[3] P. Bernays, *A system of axiomatic set theory*, part II, Journal of Symbolic Logic 6 (1941), p. 1-17.

[4] K. Gödel, *Über die Länge der Beweise*, Ergebnisse eines mathematischen Kolloquiums 7 (1931), p. 23-24.

[5] — *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*. Monatshefte für Mathematik und Physik 38 (1931), p. 173-198.

[6] Hao Wang, *Arithmetic translations of axiom systems*, Transactions of the American Mathematical Society 71 (1951), p. 283-293.

[7] — *Truth definitions and consistency proofs*, Transactions of the American Mathematical Society 73 (1952), p. 243-275.

[8] — *Between number theory and set theory*, Mathematische Annalen 126 (1953), p. 385-409.

[9] D. Hilbert und P. Bernays, *Grundlagen der Mathematik*, vol. II, Berlin 1939.

[10] S. C. Kleene, *Finite axiomatizability of theories in the predicate calculus using additional predicate symbols. Two papers on the predicate calculus*, Memoirs of the American Mathematical Society 10 (1952), p. 27-68.

[11] G. Kreisel, *A variant to Hilbert's theory of the foundations of arithmetic*, The British Journal for the Philosophy of Science IV, 14 (1953), p. 107-129.

[12] — *Note on arithmetic models for consistent formulae of the predicate calculus, II*, Proceedings of the XIth International Congress of Philosophy XIV (1953), p. 39-49.

[13] A. Mostowski, *On models of axiomatic systems*, Fundamenta Mathematicae 39 (1952), p. 133-158.

[14] — *Sentences undecidable in formalized arithmetic — An exposition of the theory of Kurt Gödel*, Amsterdam 1952.

[15] R. McNaughton, *Some formal relative consistency proofs*, Journal of Symbolic Logic 18 (1953), p. 136-144.

[16] C. Ryll-Nardzewski, *The role of the axiom of induction in elementary arithmetic*, Fundamenta Mathematicae 39 (1952), p. 239-263.

Reçu par la Rédaction le 19. 5. 1954

On manifolds and r-spaces *

by

A. Kosiński (Warszawa)

We shall say that a point p belonging to a space K is an *r-point* of K if each neighbourhood of p contains a neighbourhood U of p such that, for each $q \in U$, $\mathrm{Fr}(U)$ is a deformation retract of $U-(q)$. (See [4]; $\mathrm{Fr}(U)$ denotes the boundary of U, *i. e.* the set $\overline{U}\cdot(\overline{K-U})$.) The neighbourhood with the property just mentioned will be called a *canonical neighbourhood*. (It is worth noting that a canonical neighbourhood U of a point p is also a canonical neighbourhood for each point $q \in U$).

The space K is said to be an *r-space* if it is compact, metric, separable, finite dimensional, and if each point of K is its r-point.

It turns out that r-spaces have a very similar structure to that of topological manifolds. In particular, many of the classical theorems about the manifolds (such as for instance theorems on the invariance of domain, internal characterization of separating sets and so on) hold also in r-spaces. Moreover, among the spaces of dimension $\leqslant 2$ and among the polytopes of dimension $\leqslant 3$ connected r-spaces are identical with the manifolds. The notion of r-space gives therefore a new topological characterization of 2-manifolds. The basic problem whether there exists an r-space not homeomorphic to a manifold remains open.

This paper should be considered as a continuation of the researches of K. Borsuk on the "spheroidal spaces" ([5], [3]). In particular, the proofs of some lemmas in § 1 are suitably adapted proofs of the corresponding lemmas in [5] and [3]. These proofs are based on some auxiliary lemmas, which are modified lemmas from the papers mentioned. For the convenience of the reader all these auxiliary lemmas are gathered in the supplement at the end of the paper. (S. I, S. II etc. denote the lemmas from the supplement).

