilde{F}=F^\sigma$, and $I=I^\sigma$. In fact, (5.3f) is a consequence of (3.2d), with $F_\tau=F^\sigma$, (5.3e) follows from (6.1a,b), since $B\subset M$ and $F\subset D$, and all the other verifications are trivial (we ignore (5.3d), since we have (5.3e)). From the conclusions of this corollary, we obtain the corresponding conclusions of the present theorem.

Now from (3.2c) and (6.1c), we find that $|U|=\mathfrak{c}$; let the elements of U be enumerated as $\{u_\tau\}_{\tau<\omega_\mathfrak{c}}$. We shall reorder this sequence into a new sequence $\{u^\sigma\}_{\sigma<\omega_\mathfrak{c}}$, and we shall reorder the sequence $\{F_\tau\}_{\tau<\omega_\mathfrak{c}}$ into a new sequence $\{F^\sigma\}_{\sigma<\omega_\mathfrak{c}}$. Our purpose is to arrange them so that each of the sets $F^\sigma\cup\{u^\sigma\}$ will be $\mathfrak{c}$-homogeneous, hence, since F^σ already is, to see to it that the addition of u^σ does not introduce a jump. Now for each F_τ, there are $\mathfrak{c}$ elements u_ξ available in this way: there are $\mathfrak{c}$ such elements in $D\frown V$ (3.2c, 6.1c), for every one of these occupies a gap of F_τ, since F_τ is dense in D, and $F\frown V=0$.

On the other hand, for each u_τ, there are $\mathfrak{c}$ sets F_ξ available in this way: since every F_ξ is dense in D, there is at most one ξ such that u_τ has an immediate predecessor in F_ξ, and at most one ξ such that u_τ has an immediate successor in F_ξ; for every other value of ξ, either u_τ occupies a gap of F_ξ, or $u_\tau<F_\xi$, or $u_\tau>F_\xi$ (note that F_ξ is unbordered).

Let $\delta<\omega_\mathfrak{c}$, and suppose that u^σ and F^σ have been chosen for every $\sigma<\delta$. Put $U^\delta=\{u^\sigma\}_{\sigma<\delta}$, $\mathfrak{F}^\delta=\{F^\sigma\}_{\sigma<\delta}$, and define $\pi_\delta=$ least $\tau<\omega_\mathfrak{c}$ such that $u_\tau\notin U^\delta$, $\varrho_\delta=$ least $\tau<\omega_\mathfrak{c}$ such that $F_\tau\notin\mathfrak{F}^\delta$. In case $\varrho_\delta\leqslant\pi_\delta$, we put $F^\delta=F_{\varrho_\delta}$ and choose for u^δ the u_ξ of least index ξ such that $F^\delta\cup\{u_\xi\}$ is dense and $u_\xi\notin U^\delta$ (this is possible, since $\delta<\omega_\mathfrak{c}$); while in case $\pi_\delta<\varrho_\delta$, we put $u^\delta=u_{\pi_\delta}$ and choose for F^δ the F_ξ of least index ξ such that $F_\xi\cup\{u^\delta\}$ is dense and $F_\xi\notin\mathfrak{F}^\delta$ (again, this is possible since $\delta<\omega_\mathfrak{c}$).

This completes the constructions of the sequences $\{u^\sigma\}_{\sigma<\omega_\mathfrak{c}}$ and $\{F^\sigma\}_{\sigma<\omega_\mathfrak{c}}$. It is clear that these sequences account for every u_τ and F_τ. Now define

$$E^\sigma=F^\sigma\cup\{u^\sigma\}\qquad(\sigma<\omega_\mathfrak{c});$$

then $\bigcup_{\sigma<\omega_\mathfrak{c}}E^\sigma=M$ (6.1b). This completes our proof.

A few of the particular consequences of this theorem that can be obtained for the case $M=D=C$ are listed in the following corollary. (For the sake of simplicity, they are not stated as strongly as possible.)

