1° from equality $\dot{Z}_i^n = Z_i^{n-1} - \varphi_i$, where φ_i is a chain in F, we infer that Z_i^{n-1} is an F-cycle in M_i;

2° since $\dot{Z}^n = (Z^{n-1} - Z_1^{n-1}) - (\varphi_0 - \varphi_1)$ is a chain in F, we infer that $Z_0^{n-1} = Z_1^{n-1}$.

Let $Z^{n-1} = Z_0^{n-1} = Z_1^{n-1}$. By 1° and 2° Z^{n-1} is an F-cycle in $M_0 \cdot M_1$. Hence (b) yields an n-dimensional true chain P^n in $M_0 \cdot M_1$ such that $\dot{P}^n = Z^{n-1} - \psi$, ψ in F. Putting

(20) $$R_i^n = Z_i^n - P^n$$

we infer from equality $\dot{R}_i^n = \psi - \varphi_i$ that R_i^n is an n-dimensional true F-cycle in M_i. Thus (a) yields an $(n+1)$-dimensional true chain Q_i^{n+1} in M_i such that

(21) $$\dot{Q}_i^{n+1} = R_i^n - \chi_i, \qquad \chi_i \text{ in } F.$$

From (19), (20), (21) we have

$$\dot{Q}_0^{n+1} - \dot{Q}_1^{n+1} = (R_0^n - R_1^n) - (\chi_0 - \chi_1)$$
$$= (Z_0^n - Z_1^n) - (P^n - P^n) - (\chi_0 - \chi_1) = Z^n - (\chi_0 - \chi_1),$$

hence $Z^n \underset{F}{\widetilde{=}} 0$ in M and the proof is completed.

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A formula with no recursively enumerable model

by

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G. Kreisel [4] was the first to construct a first-order formula which has no recursive model[1]). A formula with the same property was also constructed independently by the present author in a paper read before the VIII Congress of the Polish Mathematicians in the autumn of 1953 (see Mostowski [6]). Both formulas were obtained by suitable modifications of the axioms of the set-theory proposed by Bernays [1].

The present paper contains another example of a formula which has no recursive model. This example seems to be simpler than the former ones in so far, as it makes no reference to the axiomatic set-theory and uses exclusively tools known from the theory of recursive functions.

The formula to be given below was found in the course of unsuccessful attempts to construct a formula no model of which would belong to the smallest field of sets generated by the classes $\boldsymbol{P}_1^{(n)}$ and $\boldsymbol{Q}_1^{(n)}$ [2]). It is published in the hope that it might suggest a solution of this problem.

It has been justly observed that many recent papers in the field of symbolic logic do not supply full proofs of the statements they contain. While it would certainly not be reasonable to require from all papers to give exhaustive proofs it is certainly necessary to publish full proofs from time to time. This line is followed in the present paper.

1. Post's theory of recursively enumerable sets [7]. Let G be a free semigroup (with cancellation) generated by the free generators a, b, c. Thus the elements of G are finite strings $x_1 x_2 \ldots x$ where each x_j is either a or b or c and the multiplication of strings is performed simply by juxtaposing them. The void string is not admitted in G. Elements of G will be denoted by lower case Greek letters. The length $l(\alpha)$ of a string α is defined as the number of letters it contains.

A string α is said to be (a) a *segment* of β; (b) a *part* of β; (c) a *rest* of β if either $\alpha = \beta$ or (a) there is a γ such that $\beta = \alpha\gamma$; (b) there are γ, δ

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1) Kreisel's paper contains even a slightly stronger result; cf. the theorem on p. 47 of his paper.

2) For terminology see my paper [5]. The problem was formulated by Kreisel [4]; cf. p. 47.

such that either $\beta=\gamma\alpha$ or $\beta=\alpha\delta$ or $\beta=\gamma\alpha\delta$; (c) there is a γ such that $\beta=\gamma\alpha$.

A string α is an *ingredient* of β if c is not a part of α and $c\alpha c$ is a part of β[3]. It is evident that each string has at most a finite number of ingredients.

Let α,β be ingredients of a string γ. We say that α *precedes* β in γ if there is a segment δ of γ such that $c\alpha c$ is a part of δ and $c\beta c$ is not a part of δ.

A string α is the *first (last) ingredient* of a string γ if there is a string δ containing no occurrences of a's and of b's such that $\delta\alpha c$ (resp. $c\alpha\delta$) is a segment (rest) of γ. This definition is independent of the ordering determined by the precedence-relation.

A string consisting of n consecutive a's is denoted by λ_n. A string is called *proper* if c is not its part.

Let

(1) $$B:\quad \alpha,\beta_1,\beta_1',\beta_2,\beta_2',\ldots,\beta_k,\beta_k'$$

be a sequence of $2k+1$ proper strings. Such a sequence is called a *basis*.

A string γ is said to be *B-generating* if it has the following properties:

(2) α is the first ingredient of γ;

(3) if ξ is an ingredient of γ, then either $\xi=\alpha$ or there are: an ingredient η of γ which precedes ξ in γ; a part ζ of η; and an integer $j\leqslant k$ such that $\xi=\zeta\beta_j'$ and $\eta=\beta_j\zeta$.

For any basis B let S_B be the set of integers n with the following property: *there is a B-generating string γ such that λ_n is the last ingredient of γ.*

Using these definitions we can formulate the following

THEOREM 1 (Post)[4]. *For every recursively enumerable set X there is a basis B such that $X=S_B$.*

2. The formula $\mathcal{F}_1$.

The formula $\mathcal{F}$ to be constructed will be a conjunction of two formulas $\mathcal{F}_1$ and $\mathcal{F}_2$. In the present section only the first formula will be defined. It contains 5 predicate-variables, three of which (A,B,C) have one argument, one (F) has two arguments and one (G) has three arguments.

The intended interpretation of the formula $\mathcal{F}_1$ is simply that the universe of discourse is the semigroup $\boldsymbol{G}$ defined in section 1. The interpretation of the formulas A(x), B(x), C(x) is that x is one of the generators of $\boldsymbol{G}$. The formula F(x,y) is to be interpreted as: "x and y are identic" and the formula G(x,y,z) as: "x is the result of multiplication of y and z".

[3]) The idea of "ingredients" is due to Quine. See [8], p. 296.

[4]) This theorem easily results from the theorem obtained by Post [7]; in the proof we have to use the technique developed by Quine [8], p. 296 seq.

The formula $\mathcal{F}_1$ is the conjunction of the following 10 formulas:

I. (Axioms of identity).

I1. $(x,y)\{F(x,y)\supset[A(x)\equiv A(y)].[B(x)\equiv B(y)].[C(x)\equiv C(y)]\}$;

I2. $(x,y,z,t)\{F(x,y)\supset[G(x,z,t)\equiv G(y,z,t)].[G(z,x,t)\equiv G(z,y,t)].[G(z,t,x)\equiv G(z,t,y)]\}$.

II. (Axioms of existence).

II1. $(\exists x,y,z)[A(x).B(y).C(z)]$;

II2. $(x,y)\{[A(x).A(y)\vee B(x).B(y)\vee C(x).C(y)]\supset F(x,y)\}$;

II3. $\sim\{(\exists x)[A(x).B(x)\vee B(x).C(x)\vee C(x).A(x)]\}$.