Terminology and notation. All set-theoretical topological notions used here are defined in [9]. Manifolds, pseudomanifolds, polytopes are meant in the sense defined in [2]. The homology theory here used

* Presented to the Polish Mathematical Society (Warsaw Section) at the meeting of March 12, 1954.

is that of [1]. Briefly: A sequence $Z^r=\{z_k^r\}$, $k=1,2,\ldots$, is called an *r-dimensional true chain* if there exists a sequence $\{\varepsilon_k\}$ of positive numbers such that $\varepsilon_k\to 0$ and z_k^r is an r-dimensional ε_k-chain mod m_k, $m_k\geqslant 2$. (Coefficients mod 0 will not be used.) If z_k^r are ε_k-cycles then Z^r is called an *r-dimensional true cycle*. If all the numbers m_k are powers of the same number m, then Z^r is said to be an *r-dimensional true power cycle* (Potenzzyklus).

Let F be a compact subset of K. An ε-chain z in K is said to be an *F-cycle* if $\dot z$ is an ε-chain in F. *True F-cycles* are defined in an obvious manner. A true cycle Z^r is called an *F-boundary* in K if there exists in K a true chain Q^{r+1} such that $\dot Q^{r+1}=Z^r-Z_F^r$ where Z_F^r is a true chain in F. We write $Z_1^r\sim Z_2^r$ if $Z_1^r-Z_2^r$ is homologous to zero in K (bounds in K). We write $Z_1^r\underset{F}{\sim}Z_2^r$ if $Z_1^r-Z_2^r$ is an F-boundary in K. If no subsequence of a true cycle $Z^r=\{z_k^r\}$ is homologous to zero in K, we say that Z^r is *totally non-homologous to zero in K*.

Several times we shall use the fact that the homological dimension based on a variable module, as well as on a power module, is equal to the topological dimension in the sense of Menger-Urysohn (see [1], p. 195, 209). Hence if K is compact and $\dim K=n$ then there exists in K a true n-dimensional power cycle bounding in K — but totally non-homologous to zero in a compact subset $K'\subset K$.

§ 1. Preliminary lemmas

In § 1 we shall consider a space K which is supposed to be compact, metric, separable and of finite positive dimension, and a fixed canonical neighbourhood $U\subset K$. F always denote $\mathrm{Fr}(U)$ and n denotes $\dim U$. We assume $n>0$.

LEMMA 1. *Suppose that B is a closed proper subset of $\overline{U}$. If Z^n is an n-dimensional true F-cycle in $B+F$, then $Z^n\underset{F}{\not\sim}0$ in $B+F$.*

Proof. Let $p\in U-B$. There exists a deformation f retracting $\overline{U}-(p)$ to F. Let Z_f^n be a true cycle assigned by the deformation f to the cycle Z^n. Then we have

$$(1)\qquad (Z-Z_f^n)^{\cdot}=\dot Z^n-(Z_f^n)^{\cdot}=\dot Z^n-(\dot Z^n)_f=0,$$

$$(2)\qquad (Z^n-Z_f^n)_f=Z_f^n-Z_{ff}^n=Z_f^n-Z_f^n=0.$$

Putting $Z=Z^n-Z_f^n$ we infer from (1) that Z is a true n-dimensional cycle in $B+F$. Hence, by (2) and S. I, there exists a neighbourhood $V(p)$ such that $Z\sim 0$ in $\overline{U}-V(p)$. Then by the definition of homological dimension it follows that $Z\sim 0$ in each carrier, in particular in $B+F$. However, Z_f^n is a cycle in F and the established relation $Z^n\sim Z_f^n$ in $B+F$ gives $Z^n\underset{F}{\not\sim}0$ in $B+F$. The lemma is thus established.