6.2. Corollary. *The linear continuum C has a decomposition into $\mathfrak{c}$ mutually exclusive sets E^σ, each of which is $\mathfrak{c}$-homogeneous and dense in C, having the following properties:*

(a) *No two of these sets are anti-similar, and, in fact, no non-empty interval J^σ of any E^σ (in particular, E^σ itself) is anti-similar to any subset whatsoever of any E^τ ($\tau=\sigma$ as well as $\tau\neq\sigma$) — indeed, for any $\mathfrak{m}<\mathfrak{c}$, J^σ is not anti-similar to any subset B of the union of any $\mathfrak{m}$ of the E^τ's.*

(b) *If J^σ is unbordered, then neither is it similar to any such set $B\neq J^\sigma$.*

(c) *Excluding essentially identical mappings, J^σ is not even pseudo-similar to any such set B.*

(d) *Finally, let A be any arbitrary subset of C that excludes less than $\mathfrak{c}$ points of J^σ; then J^σ may be replaced in* (a) *by A, in* (b) *by A provided that A and B meet this J^σ in distinct sets, and in* (c) *by A provided that J^σ excludes only a finite number of points of A.*

Remark. The foregoing theorem and proof can be extended from $\mathfrak{c}$ to any cardinal $2^{\aleph_\alpha}$ for which there exists an (unbordered) continuous $2^{\aleph_\alpha}$-homogeneous set C' having a subset R' of power $\aleph_\alpha$ that is dense in C'. Such sets exist with arbitrarily large α. In fact, let $C_{\alpha+1}$ denote the lexicographically ordered set of all sequences $x=\{x_\xi\}_{\xi<\omega_\alpha}$ such that: (i) for each $\xi<\omega_\alpha$, either $x_\xi=0$ or $x_\xi=1$; (ii) for each $\xi<\omega_\alpha$, there exists a $\tau<\omega_\alpha$ with $\tau>\xi$ and $x_\tau=1$; and (iii) there exists at least one $\sigma<\omega_\alpha$ with $x_\sigma=0$. Denote by R_α the set of all $x\in C_{\alpha+1}$ for which there exists a $\nu<\omega_\alpha$ such that $x_\xi=1$ for every ξ satisfying $\nu<\xi<\omega_\alpha$. Then (cf. Sierpiński [5], [6], p. 57) $C_{\alpha+1}$ is an unbordered, continuous, $2^{\aleph_\alpha}$-homogeneous set, and R_α is dense in $C_{\alpha+1}$. Now let λ be any limit ordinal, and take α to be the number $\pi(\lambda)$ defined by Tarski in [8], p. 9. There exist arbitrary large α of this form, and for any such α, we have $|R_\alpha|=\aleph_\alpha$, as required. (Under the generalized continuum hypothesis, $|R_\alpha|=\aleph_\alpha$ for *every* ordinal α.)

6.3. Lemma. *Let D be a dense set, and let X be a subset of D that is dense in D and such that no two intervals of X are similar. Then, if s,t are any two elements of $D-X$, the sets $Z=X\cup\{s\}$, $W=X\cup\{t\}$ are dissimilar.*

Proof. Say $s<t$. Let g be an assumed similarity mapping of W onto Z. Put $u=g(t)$. Then (cf. 4.1, 4.3) $Z^{(u)}\simeq W^{(t)}=X^{(t)}$ and $Z_{(u)}\simeq W_{(t)}=X_{(t)}$. If $s<u$, then $Z_{(u)}=X_{(u)}$. Hence $X_{(u)}=X_{(t)}$ (since no two intervals of X are similar). Since X is dense in the dense set D, we must then have $u=t$. But this is impossible, since $t\notin X$, while $u\in Z-\{s\}=X$. On the other hand, if $u\leqslant s$, then $Z^{(u)}=X^{(u)}$, whence $X^{(u)}=X^{(t)}$, which is impossible since $u<t$ and X is dense in the dense set D.