III. (Axioms of juxtaposition).

III1. $(x,y)(\exists z)\,G(z,x,y)$;

III2. $(x,y,z,t)\{[G(z,x,y).G(t,x,y)\vee G(x,z,y).G(x,t,y)\vee G(x,y,z).G(x,y,t)]\supset F(z,t)\}$;

III3. $(x)\{(\exists y,z)G(x,y,z)\equiv\sim[A(x)\vee B(x)\vee C(x)]\}$;

III4. $(x,y,z,t,v)[G(x,y,z).G(y,u,v).G(t,v,z)\supset G(x,u,t)]$;

III5. $(x,y,z,u,v)\{\sim F(y,u).G(x,y,z).G(x,u,v)\supset(\exists t)[G(y,u,t).G(v,t,z)\vee G(u,y,t).G(z,t,v)]\}$.

3. Auxiliary definitions.

The formula $\mathcal{F}_2$ will be written down by means of a number of auxiliary formulas listed below. In the intended interpretation described at the beginning of section 2 most of these formulas describe notions which we have introduced in section 1[5]).

$xSy\equiv F(x,y)\vee(\exists z)G(y,x,z)$ [x is a segment of y].

$xPy\equiv F(x,y)\vee(\exists z,t,u)[G(y,x,z)\vee G(y,z,x)\vee G(z,t,x).G(y,z,u)]$ [x is a part of y].

$xRy\equiv F(x,y)\vee(\exists z)G(y,z,x)$ [x is a rest of y].

$xIy\equiv(\exists z,t,u)[C(u).G(t,x,u).G(z,u,t).(zPy).\sim(uPx)]$ [x is an ingredient of y].

$V(x,y,z)\equiv(xIz).(yIz).(\exists u,v,w,t,s,r)[C(u).G(v,x,u).G(w,u,v).G(t,y,u).G(s,u,t).(rSz).(wPr).\sim(sPr)]$ [x precedes y in z].

$CS(x)\equiv(u)\{[A(u)\vee B(u)]\supset\sim(uPx)\}$ [x contains no a's and no b's].

$xFIy\equiv(\exists u,v,w,t)[CS(u).C(v).G(w,x,v).G(t,u,w).(tSy)]$ [x is the first ingredient of y].

$xLIy\equiv(\exists u,v,w,t)[CS(u).C(v).G(w,v,x).G(t,w,u).(tRy)]$ [x is the last ingredient of y].

$LN(x)\equiv(y)\{[B(y)\vee C(y)]\supset\sim(yPx)\}$ [x has the form λ_n].

[5]) The ideas underlying our construction in this section are due to Quine [8].

We shall now associate a formula $\Gamma_\alpha(\mathrm{x})$ with each string α. We shall call this formula a *description* of α. The definition is an inductive one:

$$\Gamma_a(\mathrm{x})\equiv\mathrm{A}(\mathrm{x}),\qquad \Gamma_b(\mathrm{x})\equiv\mathrm{B}(\mathrm{x}),\qquad \Gamma_c(\mathrm{x})\equiv\mathrm{C}(\mathrm{x}),$$
$$\Gamma_{\alpha a}(\mathrm{x})\equiv(\exists\mathrm{u},\mathrm{v})[\Gamma_\alpha(\mathrm{u}).\mathrm{A}(\mathrm{v}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v})],$$
$$\Gamma_{\alpha b}(\mathrm{x})\equiv(\exists\mathrm{u},\mathrm{v})[\Gamma_\alpha(\mathrm{u}).\mathrm{B}(\mathrm{v}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v})],$$
$$\Gamma_{\alpha c}(\mathrm{x})\equiv(\exists\mathrm{u},\mathrm{v})[\Gamma_\alpha(\mathrm{u}).\mathrm{C}(\mathrm{v}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v})].$$

For arbitrary strings β,β' we put

$$H_{\beta,\beta'}(\mathrm{w},\mathrm{v})\equiv(\exists\mathrm{z},\mathrm{t},\mathrm{u})[\Gamma_\beta(\mathrm{z}).\Gamma_{\beta'}(\mathrm{t}).\mathrm{G}(\mathrm{v},\mathrm{z},\mathrm{u}).\mathrm{G}(\mathrm{w},\mathrm{u},\mathrm{t})]$$

[w has the form $\zeta\beta'$ and v the form $\beta\zeta$].

Finally, for an arbitrary basis B defined in (1), we put

$$\Xi_B(\mathrm{x})\equiv(\exists\mathrm{z})[(\mathrm{zFIx}).\Gamma_\alpha(\mathrm{z})].(\mathrm{u})[(\mathrm{uIx})$$
$$\supset\{\Gamma_\alpha(\mathrm{u})\vee(\exists\mathrm{v})[\mathrm{V}(\mathrm{v},\mathrm{u},\mathrm{x}).\sum_{j=1}^{k}H_{\beta_j,\beta_j'}(\mathrm{u},\mathrm{v})]\}]\ ^{6})$$

[x is a B-generating string].

4. The formula $\mathcal{F}_2$. Let X,Y be two disjoint recursively enumerable sets of integers which cannot be separated by means of recursive sets (see [2] and [9]) and let (1) and

$$C:\quad \gamma,\delta_1,\delta_1',\dots,\delta_l,\delta_l'$$

be bases such that $X=S_B$, $Y=S_C$.

Let D and E be predicate-variables with one argument. We define $\mathcal{F}_2$ as the conjunction of the following four formulas:

IV1. $(\mathrm{x},\mathrm{y})[\Xi_B(\mathrm{y}).(\mathrm{xLIy}).\mathrm{LN}(\mathrm{x})\supset\mathrm{D}(\mathrm{x})]$,

IV2. $(\mathrm{x},\mathrm{y})[\Xi_C(\mathrm{y}).(\mathrm{xLIy}).\mathrm{LN}(\mathrm{x})\supset\mathrm{E}(\mathrm{x})]$,

IV3. $(\mathrm{x})\{\mathrm{LN}(\mathrm{x})\supset[\mathrm{D}(\mathrm{x})\vee\mathrm{E}(\mathrm{x})]\}$,

IV4. $(\mathrm{x})[\sim\mathrm{D}(\mathrm{x})\vee\sim\mathrm{E}(\mathrm{x})]$.

We denote by $\mathcal{F}$ the conjunction of $\mathcal{F}_1$ and $\mathcal{F}_2$.

5. Consistency of $\mathcal{F}$. Let *G* be the semigroup described in section 1 and let $\mathrm{M}(\mathrm{x}_1,\dots,\mathrm{x}_k)$ be a formula with the free individual variables $\mathrm{x}_1,\dots,\mathrm{x}_k$ and the predicate-variables A, B, C, F, G. If strings $\alpha_1,\dots,\alpha_k$ satisfy M when A, B, C, F, G are interpreted as described at the beginning of section 2, then we shall write $\vdash\mathrm{M}(\alpha_1,\dots,\alpha_k)$. If M has no free individual variables, then $\vdash\mathrm{M}$ means that M is true in the model defined by the intended interpretation of A, B, C, F, G. This model will be called the *natural model.*

[6]) The letter Σ is here used as the symbol of alternation with finitely many terms. An alternation with 0 terms (a void alternation) is assumed to have the truth-value 0 (falsity).