LEMMA 2. *There exists in $\overline{U}$ a true n-dimensional power F-cycle (totally) not F-homologous to zero on $\overline{U}$.*

Proof. Since $\dim U=n$, there exists a compact subset $B'\subset U$ which is also of dimension n. It follows that B' contains a compact subset B which is a carrier of an $(n-1)$-dimensional true power cycle Z, not bounding in B. Let $p\in U-B$. By S. I there exists a neighbourhood $V(p)$ of the point p in U such that $Z\underset{F}{\not\sim}0$ in $\overline{U}-V(p)$. Then S. II, combined with the standard application of the Brouwer Reduction Theorem ([7], p. 161), yields a compact subset M_0 of U irreducible with respect to the properties

$$(\mathrm{a}_0)\qquad B\subset M_0\subset\overline{U}-V(p),$$

$$(\mathrm{b}_0)\qquad Z\underset{F}{\not\sim}0 \quad\text{in}\quad M_0.$$

Observe now that $(M_0-B)\cdot U\neq 0$. For otherwise we should have $M_0-B\subset F$, and since $B\cdot F=0$, this is the same relation as $B\cdot(M_0-B)=0$. Hence B and M_0-B would be two compact disjoint sets and (b_0) would imply $Z\sim 0$ in B, contradicting our assumption. Therefore let $p_1\in(M_0-B)\cdot U$ and let $V_1(p_1)$ be a neighbourhood of p_1 in U such that $Z\underset{F}{\not\sim}0$ in $\overline{U}-V_1(p_1)$. Denote by M_1 a compact set irreducible with respect to the properties

$$(\mathrm{a}_1)\qquad B\subset M_1\subset\overline{U}-V_1(p_1),$$

$$(\mathrm{b}_1)\qquad Z\underset{F}{\not\sim}0 \quad\text{in}\quad M_1.$$

From (a_0), (a_1) it follows that $B\subset M_0\cdot M_1$ and $M_0\neq M_1$. Consequently Z is not F-homologous to 0 on $M_0\cdot M_1$. Thus we may apply S. III and conclude that there exists in M_0+M_1 an n-dimensional true cycle not F-homologous to zero in M_0+M_1. (By extraction of a subsequence we can make this cycle totally not F-homologous to 0 in M_0+M_1.) Now, lemma 1 shows that $M_0+M_1=\overline{U}$, and this completes the proof.

LEMMA 3. *Let B be a compact subset of U and Z a true $(n-1)$-dimensional cycle in B not bounding in B. Then B disconnects U.*

Proof. Let $p_1\in U-B$ and let M_1 be a compact subset of U satisfying the conditions (a_1), (b_1) from the proof of the preceding lemma.

Our lemma will be proved if we show that $\mathrm{Fr}_U(M_1)\subset B$ ($\mathrm{Fr}_U(M_1)$ denotes the boundary of M_1 relative to U).

Suppose that there exists a point $p_2\in\mathrm{Fr}_U(M_1)-B$. Let $V_2(p_2)$ be a neighbourhood of p_2 in U such that $Z\underset{F}{\not\sim}0$ in $U-\overline{V}_2(p_2)$. By S. II and the Brouwer Reduction Theorem there exists a compact set M_2 irreducible with repect to the properties

$$(\mathrm{a}_2)\qquad B\subset M_2\subset\overline{U}-V_2(p_2),$$

$$(\mathrm{b}_2)\qquad Z\underset{F}{\not\sim}0 \quad\text{in}\quad M_2.$$

By lemma 1 and (a_i), (b_i), $i=1,2$, it follows that

(c) $$M_1+M_2=\overline{U}.$$

On the other hand, the sets $V_2(p_2)$ and $\overline{U}-M_1$ are open and non--empty. Since p_2 belongs to the closure of $\overline{U}-M_1$, it follows that $V_2\cdot(\overline{U}-M_1)\neq 0$. But $V_2\subset\overline{U}-M_2$ and the last inequality gives

$$0\neq V_2\cdot(\overline{U}-M_1)\subset(\overline{U}-M_1)\cdot(\overline{U}-M_2)=\overline{U}-(M_1+M_2).$$

This contradicts (c) and hence proves the lemma.