6.4. Lemma. *Let D be a dense set, and let X be a subset of D that is dense in D and such that no two intervals of X are similar. Let Y be a subset of D similar to X, φ the (obviously unique) similarity mapping of Y onto X, and $d\in D-X$ be such that $\varphi(Y_{(d)})\neq X_{(d)}$. Then the sets $Z=X\cup\{d\}$, $W=Y\cup\{d\}$ are dissimilar.*

Proof. Let h be an assumed similarity mapping of Z onto W, and put $r=h(d)$. Then (cf. 4.1, 4.3) $W^{(r)}\simeq Z^{(d)}=X^{(d)}$ and $W_{(r)}\simeq Z_{(d)}=X_{(d)}$. Say $r\geqslant d$. Then $W_{(r)}=Y_{(r)}$, so that $Y_{(r)}\simeq X_{(d)}$. Since $\varphi(Y_{(r)})\simeq Y_{(r)}$, we have $\varphi(Y_{(r)})\simeq X_{(d)}$, whence $\varphi(Y_{(r)})=X_{(d)}$ (since similar intervals of X coincide). Thus, by hypothesis, we cannot have $r=d$. Hence $r\in Y$. Therefore $\varphi(r)$ is defined, and $\varphi(Y_{(r)})=X_{(\varphi(r))}$. Thus $X_{(d)}=X_{(\varphi(r))}$. Since X is dense in the dense set D, this implies that $\varphi(r)=d$. This, however, contradicts the relations $\varphi(r)\in X$, $d\notin X$.

6.5. THEOREM. *Let M be any ordered set of power $\mathfrak{c}$ that has a subset D^0 similar to C. Let D be any non-empty unbordered interior interval of D^0. Then M has a $\mathfrak{c}$-homogeneous decomposition $\{E_\sigma\}_{\sigma<\omega_{\mathfrak{c}}}$* (cf. 5.5) *having the following properties: Each E_σ has both a first element and a last element. Denote by E^σ the set remaining when these elements are deleted.*

There is associated with every E^σ a certain "distinguished element" $s^\sigma\in E^\sigma$ such that the set $F^\sigma=E^\sigma-\{s^\sigma\}$ is a $\mathfrak{c}$-homogeneous subset of D dense in D. The sets E_σ are mutually dissimilar. (Hence the sets E^σ are mutually dissimilar.) The sets F^σ, however, are mutually similar. In fact, there is a similarity mapping ψ of D onto C, under which the sets $\psi(F^\sigma)$ are translates of one another. On the other hand, there is, for each σ,τ, only one similarity mapping $g^{\sigma\tau}$ of F^σ onto F^τ.

Let A,B be any subsets of M such that

$$|B-F^0|<\mathfrak{c}, \tag{6.5a}$$

and let I^0 be any non-empty interval of F^0. Then the proposition $\mathbf{P}(\mathfrak{c};A,B,D,I^0)$ (see 5.1) *holds.*

Remark 1. In case D^0 is coinitial resp. cofinal with M, the set D may be taken as an initial resp. final interval of D^0, and the sets E_σ constructed so as to have no first resp. last element. If D^0 is both coinitial and cofinal with M, then D may be taken as D^0 itself, and every E_σ unbordered (*i. e.*, $E_\sigma=E^\sigma$).

Remark 2. The conclusions concerning the sets A and B may be extended by making use of the mutual similarity of the sets F^σ, in an obvious way.

Proof of Theorem 6.5. Designate the family $\mathcal{F}(D,D)$ (which is of power $\mathfrak{c}$) as $\{f_\xi\}_{\xi<\omega_{\mathfrak{c}}}$. By Corollary 4.18, these transformations are non-trivial ($\mathfrak{c}$). Let ψ be a fixed similarity transformation of D onto C. Let $\{\varphi'_\tau\}_{\tau<\omega_{\mathfrak{c}}}$ be an enumeration of all the translations of C, with φ'_0 the identity, and put

$$\varphi_\tau=\psi^{-1}\varphi'_\tau\psi \qquad (\tau<\omega_{\mathfrak{c}}).$$

Then $\{\varphi_\tau\}_{\tau<\omega_{\mathfrak{c}}}$ is a family of similarity transformations of D onto D, with φ_0 the identity. Moreover, this family are completely distinct ($\mathfrak{c}$), since the family of all translations of C are completely distinct ($\mathfrak{c}$) (cf. 3.3).