The following lemmas are obvious:

L.1. $\vdash\mathcal{F}_1$.

L.2. *For arbitrary strings α,β,γ the following equivalences hold* [7]):

$\vdash\alpha\mathrm{S}\beta\equiv\alpha$ *is a segment of* β,

$\vdash\alpha\mathrm{P}\beta\equiv\alpha$ *is a part of* β,

$\vdash\alpha\mathrm{R}\beta\equiv\alpha$ *is the rest of* β,

$\vdash\alpha\mathrm{I}\beta\equiv\alpha$ *is an ingredient of* β,

$\vdash\mathrm{V}(\alpha,\beta,\gamma)\equiv\alpha$ *and* β *are ingredients of* γ *and* α *precedes* β *in* γ,

$\vdash\mathrm{CS}(\alpha)\equiv$ *neither* a *nor* b *occurs in* α,

$\vdash\alpha\mathrm{FI}\beta\equiv\alpha$ *is the first ingredient of* β,

$\vdash\alpha\mathrm{LI}\beta\equiv\alpha$ *is the last ingredient of* β,

$\vdash\mathrm{LN}(\alpha)\equiv\alpha$ *has the form* λ_n.

L.3. $\vdash\Gamma_\alpha(\beta)\equiv(\beta=\alpha)$.

L.4. *If* B *is the basis* (1), *then* $\vdash\Xi_B(\gamma)\equiv\gamma$ *is a* B*-generating string.*

L.5. *If* B *is the basis* (1), *then the following conditions are equivalent:*

(4) $$\vdash\Xi_B(\gamma).(\delta\mathrm{LI}\gamma).\mathrm{LN}(\delta),$$

(5) $\{\gamma$ *is a* B*-generating string*$\}.(\exists n)[(n\,\epsilon\,S_B).(\delta=\lambda_n).$
$.(\delta$ *is the last ingredient of* $\gamma)]$.

Proof. If (4) is satisfied, then by L.4 γ is a B-generating string and by L.2 δ is the last ingredient of γ and has the form λ_n. Hence $n\,\epsilon\,S_B$ and (5) is satisfied. The converse implication results immediately from L.2 and L.4.

Now let B and C be bases as described at the beginning of section 4. We extend the natural model of $\mathcal{F}_1$ to a model of $\mathcal{F}_2$ by interpreting D as the set of strings λ_n such that $n\,\epsilon\,X$, and E as the set of strings λ_n such that n non $\epsilon\,X$. If $\mathrm{M}(\mathrm{x}_1,\dots,\mathrm{x}_n)$ is a formula containing the individual variables $\mathrm{x}_1,\dots,\mathrm{x}_n$ and the predicate-variables A, B, C, D, E, F, G, then we shall continue to use the symbol $\vdash\mathrm{M}(\alpha_1,\dots,\alpha_n)$ to express the fact that $\alpha_1,\dots,\alpha_n$ satisfy $\mathrm{M}(\mathrm{x}_1,\dots,\mathrm{x}_n)$ in the extended natural model.

L.6. *Formulas* IV3 *and* IV4 *are true in the extended natural model.*

Proof. Obvious.

L.7. *Formulas* IV1 *and* IV2 *are true in the extended natural model.*

Proof. Let γ and δ be strings satisfying (4). We have to prove that $\vdash\mathrm{D}(\delta)$, *i. e.* that there is an integer n such that $\delta=\lambda_n$ and $n\,\epsilon\,X=S_B$.

[7]) We use the same logical symbols in the informal discussion of formulas as in the formulas themselves. No confusion will arise when one observes the rule that variables of the formal calculus are printed in the ordinary (Roman) type whereas variables and constants used in the informal discussion are printed in italics.

Now it follows from L.5 that such an integer exists. Hence formula IV1 is true in the extended natural model.

In order to show this for the formula IV2 we have to prove that if γ and δ satisfy the condition

$$\vdash \Xi_C(\gamma).(\delta \mathrm{LI} \gamma).\mathrm{LN}(\delta),$$

then $\vdash \mathrm{E}(\delta)$. Now it follows from L.5 that there is an n such that $\delta = \lambda_n$ and $n \in \mathcal{S}_C = Y$ whence n non $\in X$, *i. e.*, $\vdash \mathrm{E}(\lambda_n)$ which gives $\vdash \mathrm{E}(\delta)$. Lemma 7 is thus proved.

THEOREM 2. *Formula $\mathcal{F}$ is true in the extended natural model and hence consistent.*

Proof. Immediate by L.1, L.6, L.7.

6. Formal provability of properties of strings. Let $\mathrm{M}(x_1, \ldots, x_k)$ be a formula with the free variables $x_1, \ldots, x_k$, A, B, C, F, G. It follows immediately from L.3 that $\vdash \mathrm{M}(\alpha_1, \ldots, \alpha_k)$ is equivalent to

$$\vdash (x_1, \ldots, x_k)[\Gamma_{\alpha_1}(x_1) \ldots \Gamma_{\alpha_k}(x_k) \supset \mathrm{M}(x_1, \ldots, x_k)].$$

In the present section we shall examine the question whether the formula following the assertion-symbol $\vdash$ above is deducible from $\mathcal{F}_1$. We shall write $\mathcal{F}_1 \to \mathrm{M}(x_1, \ldots, x_k)$ instead of "$\mathrm{M}(x_1, \ldots, x_k)$ is deducible from $\mathcal{F}_1$".

Definition. A formula $\mathrm{M}(x_1, \ldots, x_k)$ is *normal* if

$$\mathcal{F}_1 \to [\Gamma_{\alpha_1}(x_1) \ldots \Gamma_{\alpha_k}(x_k) \supset \mathrm{M}(x_1, \ldots, x_k)] \tag{6}$$

whenever $\vdash \mathrm{M}(\alpha_1, \ldots, \alpha_k)$ and

$$\mathcal{F}_1 \to [\Gamma_{\alpha_1}(x_1) \ldots \Gamma_{\alpha_k}(x_k) \supset \sim \mathrm{M}(x_1, \ldots, x_k)] \tag{7}$$

whenever non $\vdash \mathrm{M}(\alpha_1, \ldots, \alpha_k)$.

L.8. *If M_1, M_2 are normal formulas, then so are $\sim M_1$ and $M_1 . M_2$.*

Proof. Obvious.

L.9. *Formulas* A(x), B(x), C(x) *are normal.*

Proof. It will be sufficient to consider the formula A(x). If $\vdash \mathrm{A}(\alpha)$, then $\alpha = a$, $\Gamma_a(x) = \mathrm{A}(x)$ and hence $\mathcal{F}_1 \to [\Gamma_a(x) \supset \mathrm{A}(x)]$. If non $\vdash \mathrm{A}(\alpha)$, then $\alpha \neq a$ and hence either $\alpha = b$ or $\alpha = c$ or α has the form $\beta\gamma$ with $\gamma = a$ or $\gamma = b$ or $\gamma = c$. In the first two cases we apply II3 and get

$$\mathcal{F}_1 \to [\Gamma_\alpha(x) \supset \sim \mathrm{A}(x)]. \tag{8}$$

In the remaining case we have $\Gamma_\alpha(x) = (\exists u, v)[\Gamma_\beta(u) . \Gamma_\gamma(v) . \mathrm{G}(x, u, v)]$ and hence by III3 we get again formula (8).