LEMMA 4. *If B is a compact subset of U and every $(n-1)$-dimensional true power cycle in B bounds in B, then $U-B$ is connected.*

Proof. Suppose the contrary: $U-B=S_1+S_2$ with S_i closed, non--empty and disjoint. Let $M_i=S_i+B+F$, $i=1,2$. Then M are compact proper subsets of U with $M_1\cdot M_2=B+F$ and $M_1+M_2=\overline{U}$. By lemma 1 each n-dimensional true power F-cycle in M_i is F-homologous to 0 in M_i. By our hypothesis such is also every $(n-1)$-dimensional true power cycle in B. Hence S. IV applies, and we conclude that every n-dimensional true power F-cycle in $M_1+M_2=\overline{U}$ is F-homologous to 0 in $\overline{U}$. This, however, is impossible because it contradicts lemma 2.

LEMMA 5. *Let $U=F_1+F_2$, F_1,F_2 being closed proper subsets of U. Then* $\dim F_1\cdot F_2>n-2$.

Proof. Let $M_i=\overline{F}_i$ (closure relatively $\overline{U}$), $i=1,2$. Since $M_1\cdot M_2-F=$ $=F_1\cdot F_2$, the assumption $\dim F_1\cdot F_2\leqslant n-2$ would imply that each true $(n-1)$-dimensional F-cycle in $M_1\cdot M_2$ is F-homologous to zero in $M_1\cdot M_2$. Hence S. IV applies, and the conclusion is as in the proof of lemma 4.

LEMMA 6. *U is connected and cannot be disconnected by a subset of dimension $\leqslant n-2$ In particular, U is homogeneously n-dimensional. If $n>1$ then $F=\mathrm{Fr}(U)$ is also connected.*

Proof. The equivalence between lemma 5 and the first part of our lemma is proved in [7], p. 47. The last part is obvious: if $n>1$ then by the preceding argument no point disconnects U, and F, as a deformation retract of a connected set, is also connected.

LEMMA 7. *Let $U,V,U\supset\overline{V}$, be two canonical neighbourhoods of p and let B be a compact subset of V. Then B disconnects V if and only if it separates* $\mathrm{Fr}(V)$ *from a point $p\in V$ (we suppose that $n>1$).*

Proof. Let us observe first that $U-V$ is connected. For let C_1,C_2 be two components of $U-V$. U is connected; it follows hence that $C_i\cdot\mathrm{Fr}(V)\neq 0$ for $i=1,2$ and since $\mathrm{Fr}(V)$ is connected, so is also $C_1+\mathrm{Fr}(V)+C_2$. Hence $C_1=C_2$.

Suppose now that no component of $V-B$ is separated from $\mathrm{Fr}(V)$. It follows from the foregoing remark that $U-B$ is connected, and we conclude by lemmas 3 and 4 that so is also $V-B$.

The converse being obviously true, the lemma is established.

LEMMA 8. *Let U be a canonical neighbourhood of p and let V be any neighbourhood of p contained with closure in U. Then there exists in $G=\mathrm{Fr}(V)$ a true $(n-1)$-dimensional cycle Z^{n-1} bounding in V but totally non-homologous to 0 on G. If V is also a canonical neighbourhood, then Z^{n-1} bounds irreducibly on $\overline{V}$.*

Proof. Lemma 2 yields an n-dimensional true F-cycle Z^n in $\overline{U}$ which is totally not F-homologous to 0 in U. By lemma 12 from [3] we may assume that if $Z^n=\{z_k^n\}$, then each simplex of z_k^n lies wholly in one of the sets $\overline{V},\overline{U}-V$. Denote $\dot{Z}^n=Z_F^{n-1}$, thus Z_F^{n-1} is a cycle in F. Denote by z_{k1}^n the chain composed of those simplexes from z_k^n which are in $\overline{V}$. Thus $Z_1^n=\{z_{k1}^n\}$ is a true chain in $\overline{V}$ and $Z^n=Z_1^n-Z_2^n$. Let