The hypotheses of Theorem 3.4, with $\aleph_\alpha=\mathfrak{c}$, $R=D$, and the sets D_ξ the non-empty unbordered intervals (with repetitions) of D, are now satisfied. Let V, F and $\{F_\tau\}_{\tau<\omega_{\mathfrak{c}}}$ be as in the conclusion of that theorem. From (3.4e), we have $|f_\xi(D_\xi\cap F_0)|=\mathfrak{c}$ $(\xi<\omega_{\mathfrak{c}})$, whence $|D_\xi\cap F_0|=\mathfrak{c}$ $(\xi<\omega_{\mathfrak{c}})$, since every f_ξ is single-valued; thus F_0 is a $\mathfrak{c}$-homogeneous subset of D dense in D. Obviously, $\psi(F_0)$ has the corresponding property as a subset of C, and clearly every translate $\varphi'_\tau(\psi(F_0))$ has it likewise. Consequently, each of the mutually similar sets $F_\tau=\varphi_{\gamma_\tau}(F_0)$ (see (3.4c)) is a $\mathfrak{c}$-homogeneous subset of D dense in D.

Now put

$$U=V\cup(M-D^0).$$

Let A, B and I^0 be as in (6.5a) ff., with $F^0=F_0$. We verify the hypotheses of Corollary 5.3, with $\aleph_\alpha$, A, B, D, M, U and V as above, $\widetilde{F}=F^0$, $I=I^0$, and $L=D$. In fact, (5.3f) is a consequence of (3.4e), and the other verifications are all trivial (here we obtain (5.3d), and therefore ignore (5.3e)). From the conclusions of this corollary, we obtain the corresponding conclusions of our present theorem.

In particular, no two intervals of F_0 are similar. Now $F_\tau\simeq F_0$ $(\tau<\omega_{\mathfrak{c}})$; therefore no two intervals of F_τ are similar. Clearly, any similarity transformation of such a set onto any other fixed set is unique (cf. the statement of Lemma 6.4).

Now put $N=$ smallest interval of M that contains D, and define

$$S=N-F; \tag{6.5b}$$

from (3.4b), we have

$$|D-F|=\mathfrak{c}. \tag{6.5c}$$

Hence $|S|=\mathfrak{c}$; let the elements of S be $\{s_\tau\}_{\tau<\omega_{\mathfrak{c}}}$. We shall rearrange this sequence into a new one, $\{s^\sigma\}_{\sigma<\omega_{\mathfrak{c}}}$, and we shall reorder the sequence $\{F_\tau\}_{\tau<\omega_{\mathfrak{c}}}$ into a new sequence $\{F^\sigma\}_{\sigma<\omega_{\mathfrak{c}}}$. We do this in such a way that the sets $F^\sigma\cup\{s^\sigma\}$ will be $\mathfrak{c}$-homogeneous and mutually dissimilar. For the first property, since F^σ is already $\mathfrak{c}$-homogeneous, we need only avoid introducing a jump upon adding s^σ.

Let $\delta<\omega_{\mathfrak{c}}$, and suppose that s^σ and F^σ have been chosen for all $\sigma<\delta$. Put $S^\delta=\{s^\sigma\}_{\sigma<\delta}$, $\mathfrak{F}^\delta=\{F^\sigma\}_{\sigma<\delta}$; and define $\pi_\delta=$ least $\tau<\omega_{\mathfrak{c}}$ such that $s_\tau\notin S^\delta$, $\varrho_\delta=$ least $\tau<\omega_{\mathfrak{c}}$ such that $F_\tau\notin\mathfrak{F}^\delta$. In case $\varrho_\delta\leqslant\pi_\delta$, put $F^\delta=F_{\varrho_\delta}$, and consider the sets $F^\delta\cup\{s_\tau\}$ for various τ. Since F^δ is dense in D, and $D-F\subset N-F=S$, we see from (6.5c) that there are $\mathfrak{c}$ elements $s_{\tau'}\in S\cap D$ that occupy gaps of F^δ, and hence do not create jumps if added to F^δ. Since no two intervals of F^δ are similar, we see further from Lemma 6.3 that the corresponding $\mathfrak{c}$ sets $F^\delta\cup\{s_{\tau'}\}$ are mutually dissimilar. Therefore, since $\delta<\omega_{\mathfrak{c}}$, we can choose s^δ as the s_τ of least index τ such that $s_\tau\notin S^\delta$

and such that $F^{\delta} \cup \{s_{\tau}\}$ is dense and is dissimilar to every $F^{\sigma} \cup \{s^{\sigma}\}$ with $\sigma < \delta$.