Lemma 9 is thus proved.

L.10. *For an arbitrary string α*

$$\mathcal{F}_1 \to (\exists x)\Gamma_\alpha(x), \tag{9}$$

$$\mathcal{F}_1 \to [\Gamma_\alpha(x) . \Gamma_\alpha(y) \supset \mathrm{F}(x, y)]. \tag{10}$$

Proof. We use induction with respect to the length of α. If $l(\alpha) = 1$, then (9) and (10) immediately result from II1 and II2. Assume now that L.10 holds for strings of length $< n$ and let $l(\alpha) = n$. Hence $\alpha = \beta\xi$ where $l(\beta) = n - 1$, $l(\xi) = 1$, and

$$\Gamma_\alpha(x) = (\exists y, z)[\Gamma_\beta(y) . \Gamma_\xi(z) . \mathrm{G}(x, y, z)].$$

Using this, we immediately find that (9) follows from III1 and the inductive assumption, and that (10) follows from the inductive assumption III2, and I2. Lemma 10 is thus proved.

L.11. *The formula* F(x, y) *is normal.*

Proof. Let α_1, α_2 be such that $\vdash \mathrm{F}(\alpha_1, \alpha_2)$. Hence $\alpha_1 = \alpha_2$ and therefore by L.10

$$\mathcal{F}_1 \to [\Gamma_{\alpha_1}(x_1) . \Gamma_{\alpha_2}(x_2) \supset \mathrm{F}(x_1, x_2)]. \tag{11}$$

We still have to prove that if non $\vdash \mathrm{F}(\alpha_1, \alpha_2)$, *i. e.*, if $\alpha_1 \neq \alpha_2$, then

$$\mathcal{F}_1 \to [\Gamma_{\alpha_1}(x_1) . \Gamma_{\alpha_2}(x_2) \supset \sim \mathrm{F}(x_1, x_2)]. \tag{12}$$

If $l(\alpha_1) = l(\alpha_2) = 1$, then (12) follows from I1 and II3. If $l(\alpha_1) = 1$ and $l(\alpha_2) > 1$, then (12) follows from I1 and III3. The same is true if $l(\alpha_1) > 1$ and $l(\alpha_2) = 1$. We assume now the validity of (12) for strings α_1, α_2 such that $\min\big(l(\alpha_1), l(\alpha_2)\big) \leqslant k$ and let β_1, β_2 be two different strings such that $\min\big(l(\beta_1), l(\beta_2)\big) \leqslant k + 1$. Since the case when the length of one of the strings is 1 has already been dealt with, we may assume that $\beta_i = \alpha_i \xi_i$ $(i = 1, 2)$ where $l(\xi_1) = l(\xi_2) = 1$ and $\min\big(l(\alpha_1), l(\alpha_2)\big) \leqslant k$. Hence we have formula (12) or (11) according to whether $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2$. Using the definitions of $\Gamma_{\beta_i}(x_i)$ $(i = 1, 2)$ we easily reduce the formula to be proved to the following one:

$$\mathcal{F}_1 \to [\Gamma_{\alpha_1}(x_1) . \Gamma_{\xi_1}(y_1) . \mathrm{G}(z_1, x_1, y_1) . \Gamma_{\alpha_2}(x_2) . \Gamma_{\xi_2}(y_2) . \mathrm{G}(z_2, x_2, y_2) \supset \sim \mathrm{F}(z_1, z_2)]. \tag{13}$$

If $\alpha_1 \neq \alpha_2$, this formula follows from (12), III5, III3, and I2. If $\alpha_1 = \alpha_2$, then $\xi_1 \neq \xi_2$ (since $\beta_1 \neq \beta_2$) and (13) follows from (11), I2, II3, and III2. Lemma 11 is thus proved.

L.12. *The formula* G(x, y, z) *is normal.*

Proof. Let α, β, γ be strings such that $\vdash \mathrm{G}(\alpha, \beta, \gamma)$, *i. e.* $\alpha = \beta\gamma$. We have to prove first that

$$\mathcal{F}_1 \to [\Gamma_\alpha(x) . \Gamma_\beta(y) . \Gamma_\gamma(z) \supset \mathrm{G}(x, y, z)]. \tag{14}$$

We proceed by induction with respect to $l(\gamma)$. If $l(\gamma)=1$, then we use the definition of $\Gamma_\alpha(\mathrm{x})$ and reduce (14) to the proof of

$$\mathcal{F}_1\to[\Gamma_\beta(\mathrm{u}).\Gamma_\beta(\mathrm{y}).\Gamma_\gamma(\mathrm{v}).\Gamma_\gamma(\mathrm{z}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v})\supset\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})].$$

This is an immediate consequence of L.10 and I2. Let us now assume that formula (14) has been proved for strings γ satisfying $l(\gamma)<n$ and let $l(\gamma)=n$. We may assume that $\gamma=\delta\xi$ where $l(\xi)=1$. Hence $\alpha=(\beta\delta)\xi$ and

$$\Gamma_\alpha(\mathrm{x})=(\exists\mathrm{u},\mathrm{v})[\Gamma_{\beta\delta}(\mathrm{u}).\Gamma_\xi(\mathrm{v}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v})],$$

$$\Gamma_\gamma(\mathrm{z})=(\exists\mathrm{s},\mathrm{t})[\Gamma_\delta(\mathrm{s}).\Gamma_\xi(\mathrm{t}).\mathrm{G}(\mathrm{z},\mathrm{s},\mathrm{t})].$$

This shows that (14) is equivalent to

$$(15)\quad \mathcal{F}_1\to[\Gamma_{\beta\delta}(\mathrm{u}).\Gamma_\beta(\mathrm{y}).\Gamma_\delta(\mathrm{s}).\Gamma_\xi(\mathrm{v}).\Gamma_\xi(\mathrm{t}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v}).\mathrm{G}(\mathrm{z},\mathrm{s},\mathrm{t})\supset\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})].$$

In order to prove this formula we observe that, in view of the inductive assumption

$$\mathcal{F}_1\to[\Gamma_{\beta\delta}(\mathrm{u}).\Gamma_\beta(\mathrm{y}).\Gamma_\delta(\mathrm{s})\supset\mathrm{G}(\mathrm{u},\mathrm{y},\mathrm{s})]$$

whence

$$\mathcal{F}_1\to\{\Gamma_{\beta\delta}(\mathrm{u}).\Gamma_\beta(\mathrm{y}).\Gamma_\delta(\mathrm{s}).\Gamma_\xi(\mathrm{v})$$
$$.\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v}).\mathrm{G}(\mathrm{z},\mathrm{s},\mathrm{v})\supset[\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v}).\mathrm{G}(\mathrm{u},\mathrm{y},\mathrm{s}).\mathrm{G}(\mathrm{z},\mathrm{s},\mathrm{v})]\}.$$

In view of III4 this formula gives

$$\mathcal{F}_1\to[\Gamma_{\beta\delta}(\mathrm{u}).\Gamma_\beta(\mathrm{y}).\Gamma_\delta(\mathrm{s}).\Gamma_\xi(\mathrm{v}).\mathrm{G}(\mathrm{x},\mathrm{u},\mathrm{v}).\mathrm{G}(\mathrm{z},\mathrm{s},\mathrm{v})\supset\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})]$$

which in view of I2 and L.10 implies (15). Formula (14) is thus proved.