(3) $$\dot{Z}_1^n=Z_1^{n-1},\qquad \dot{Z}_2^n=Z_2^{n-1}-Z_F^{n-1}.$$

Since Z_F^{n-1} is a cycle, so are also Z_i^{n-1} (in $\overline{V},\overline{U}-V$ respectively). But $Z_F^{n-1}=\dot{Z}^n=Z_1^{n-1}-(Z_2^{n-1}-Z_F^{n-1})$. Thus $Z^{n-1}=Z_1^{n-1}=Z_2^{n-1}$ is a true cycle in $\overline{V}\cdot(\overline{U}-V)=G$ bounding in $\overline{V}$. In order to demonstrate the first part of our lemma it suffices to show that Z^{n-1} is (totally) non-homologous to 0 in G. Suppose that

(4) $$\dot{P}^n=Z^{n-1}$$

where P^n is a true chain in G. Thus by (3) and (4)

$$\dot{Z}_1^n-\dot{P}^n=0,\qquad \dot{P}^n-\dot{Z}_2^n=Z_F^{n-1},$$

which shows that $Z_i^n-P^n$ are true n-dimensional F-cycles. Hence by lemma 1 they are F-homologous to 0 in $\overline{V}$ and $\overline{U}-V$ respectively. Thus $(Z_1^n-P^n)-(Z_2^n-P^n)=Z_1^n-Z_2^n=Z^n$ is F-homologous to zero in $\overline{V}+(\overline{U}-V)=\overline{U}$. If a certain subsequence of Z^{n-1} were homologous to 0 in G we should conclude in the same manner that some subsequence of Z^n was F-homologous to 0 in $\overline{U}$. But this contradicts our assumption about the cycle Z^n. Hence Z^{n-1} is totally non-homologous to zero in $\overline{U}$, which proves the first part of the lemma.

Now let V be a canonical neighbourhood. Since every true cycle in $\mathrm{Fr}(V)$ bounding in a proper closed subset of $\overline{V}$ bounds also in $\mathrm{Fr}(V)$, Z^{n-1} is irreducibly homologous to 0 in V. This completes the proof.

Remark. It follows from lemma 8 that there exists no continuous transformation f of a canonical neighbourhood $\overline{V}$ into a proper closed subset of itself, such that f is an identity on $\mathrm{Fr}(V)$. For let Q be a true chain in $\overline{V}$ and let $\dot{Q}=Z$, where Z is a true cycle in $\mathrm{Fr}(V)$. Suppose that there exists such a transformation f and denote by Q_f a chain assigned by this transformation to the chain Q. Then $(Q_f)^{\cdot}=(\dot{Q}_f)=Z$. Since Q_f is a true chain in a closed proper subset $f(\overline{V})$ of $\overline{V}$, it follows that Z is homologous to zero in a closed proper subset of $\overline{V}$. Thus the existence of such a transformation contradicts the existence of a true cycle in $\mathrm{Fr}(V)$ irreducibly bounding in $\overline{V}$, hence it contradicts lemma 8.

In particular, it follows that all points of V are stable in $\overline{V}$ (in the Hopf-Pannwitz sense, see [2], p. 523).

It is worth noting that by lemma 8 and "1. Konvergenzsatz" from [1] each point of a canonical neighbourhood V is a "directer Kernpunkt", hence $\overline{V}$ is a "Kernmenge"; both notions being taken in the sense adopted in [1], p. 213-214.

§ 2. *R*-spaces

THEOREM 1. *A connected r-space K is a locally connected Cantor-manifold without local cut points. All points of K are stable in K.*

Proof. By lemma 6 K is a Peano continuum without local cut points. By the same lemma and the connectedness of K, K is homogeneously n-dimensional. Hence lemma 6 applies with the same n to each canonical neighbourhood in K, and we prove Cantor-manifold property in exactly the same manner as it is proved for manifolds in [2], p. 48. The stability of all points of K was proved in the remark on lemma 8.