If, on the other hand, $\pi_{\delta} < \varrho_{\delta}$, we put $s^{\delta} = s_{\pi_{\delta}}$, and consider the sets $F_{\tau} \cup \{s^{\delta}\}$ for various τ. Since every F_{τ} is dense in D, there is at most one τ such that s^{δ} has an immediate predecessor in F_{τ}, and at most one τ such that s^{δ} has an immediate successor in F_{τ}. Hence there are $\mathfrak{c}$ sets $F_{\tau'}$ in each of which s^{δ} occupies a gap. Let F' denote the union of these sets $F_{\tau'}$; then s^{δ} occupies a gap of F'. Let d^{δ} be that element of D that occupies this same gap (if $s^{\delta} \in D$, then $d^{\delta} = s^{\delta}$). Then $\psi(d^{\delta})$ occupies a gap of $\psi(F')$, ψ being the given similarity mapping of D onto C.

Since the sets $\psi(F_{\tau'})$ are translates of one another, the element $\psi(d^{\delta})$ cannot occupy corresponding gaps of any two such sets; therefore d^{δ} cannot occupy corresponding gaps of any two of our sets $F_{\tau'}$ ("corresponding" referring, in each case, to the unique similarity transformation between the sets in question). It now follows from Lemma 6.4 that no two of the $\mathfrak{c}$ dense sets $F_{\tau'} \cup \{d^{\delta}\}$ are similar. Hence, since $\delta < \omega_{\mathfrak{c}}$, we can choose F^{δ} as the F_{τ} of least index τ such that $F_{\tau} \notin \mathfrak{F}^{\delta}$ and such that $F_{\tau} \cup \{s^{\delta}\}$ is dense and is dissimilar to every $F^{\sigma} \cup \{s^{\sigma}\}$ with $\sigma < \delta$.

This completes the construction of the sequences $\{s^{\sigma}\}_{\sigma<\omega_{\mathfrak{c}}}$ and $\{F^{\sigma}\}_{\sigma<\omega_{\mathfrak{c}}}$. Clearly, these sequences account for every s_{τ} and every F_{τ}. Now define

$$E^{\sigma} = F^{\sigma} \cup \{s^{\sigma}\} \qquad (\sigma < \omega_{\mathfrak{c}}).$$

These sets are mutually dissimilar. By (6.5b), their union is N.

Finally, write

$$D^0 = D^1 + D + D^2, \qquad M = N^1 + N + N^2.$$

We have $N^1 \supset D^1$, $N^2 \supset D^2$, and $N^1 < D < N^2$. Since D is an interior interval of the set D^0, and $D^0 \simeq C$, we have $|D^1| = |D^2| = \mathfrak{c}$, whence $|N^2| = |N^1| = \mathfrak{c}$ (with the appropriate modification in case D is not an interior interval of D^0; see Remark 1). Write

$$N^1 = \{n^{1,\sigma}\}_{\sigma<\omega_{\mathfrak{c}}}, \qquad N^2 = \{n^{2,\sigma}\}_{\sigma<\omega_{\mathfrak{c}}}.$$

Now $N^1 < E^{\sigma} < N^2$, since $E^{\sigma} \subset N$ $(\sigma < \omega_{\mathfrak{c}})$. We may thus define

$$E_{\sigma} = \{n^{1,\sigma}\} + E^{\sigma} + \{n^{2,\sigma}\} \qquad (\sigma < \omega_{\mathfrak{c}}),$$

and now our proof is complete.

A few of the particular consequences of this theorem that can be obtained for the case $M = D = C$ (see Remark 1) are listed in the following corollary. (For the sake of simplicity, they are not stated as strongly as possible.)