We still have to prove that if $\alpha\neq\beta\gamma$, then

$$(16)\quad \mathcal{F}_1\to[\Gamma_\alpha(\mathrm{x}).\Gamma_\beta(\mathrm{y}).\Gamma_\gamma(\mathrm{z})\supset\sim\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})].$$

To show this, we use (14) and obtain

$$\mathcal{F}_1\to[\Gamma_\beta(\mathrm{y}).\Gamma_\gamma(\mathrm{z}).\Gamma_{\beta\gamma}(\mathrm{s})\supset\mathrm{G}(\mathrm{s},\mathrm{y},\mathrm{z})]$$

whence by III2

$$(17)\quad \mathcal{F}_1\to[\Gamma_\alpha(\mathrm{x}).\Gamma_\beta(\mathrm{y}).\Gamma_\gamma(\mathrm{z}).\Gamma_{\beta\gamma}(\mathrm{s}).\sim\mathrm{F}(\mathrm{x},\mathrm{s})\supset\sim\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})].$$

Since $\mathcal{F}_1\to[\Gamma_\alpha(\mathrm{x}).\Gamma_{\beta\gamma}(\mathrm{s})\supset\sim\mathrm{F}(\mathrm{x},\mathrm{s})]$ by L.11 and $\mathcal{F}_1\to(\exists\mathrm{s})\Gamma_{\beta\gamma}(\mathrm{s})$ by L.10 we obtain (16) immediately from (17). Lemma 12 is thus proved.

L.13. *Formulas without quantifiers are normal.*

Proof. By L.8, L.9, L.11, and L.12.

In the next few lemmata we shall establish the normality of certain formulas containing quantifiers.

Definition. Let $\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})$ be a formula containing the free individual variables $\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y}$ and the predicate-variables A, B, C, F, G.

We shall say that M *satisfies the finiteness condition with respect to the variable* y if for arbitrary strings $\alpha_1,\ldots,\alpha_k$ there is an integer $l\geqslant 0$ and l strings $\beta_1,\ldots,\beta_l$ such that

$$(18)\quad \mathcal{F}_1\to[\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k).\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})\supset\sum_{j=1}^{l}\Gamma_{\beta_j}(\mathrm{y})].$$

L.14. *If* M *satisfies the finiteness condition with respect to the variable* y, *then so does the conjunction* M.N *where* N *is an arbitrary formula.*

L.15. *A normal formula* $\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})$ *satisfies the finiteness condition with respect to the variable* y *if and only if arbitrary strings* $\alpha_1,\ldots,\alpha_k$ *determine an integer* $h\geqslant 0$ *and strings* $\gamma_1,\ldots,\gamma_h$ *such that*

$$(19)\quad \mathcal{F}_1\to\{\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k)\supset[\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})\equiv\sum_{j=1}^{h}\Gamma_{\gamma_j}(\mathrm{y})]\}.$$

Proof. If (19) is satisfied, then so is (18) with $l=h$ and $\beta_j=\gamma_j$ $(j=1,2,\ldots,l)$. Let us now assume (18). The formula M being normal we have either

$$(20)\quad \mathcal{F}_1\to[\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k).\Gamma_{\beta_j}(\mathrm{y})\supset\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})]$$

or

$$\mathcal{F}_1\to[\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k).\Gamma_{\beta_j}(\mathrm{y})\supset\sim\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})]$$

for each $j=1,2,\ldots,l$. If we denote by $\gamma_1,\ldots,\gamma_h$ those of the β_j's for which (20) is true, we immediately obtain formula (19).

L.16. *If* M *is a normal formula and* (19) *is true, then* $\vdash\mathrm{M}(\alpha_1,\ldots,\alpha_k,\beta)$ *holds if and only if* β *is identic with one of the strings* $\gamma_1,\ldots,\gamma_h$.

Proof. If follows from (19) that $\vdash\mathrm{M}(\alpha_1,\ldots,\alpha_k,\beta)$ holds if and only if one of the formulas $\vdash\Gamma_{\gamma_j}(\beta)$ is true whence the lemma follows by L.3.

L.17. *If* $\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})$ *is a normal formula satisfying the finiteness condition with respect to the variable* y *and if* $\mathrm{N}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})$ *is an arbitrary formula, then the formula* $(\exists\mathrm{y})[\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y}).\mathrm{N}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})]$ *is normal.*

Proof. Let $\alpha_1,\ldots,\alpha_k$ be arbitrary strings such that

$$(21)\quad \vdash(\exists\mathrm{y})[\mathrm{M}(\alpha_1,\ldots,\alpha_k,\mathrm{y}).\mathrm{N}(\alpha_1,\ldots,\alpha_k,\mathrm{y})].$$

This means that there is a string β such that

$$\vdash\mathrm{M}(\alpha_1,\ldots,\alpha_k,\beta)\quad\text{and}\quad\vdash\mathrm{N}(\alpha_1,\ldots,\alpha_k,\beta).$$

The formulas M and N being normal, we obtain

$$\mathcal{F}_1\to[\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k).\Gamma_\beta(\mathrm{y})\supset\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y}).\mathrm{N}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})],$$

whence by L.10 we get

$$\mathcal{F}_1\to\{\Gamma_{\alpha_1}(\mathrm{x}_1)\ldots\Gamma_{\alpha_k}(\mathrm{x}_k)\supset(\exists\mathrm{y})[\mathrm{M}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y}).\mathrm{N}(\mathrm{x}_1,\ldots,\mathrm{x}_k,\mathrm{y})]\}.$$

Let us now assume that (21) does not hold. Since M satisfies the finiteness condition, we may assume that (19) holds. Hence on using L.16 we infer that $\vdash \sim N(a_1, \ldots, a_k, \gamma_j)$ for $j=1,2,\ldots,h$ whence, N being normal,

(22) $\mathcal{F}_1 \to [\Gamma_{a_1}(x_1)\ldots\Gamma_{a_k}(x_k).\Gamma_{\gamma_j}(y) \supset \sim N(x_1,\ldots,x_k,y)], \qquad j=1,2,\ldots,h.$

In order to prove L.17 we have to show that

$$\mathcal{F}_1 \to \{\Gamma_{a_1}(x_1)\ldots\Gamma_{a_k}(x_k) \supset \sim(\exists y)[M(x_1,\ldots,x_k,y).N(x_1,\ldots,x_k,y)]\}$$

or, what amounts to the same,

$$\mathcal{F}_1 \to [\Gamma_{a_1}(x_1)\ldots\Gamma_{a_k}(x_k).M(x_1,\ldots,x_k,y) \supset \sim N(x_1,\ldots,x_k,y)].$$

Using (19) we reduce this formula to the following one:

$$\mathcal{F}_1 \to [\Gamma_{a_1}(x_1)\ldots\Gamma_{a_k}(x_k).\sum_{j=1}^{h}\Gamma_{\gamma_j}(y) \supset \sim N(x_1,\ldots,x_k,y)].$$

Since this last formula is a direct consequence of (22), lemma 17 is proved.