THEOREM 2. *Let K be a connected n-dimensional r-space. There exists $\varepsilon>0$ such that every compact subset B of diameter less than ε cuts K if and only if there exists in B a true $(n-1)$-dimensional cycle not bounding in B.*

Proof. The compactness of K insures the existence of such a positive number ε that every compact subset $B\subset K$ of diameter less than ε is contained in a canonical neighbourhood V, contained in turn in a second canonical neighbourhood U. Thus by lemma 3 and 4 the condition of the theorem is necessary and sufficient in order that B cut V. But by lemma 7 B cuts V if and only if it cuts K. This completes the proof.

THEOREM 3. *A connected r-space of dimension at most two is a topological manifold.*

Proof. Suppose first that $\dim K=1$. If $\mathrm{ord}_p K=1$ then obviously p is not an r-point. Hence we may assume that a and b belong to different components of $\mathrm{Fr}(U)$, where U is a canonical neighbourhood of p. But U is connected and it follows directly from the definition of an r-point that each point of U separates a from b in U. By Lennes Theorem ([9], § 41, IX, 6) $U+(a)+(b)$ is homeomorphic to an interval. Thus $\mathrm{ord}_p K=2$.

If K is a 2-dimensional r-space, then theorem 2 states that there exists $\varepsilon>0$ such that every simple closed curve of diameter $<\varepsilon$ cuts K and no arc of diameter less than ε does so. If in addition K is connected, then by theorem 1 it is a Peano continuum, so that theorem IV from [8] applies and the proof is complete.

We shall now demonstrate two theorems showing a deep similarity in structure of r-spaces and topological manifolds.

THEOREM 4. *Let K be an n-dimensional connected r-space. A necessary and sufficient condition that a closed subset E of K be n-dimensional is that E contains a non-empty subset which is open in K.*

Proof. The condition is necessary. For suppose that $E=\overline{E}\subset K$ and $\dim_p E=n$. We shall show that E contains interior points.

Let U be a canonical neighbourhood of p in K and let A be a compact subset of E contained in U and containing a neighbourhood of p in E. Hence $\dim A=n$ and there exists an $(n-1)$-dimensional true cycle Z bouding in A but non-homologous to 0 in a closed subset $A_0\subset A$. Let A_1 be a subset of A in which Z bounds irreducibly and let $q\in A_1-A_0$. Then by S. I there exists a neighbourhood V of the point q in $U-A_0$ such that $Z\underset{F}{\sim}0$ in $\overline{U}-V$. Since $A_1\cdot(\overline{U}-V)$ is a closed proper subset of A_1, $Z\nsim 0$ in $A_1\cdot(\overline{U}-V)$. Moreover, since $F\cdot A_1=0$ then also Z is not F-homologous to 0 in $A_1\cdot(\overline{U}-V)+F$, F denotes as usual $\mathrm{Fr}(U)$. Then S. III furnishes an n-dimensional true F-cycle in $A_1+(\overline{U}-V)$ not F-homologous to 0 in $A_1+(\overline{U}-V)$. Applying lemma 1 to this cycle we conclude that $A_1+(\overline{U}-V)=\overline{U}$; thus $V\subset A_1\subset A\subset E$ and the necessity is established.

By theorem 1 the condition is sufficient and thus the proof is completed.

COROLLARY. *Let U be a canonical neighbourhood in a connected n-dimensional r-space. Then $\mathrm{Fr}(U)$ is an $(n-1)$-dimensional continuum.*

Proof. $F=\mathrm{Fr}(U)$ contains no interior points, hence $\dim F\leqslant n-1$. But K is a Cantor-manifold, hence $\dim F\geqslant n-1$. (Obviously F need not be a Peano continuum.)