6.6. COROLLARY. *The linear continuum C has a decomposition into $\mathfrak{c}$ mutually exclusive sets E^{σ}, each of which is $\mathfrak{c}$-homogeneous and dense in C, having the following properties:*

(a) *No two of the sets E^{σ} are similar. However, upon deleting a properly chosen element from each E^{σ}, the resulting sets F^{σ} are not only mutually similar, but congruent, i. e., any F^{σ} is the image of any F^{τ} under a suitable translation of C onto itself (and this translation is the unique similarity mapping of F^{τ} onto F^{σ}).*

(b) *No non-empty interval J^{σ} of any E^{σ} (in particular, E^{σ} itself) is anti-similar to any subset whatsoever of E^{σ}, or, in fact, to any subset B of C that has less than $\mathfrak{c}$ elements not in E^{σ}.*

(c) *If J^{σ} is unbordered, then neither is it similar to any such set $B \neq J^{\sigma}$.*

(d) *Excluding essentially identical mappings, J^{σ} is not even pseudo-similar to any such set B.*

(e) *Finally, let A be any arbitrary subset of C that excludes less than $\mathfrak{c}$ points of J^{σ}; then J^{σ} may be replaced in* (b) *by A, in* (c) *by A provided that A and B meet this J^{σ} in distinct sets, and in* (d) *by A provided that J^{σ} excludes only a finite number of points of A.*

The only property listed here that is not implicit in the statement of the theorem is the mutual congruence of the sets F^{σ}; to achieve this, we simply choose the mapping ψ in the proof of the theorem to be the identity.

6.7. THEOREM. *Let M be any ordered set of power $\mathfrak{c}$ that contains a subset D similar to C and both coinitial and cofinal with M. Then M has a $\mathfrak{c}$-homogeneous decomposition $\{F^{\tau}\}_{\tau<\omega_{\mathfrak{c}}}$* (cf. 5.5) *with the following properties:*

The sets F^{τ} are mutually similar [7]*. On the other hand, for all σ, τ, there is just one similarity mapping $g^{\sigma\tau}$ of F^{σ} onto F^{τ}.*

Let A, B be any subsets of M such that

(6.7a) $$|B - F^0| < \mathfrak{c},$$

and let I^0 be any non-empty interval of F^0. Then the proposition $\mathbf{P}(\mathfrak{c}; A, B, D, I^0)$ (see 5.1) *holds.*

Remark. The conclusions concerning the sets A and B may be extended by making use of the mutual similarity of the sets F^{τ}, in an obvious way (cf. Theorem 6.5, Remark 2).

Proof of Theorem 6.7. Designate the family $\mathcal{F}(D, D)$ as $\{f_{\xi}\}_{\xi<\omega_{\mathfrak{c}}}$ (this family being of power $\mathfrak{c}$). By Corollary 4.18, these transformations are non-trivial ($\mathfrak{c}$).

[7]) See footnote [1]).

Next, let x be any element of D and let y be any element of M. Say $x \leqslant y$ (the opposite case being treated analogously). Since $x \in D$, and since D is similar to C and both coinitial and cofinal with M, the intervals $D^{(x)}$, $D_{(x)}$, and $i(D_{(y)})$ (see 4.1, 4.3, and 4.5) are all similar to C (if $y \in D$, then $D_{(y)} \simeq C$, but if $y \notin D$, then $D_{(y)}$ may have a first element). Let $\psi^x, \psi_y, \psi_{x,y}$ be any fixed similarity transformations of $D^{(x)}$ onto C, $i(D_{(y)})$ onto C, $D_{(x)}$ onto $i(D_{(y)})$, respectively. Denote the family of all translations of C by $\{\varphi'_\sigma\}_{\sigma<\omega_c}$, with φ'_0 the identity. For each σ, with $0<\sigma<\omega_c$, define a similarity transformation $\varphi_{\sigma;x,y}$ from D into $D \cup \{y\}$, as follows:

$$(6.7\text{b}) \qquad \begin{aligned} \varphi_{\sigma;x,y} &= (\psi^x)^{-1} \varphi'_\sigma \psi^x \quad \text{on} \quad D^{(x)}, \\ \varphi_{\sigma;x,y} &= \psi_y^{-1} \varphi'_\sigma \psi_y \psi_{x,y} \quad \text{on} \quad D_{(x)}, \\ \varphi_{\sigma;x,y}(x) &= y. \end{aligned}$$