L.18. *For every string* a

(23) $$\mathcal{F}_1 \to [\Gamma_a(x).G(x,y,z) \supset \sum_{\beta,\gamma}\Gamma_\beta(y).\Gamma_\gamma(z)]$$

with summation over strings β,γ *satisfying the equation* $a=\beta\gamma$.

Proof. If $l(a)=1$, then the summation $\sum_{\beta,\gamma}$ is void and (23) is equivalent to the formula $\mathcal{F}_1 \to [\Gamma_a(x) \supset \sim G(x,y,z)]$ which follows from III3.

Let us now assume L.18 for a string a and let $\eta=a\xi$ where $l(\xi)=1$. In view of the definition of $\Gamma_\eta(x)$ the formula to be proved is equivalent to the following one:

(24) $$\mathcal{F}_1 \to [\Gamma_a(u).\Gamma_\xi(v).G(x,u,v).G(x,y,z) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)]$$

with summation over strings δ,ε satisfying the equation $\eta=\delta\varepsilon$.

Since

$$\mathcal{F}_1 \to \{G(x,u,v).G(x,y,z) \supset F(y,u).F(v,z) \vee (\exists t)[G(y,u,t).G(v,t,z) \vee G(u,y,t).G(z,t,v)]\}$$

by III5 and III2, the proof of (24) can be reduced to the proof of the following three formulas:

(25) $$\mathcal{F}_1 \to [\Gamma_a(u).\Gamma_\xi(v).F(y,u).F(v,z) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)],$$

(26) $$\mathcal{F}_1 \to [\Gamma_a(u).\Gamma_\xi(v).G(y,u,t).G(v,t,z) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)],$$

(27) $$\mathcal{F}_1 \to [\Gamma_a(u).\Gamma_\xi(v).G(u,y,t).G(z,t,v) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)].$$

Of these three formulas (25) is an immediate corollary of I1, I2 and the observation that a,ξ represent one of the possible sets of values for δ,ε; (26) is obvious in view of III3 and the observation that $l(\xi)=1$. It remains to prove (27).

By means of (23) we reduce (27) to the conjunction of formulas of the form

$$\mathcal{F}_1 \to [\Gamma_\beta(y).\Gamma_\gamma(t).\Gamma_\xi(v).G(z,t,v) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)]$$

where $\beta\gamma=a$. In view of the definition of $\Gamma_{\gamma\xi}(z)$ this formula is equivalent to

$$\mathcal{F}_1 \to [\Gamma_\beta(y).\Gamma_{\gamma\xi}(z) \supset \sum_{\delta,\varepsilon}\Gamma_\delta(y).\Gamma_\varepsilon(z)]$$

and this formula is obvious since $\beta,\gamma\xi$ represent one of the possible sets of values for δ,ε. (24) is thus proved and hence L.18 is proved by induction.

L.19. *The formula* $G(x,y,z)$ *satisfies the finiteness condition with respect to the variable* y *and with respect to the variable* z; *the formulas* xSy, xRy, *and* xPy *satisfy the finiteness condition with respect to the variable* x.

Proof. Immediate from L.18.

L.20. *Formulas* xSy, xRy, *and* xPy *are normal.*

Proof. Immediate from L.19 and L.17.

L.21. *Formula* xIy *is normal.*

Proof. The formula $(\exists t)[G(z,u,t).G(t,x,u)]$ is normal in view of L.19, L.17, and L.12. Applying again L.17 and L.19 we find that the formula $(\exists z)\{(zPy).(\exists t)[G(z,u,t).G(t,x,u)]\}$ is normal. Finally we note that the formula C(u) satisfies the finiteness condition with respect to the variable u and using L.14 we infer in the same way that the formula

$$(\exists u)\big(C(u).\sim(uPx).(\exists z)\{(zPy).(\exists t)[G(z,u,t).G(t,x,u)]\}\big)$$

is normal which proves the lemma.

L.22. *Formula* xIy *satisfies the finiteness condition with respect to the variable* x.

Proof. By L.19 each string a determines a finite number of strings β_j such that

$$\mathcal{F}_1 \to [\Gamma_a(y).(zPy) \supset \sum_j \Gamma_{\beta_j}(z)].$$

By L.18 each β_j determines a finite number of strings γ_{jk} such that

$$\mathcal{F}_1 \to [\Gamma_{\beta_j}(z).G(z,u,t) \supset \sum_k \Gamma_{\gamma_{jk}}(t)].$$

Finally each γ_{jk} determines a finite number of strings δ_{jkl} such that

$$\mathcal{F}_1 \to [\Gamma_{\gamma_{jk}}(t).G(t,x,u) \supset \sum_l \Gamma_{\delta_{jkl}}(x)].$$

Combining the last three formulas we get

$$\mathcal{F}_1 \to [\Gamma_\alpha(y).(zPy).G(z,u,t).G(t,x,u) \supset \sum_{jkl} \Gamma_{\delta_{jkl}}(x)]$$

from which we immediately obtain the lemma.

L.23. *Formula* V(x,y,z) *is normal.*

Proof. First we observe that the formula

$$(\exists w,v,r)[(wPr).G(w,u,v).G(v,x,u).(rSz).\sim(sPr)]$$

is normal in view of L.19, L.17, and L.14. The formula C(u).G(t,y,u) satisfies the finiteness condition with respect to the variable t since $\mathcal{F}_1 \to [\Gamma_\alpha(y).\Gamma_\beta(u).C(u).G(t,y,u) \supset \Gamma_{\alpha c}(t)]$ in view of the definition of $\Gamma_{\alpha c}(t)$. In the same way we show that the formula C(u).G(s,u,t) satisfies the finiteness condition with respect to the variable s. Hence by L.17 the formula

$$(\exists t)\big[C(u).G(t,y,u).(\exists s)\{C(u).G(s,u,t).(\exists w,v,r)[(wPr).G(w,u,v) \\ .G(v,x,u).(rSz).\sim(sPr)]\}\big]$$

is normal. Prefixing this formula with the quantifier $(\exists u)$ we still obtain a normal formula since C(u) satisfies the finiteness condition with respect to the variable u. By L.21 and L.8 we infer that the formula V(x,y,z) is normal.

L.24. *The formula* V(x,y,z) *satisfies the finiteness condition with respect to the variable* x *and with respect to the variable* y.

Proof. Immediate from L.14 and L.22.

L.25. *The formulas* xFIy *and* xLIy *are normal and satisfy the finiteness condition with respect to the variable* x.

Proof. First we observe that

$$\mathcal{F}_1 \to \big(\sim CS(x) \equiv (\exists u)\{(uPx).[A(u) \vee B(u)]\}\big)$$

whence it follows by L.19, L.17, L.9, L.8 that the formula CS(x) is normal. By L.19, L.12, and L.17 the formula $(\exists t)[G(t,u,w).(tSy)]$ is normal and satisfies the finiteness condition with respect to the variables w and u. It follows by L.12 and the above observation concerning CS(x) that the formula

$$(\exists u,w)\{(\exists t)[G(t,u,w).(tSy)].G(w,x,v).CS(u)\}$$

is normal. Finally we use L.9, L.17, and the fact that the formula C(v) satisfies the finiteness condition with respect to the variable v and infer that the formula

$$(\exists v)\big(C(v).(\exists u,w)\{(\exists t)[G(t,u,w).(tSy)].G(w,x,v).CS(u)\}\big)$$

is normal. This implies the normality of xFIy.

From L.18 we easily see that xFIy satisfies the finiteness condition with respect to the variable x. This concludes the proof of L.25 for the formula xFIy.

For the formula xLIy the proof is similar.

L.26. *The formula* LN(x) *is normal.*

Proof. Immediate from L.19, L.17, L.8, and the remark that

$$\mathcal{F}_1 \to \big(LN(x) \equiv \sim(\exists y)\{[B(y) \vee C(y)].(yPx)\}\big).$$

L.27. *For arbitrary* α *the formula* $\Gamma_\alpha(x)$ *is normal and satisfies the finiteness condition with respect to the variable* x.

Proof. The lemma being obvious if $l(\alpha)=1$, we may suppose that it holds for a string α. If $l(\xi)=1$, then

$$\mathcal{F}_1 \to \big(\Gamma_{\alpha\xi}(x) \equiv (\exists u)\{\Gamma_\alpha(u).(\exists v)[\Gamma_\xi(v).G(x,u,v)]\}\big).$$

The inductive assumption together with L.17 and L.12 implies that the formula $(\exists u)\{\Gamma_\alpha(u).(\exists v)[\Gamma_\xi(v).G(x,u,v)]\}$ is normal whence it follows that so is the formula $\Gamma_{\alpha\xi}(x)$. The proof that this formula satisfies the finiteness condition is obvious.

L.28. *The formulas* $H_{\beta,\beta'}(w,v)$ *and* $\Xi_B(x)$ *are normal for an arbitrary basis* B *and for arbitrary strings* β,β'.

Proof. Immediate from L.8, L.17, L.12, L.27, L.21, L.22, L.23, L.25, and L.24.

7. Proof that $\mathcal{F}$ has no recursively enumerable model.

Let A,B,C,D,E be recursively enumerable sets of integers, F a recursively enumerable binary relation, and G a recursively enumerable ternary relation (the fields of F and G are subsets of the set of all integers). If $M(x_1,\dots,x_k)$ is a first order formula with the free individual variables $x_1,\dots,x_k$ and the free predicate variables A,B,C,D,E,F,G and if the integers $p_1,\dots,p_k$ satisfy the formula M when the universe of discourse is interpreted as the set of integers and A,B,...,G are interpreted as $A,B,\dots,G$, then we shall write $\models M(p_1,\dots,p_k)$. Let $n \leftrightarrow (k_1,k_2,\dots,k_s)$ be a one-to-one correspondence between positive integers and finite sequences of such integers. It is well known that this correspondence can be chosen in such a way that s and k_j be primitive recursive functions of n. We shall write $s=L(n)$, $k_j=K(n,j)$ for $j=1,2,\ldots,s$; hence L and K are primitive recursive functions.

L.29. *If* $\models\mathcal{F}$, *then* $\models\Gamma_{\lambda_n}(p)\equiv(\exists g)\Big[\big(L(g)=n\big).\big(K(g,1)\in A\big)$
$$.(j)_n G\big(K(g,j+1),K(g,j),K(g,1)\big).\big(K\big(g,L(g)\big)=p\big)\Big]$$ [8].

Proof. The lemma says that p satisfies the formula $\Gamma_{\lambda_n}(\mathrm{x})$ if and only if there exists a finite sequence $k_1,k_2,\ldots,k_n$ such that $k_1\in A$, $k_n=p$, and $G(k_{j+1},k_j,k_1)$ for $j=1,2,\ldots,n-1$. This is evident if $n=1$. If $n>1$, then $\models\Gamma_{\lambda_n}(p)$ is equivalent to the existence of integers q,r such that $\models\mathrm{G}(p,q,r)$, $\models\mathrm{A}(r)$, and $\models\Gamma_{\lambda_{n-1}}(q)$. Proceeding by induction we assume the existence of a sequence $k_1,k_2,\ldots,k_{n-1}$ such that $k_1\in A$, $k_{n-1}=q$, and $G(k_{j+1},k_j,k_1)$ for $j=1,2,\ldots,n-2$. We contend that the sequence $k_1,k_2,\ldots,k_{n-1},p$ is the required one. Indeed, the first term of this sequence is an element of A, the last term is identic with p, and $G(k_{j+1},k_j,k_1)$ holds for $j=1,2,\ldots,n-2$. Hence it remains to show that $G(p,k_{n-1},k_1)$, *i. e.*, that $G(p,q,k_1)$. This is done as follows. From II2 and I2 we obtain

$$\mathcal{F}\to[\mathrm{A}(\mathrm{x}).\mathrm{A}(\mathrm{y})\supset\mathrm{F}(\mathrm{x},\mathrm{y})],\qquad \mathcal{F}\to[\mathrm{F}(\mathrm{x},\mathrm{y}).\mathrm{G}(\mathrm{z},\mathrm{t},\mathrm{x})\supset\mathrm{G}(\mathrm{z},\mathrm{t},\mathrm{y})].$$

Since $\models\mathcal{F}$, it follows that if $k_1\in A$ and $r\in A$, then $F(r,k_1)$ and if $F(r,k_1)$ and $G(p,q,r)$, then $G(p,q,k_1)$. Since we assume that k_1 and r are elements of A and $G(p,q,r)$, we obtain $G(p,q,k_1)$, q. e. d.

L.30. *If* $\models\mathcal{F}$, *then the sets*

$$X^*=\underset{n}{E}\{(\exists p)\models[\Gamma_{\lambda_n}(p).\mathrm{D}(p)]\},\qquad Y^*=\underset{n}{E}\{(\exists p)\models\Gamma_{\lambda_n}(p).\mathrm{E}(p)]\}$$

are recursive and disjoint.

Proof. The recursive enumerability of the sets X^*, Y^* follows from lemma 29 and the assumption that the sets D and E are recursively enumerable. The assumption that sets D and E satisfy the axiom IV4 proves that the intersection of X^* and Y^* is void.

We shall show that every integer is either an element of X^* or an element of Y^*. Indeed, from L.10 we find that $\mathcal{F}\to(\exists\mathrm{x})\Gamma_{\lambda_n}(\mathrm{x})$ and hence there is an integer p such that $\models\Gamma_{\lambda_n}(p)$. Since by L.26

$$\mathcal{F}\to[\Gamma_{\lambda_n}(\mathrm{x})\supset\mathrm{LN}(\mathrm{x})],$$

we obtain further that if $\models\Gamma_{\lambda_n}(p)$, then $\models\mathrm{LN}(p)$. Since the axiom IV3 is satisfied we obtain further that if $\models\mathrm{LN}(p)$, then either $p\in D$ or $p\in E$. If $p\in D$, then $\models[\Gamma_{\lambda_n}(p).\mathrm{D}(p)]$ whence $n\in X^*$; if $p\in E$, then, similarly, $n\in Y^*$. The sets X^*,Y^* are thus recursively enumerable, disjoint, and each is the complement of the other. This proves that both sets are recursive [9]), q. e. d.