Clearly, for each such x and y, the family

$$\{\varphi_{\sigma;x,y} \mid (D-\{x\})\}_{0<\sigma<\omega_c}$$

are completely distinct ($\mathfrak{c}$) (cf. 3.3) from $D-\{x\}$ into M. For at least one $x \in D$, let us choose $\psi_{x,x}$ to be the identity (on $D_{(x)}$). Then, for this x, the transformation $\varphi_0 = \varphi_{0;x,x}$ will be the identity on D (by (6.7b), since φ'_0 is the identity on C). Accordingly, for this x, the enlarged family

$$\{\varphi_{\sigma;x,x} \mid (D-\{x\})\}_{0<\sigma<\omega_c} \cup \{\varphi_0 \mid (D-\{x\})\}$$

will certainly still be completely distinct ($\mathfrak{c}$) from $D-\{x\}$ into M. Consequently, the family

$$\Phi = \{\varphi_{\sigma;x,y}\}_{0<\sigma<\omega_c, x\in D, y \in M} \cup \{\varphi_0\}$$

are a family of pseudo-distinct ($\mathfrak{c}$) similarity transformations (cf. 3.5) from D into M. Clearly Φ is of power $\mathfrak{c}$; we write $\Phi = \{\varphi_\tau\}_{\tau<\omega_c}$ (φ_0 = identity).

The hypotheses of Theorem 3.6, with $\aleph_\alpha = \mathfrak{c}$, $R = M$, and the sets D_ξ the non-empty unbordered intervals (with repetitions) of D, are now satisfied. Let $\{F^\tau\}_{\tau<\omega_c}$ be as in the conclusion of that theorem. From (3.6c), we have

$$|f_\xi(D_\xi \cap F^0)| = \mathfrak{c} \quad (\xi<\omega_c), \qquad \text{whence} \qquad |D_\xi \cap F^0| = \mathfrak{c} \quad (\xi<\omega_c),$$

since every f_ξ is single-valued. Since $F^0 \subset D$ (3.6a), this shows that F^0 is $\mathfrak{c}$-homogeneous and dense in D. Therefore, since the sets F^τ ($\tau<\omega_c$) are mutually similar (3.6b), every F^τ is $\mathfrak{c}$-homogeneous.

Put

$$V = M - F^0, \qquad U = V \cup (M-D).$$

Let A, B and I^0 be as in (6.7a) ff. We verify the hypotheses of Corollary 5.3, with $\aleph_\alpha$, A, B, D, M, U and V as above, $F = F^0$, $I = I^0$, and $L = D$. In fact, (5.3f) is a consequence of (3.6c), and all the other verifications are trivial (here we obtain (5.3d), so we ignore (5.3e)). The conclusions of this corollary imply the corresponding conclusions of the present theorem. In particular, no two intervals of F^0 are similar. Hence no two intervals of F^τ are similar, since $F^\tau \simeq F^0$ ($\tau<\omega_c$). Therefore any similarity transformation of any F^σ onto any F^τ is necessarily unique. This completes the proof of the theorem.

A few of the particular consequences of this theorem that can be obtained for the case $M = D = C$ are listed in the following corollary. (For the sake of simplicity, they are not stated as strongly as possible.)

6.8. Corollary. *The linear continuum C has a decomposition into $\mathfrak{c}$ mutually exclusive, mutually similar, $\mathfrak{c}$-homogeneous sets F^τ, having the following properties:*

(a) *No non-empty interval I^τ of any F^τ (in particular, F^τ itself) is anti-similar to any subset whatsoever of F^τ, or, in fact, to any subset B of C that has less than $\mathfrak{c}$ elements not in F^τ.*

(b) *If I^τ is unbordered, then neither is it similar to any such set $B \neq I^\tau$.*

(c) *Excluding essentially identical mappings, I^τ is not even pseudo-similar to any such set B.*

(d) *Finally, let A be any arbitrary subset of C that excludes less than $\mathfrak{c}$ points of I^τ; then I^τ may be replaced in* (a) *by A, in* (b) *by A provided that A and B meet this I^τ in distinct sets, and in* (c) *by A provided that I^τ excludes only a finite number of points of A.*

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