[8]) $(j)_n$ stands here for the expression: for each j satisfying the inequality $j<n$.
[9]) This is a well-known result of Kleene. Cf. for instance [3], p. 307.

L.31. *If* $\models\mathcal{F}$, *then* $X\subset X^*$ *and* $Y\subset Y^*$.

Proof. Let us assume that $n\in X$, *i. e.* that $\lambda_n\in S_B$. Hence there is a B-generating string γ such that λ_n is the last ingredient of γ. Hence $\vdash\Xi_B(\gamma).(\lambda_n\mathrm{LI}\gamma)$ and therefore by L.25 and L.28

$$\mathcal{F}_1\to[\Gamma_\gamma(\mathrm{x}).\Gamma_{\lambda_n}(\mathrm{y})\supset\Xi_B(\mathrm{x}).(\mathrm{y\,LI\,x})].$$

Using L.10 as well as the formula $\mathcal{F}_1\to[\Gamma_{\lambda_n}(\mathrm{y})\supset\mathrm{LN}(\mathrm{y})]$ resulting from L.26 we obtain

$$\mathcal{F}_1\to\{\Gamma_{\lambda_n}(\mathrm{y})\supset(\exists\mathrm{x})[\Gamma_\gamma(\mathrm{x}).\Xi_B(\mathrm{x}).\mathrm{LN}(\mathrm{y}).(\mathrm{yLIx})]\}$$

whence by IV1

$$\mathcal{F}\to[\Gamma_{\lambda_n}(\mathrm{y})\supset\mathrm{D}(\mathrm{y})]$$

and finally by L.10

$$\mathcal{F}\to(\exists\mathrm{y})[\Gamma_{\lambda_n}(\mathrm{y}).\mathrm{D}(\mathrm{y})].$$

This proves that

$$\models(\exists\mathrm{y})[\Gamma_{\lambda_n}(\mathrm{y}).\mathrm{D}(\mathrm{y})]$$

and hence that there is an integer p such that $\models[\Gamma_{\lambda_n}(p).\mathrm{D}(p)]$, *i. e.* that $n\in X^*$. This establishes the inclusion $X\subset X^*$. The proof that $Y\subset Y^*$ is similar.

THEOREM 3. *There are no recursively enumerable sets* A,B,C,D,E *and no recursively enumerable relations* F,G *such that* $\models\mathcal{F}$.

Proof. It follows from lemmata 30 and 31 that if there were such sets and relations, then X and Y would be separable by means of recursive sets, which contradicts our choice of sets X and Y.

8. Formulas no model of which belongs to $P_1\cup Q_1$. There exists a simple procedure by means of which it is possible to obtain from $\mathcal{F}$ a formula $\mathcal{F}'$ such that no model of $\mathcal{F}'$ belongs to the class $P_1\cup Q_1$. It is sufficient to take as $\mathcal{F}'$ the conjunction

$$\mathcal{F}.(\mathrm{x},\mathrm{y},\mathrm{z})\{[\mathrm{A}'(\mathrm{x})\equiv\sim\mathrm{A}(\mathrm{x})].[\mathrm{B}'(\mathrm{y})\equiv\sim\mathrm{B}(\mathrm{y})].[\mathrm{C}'(\mathrm{y})\equiv\sim\mathrm{C}(\mathrm{z})]$$
$$.[\mathrm{F}'(\mathrm{x},\mathrm{y})\equiv\sim\mathrm{F}(\mathrm{x},\mathrm{y})].[\mathrm{G}'(\mathrm{x},\mathrm{y},\mathrm{z})\equiv\sim\mathrm{G}(\mathrm{x},\mathrm{y},\mathrm{z})]\},$$

where $\mathrm{A}',\mathrm{B}',\ldots,\mathrm{G}'$ are new predicate-variables.

If $A,A',\ldots,G,G'$ are recursively enumerable sets and relations and if $\mathcal{F}'$ is true when A is interpreted as $A,\ldots,\mathrm{G}'$ is interpreted as G', then we say that $A,A',\ldots,G,G'$ *define a model of class* P_1. If $A,A',\ldots,G,G'$ are complements of recursively enumerable sets and relations, then we say that they *define a model of class* Q_1.

THEOREM 4. *The formula* $\mathcal{F}'$ *has no model of class* P_1 *and no model of class* Q_1.

Proof. There can be no model of class $\boldsymbol{P}_1$ since $\mathcal{F}$ has no such model. If there were a model of class $\boldsymbol{Q}_1$, then $A', B', \ldots, G'$ would be complements of recursively enumerable sets and relations and (since $n \in A \equiv n \notin A', \ldots, G(p,q,r) \equiv \text{non}\, G'(p,q,r)$) the sets and relations $A, B, \ldots, G$ would be recursive. Theorem 4 follows thus from theorem 3.

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Generalized dissimilarity of ordered sets *

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1. Introduction. The present paper arose from an attempt to solve the following problem: does there exist a (simply) ordered set E of more than one element, such that, for every pair of distinct elements a and b of E, the sets $E-\{a\}$ and $E-\{b\}$ are dissimilar (*i. e.*, there is no one-one order-preserving correspondence between the two sets)? An easy argument shows that there is no such set E of power $\aleph_0$; we shall prove, however, that there does exist a subset of the continuum, of power $\mathfrak{c}=2^{\aleph_0}$, possessing the property in question. Generalizations in various directions will also be obtained. In order to motivate these generalizations as they appear in the formal statements of the theorems in section 6 below, we shall give here a rough indication of their underlying ideas.

First of all, it is possible to find a subset E of the continuum such that not only is there no similarity transformation between $E-\{a\}$ and $E-\{b\}$, but there is not even a non-trivial "pseudo-similarity" transformation of $E-\{a\}$ onto $E-\{b\}$ (cf. Corollary 6.2 (d)), where we define (cf. 4.8) a pseudo-similarity transformation of an ordered set M to be a single-valued function (not necessarily one-one) defined on M that is, with respect to some decomposition of some dense subset of M into mutually exclusive subintervals of M, a similarity or anti-similarity on the interior of each of these subintervals. (This is clearly a more general kind of transformation than the "semi-similarity" introduced by Aronszajn [1]. In fact, there are $2^{\mathfrak{c}}$ pseudo-similarity transformations of the continuum into itself.) Then, it is not necessary that only single elements, a and b, be removed from E in order to obtain, say, dissimilar subsets of E; these single elements may be replaced by arbitrary distinct subsets of E of power less than $\mathfrak{c}$ (cf. Corollary 6.2 (d)). Another generalization is concerned with replacing the continuum by any ordered set M of power $\mathfrak{c}$ containing a subset of power $\mathfrak{c}$ that can be imbedded in the continuum; for any such M, we obtain a decomposition into $\mathfrak{c}$ mutually exclusive subsets (the sets E^{σ} in Theorem 6.1), each of which has the

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