THEOREM 5. *Let K be an n-dimensional connected r-space and E an arbitrary subset of K. Then $p\in E$ is an interior point of E if and only if p has arbitrarily small neighbourhoods V in E with a compact closure (relatively E) and such that there exists in $G=\mathrm{Fr}_E V$ an $(n-1)$-dimensional true cycle Z irreducibly bounding in $\overline{V}$.*

Proof. To prove the sufficiency of the condition let p be a boundary point of E. Let U be a canonical neighbourhood of p in K and V a neighbourhood of p in E having a compact closure and contained with it in U. Let as usual $G=\mathrm{Fr}_E V$, $F=\mathrm{Fr}(U)$ and let Z be a true $(n-1)$-dimensional cycle in G bounding in $\overline{V}$. We shall prove that Z bounds also in a proper closed subset of $\overline{V}$.

Denote by U_1 a neighbourhood of p in U so small that $G\cdot U_1=0$. Since $p\in \mathrm{Fr}(E)$,

$$U_1\cdot(\overline{U}-\overline{V})\neq 0 \tag{5}$$

and

$$U_1\cdot\overline{V}\neq 0. \tag{6}$$

Since U is a canonical neighbourhood,

$$Z \underset{F}{\not\sim} 0 \quad \text{in} \quad \overline{U}-U_1 \tag{7}$$

and by hypothesis

$$Z\sim 0 \quad \text{in} \quad \overline{V}. \tag{8}$$

It follows from (5) that $(\overline{U}-U_1)+\overline{V}$ is a closed proper subset of U. Hence from (7), (8), S. III and lemma 1 we conclude that

$$Z \underset{F}{\not\sim} 0 \quad \text{in} \quad (\overline{U}-U_1)\cdot V. \tag{9}$$

But Z is a cycle in $\overline{V}$ and $\overline{V}\cdot F=0$, therefore (9) implies]

$$Z\sim 0 \quad \text{in} \quad (\overline{U}-U_1)\cdot V,$$

and since, by (6), $(\overline{U}-U_1)\cdot\overline{V}$ is a closed proper subset of V, the proof is completed.

To prove the necessity let p be an interior point of E. Then a sufficiently small neighbourhood of p in K is also a neighbourhood in E. Therefore let $U,V,U\supset\overline{V}$, be two canonical neighbourhoods of p in K which are simultaneously neighbourhoods of p in E. Application of lemma 8 completes the proof.

COROLLARY. *Let K, K' be two n-dimensional connected r-spaces. If $K\supset K'$, then $K=K'$.*

Proof. By lemma 8 and theorem 5 all points of K' are interior points of K.

§ 3. *R*-polytopes

In this paragraph we investigate triangulable r-spaces which will be called *r-polytopes*. (By a *simplex* we shall always mean an open simplex.)

Let R be an $(n-1)$-dimensional closed pseudomanifold such that if $K=K(R)$ is a cone on R, then, for each $p\in K-R$, R is a retract of $K-(p)$. Such a pseudomanifold is called *s-pseudomanifold.*

LEMMA 9. *An n-dimensional s-pseudomanifold R is of the same homotopy type as the n-dimensional sphere.*

Proof. Let $K=K(R)$ be a cone on R. Let $p\in\Delta$, where Δ is an $(n+1)$-dimensional simplex of K and put $K'=K-\Delta$. Obviously

$$B^r(K')=B^r(K) \quad \text{if} \quad r\leqslant n-1. \tag{10}$$

Since K is an absolute retract,

$$B^r(K)=0 \quad \text{if} \quad r=1,2,3,\ldots \tag{11}$$

R is a retract of $K-(p)$, hence also of K', and by [4] the retraction induces a homomorphism of $B^r(K')$ onto $B^r(R)$. onto $B^r(R)$. Hence by (10) and (11)

$$B^r(R)=0 \quad \text{if} \quad 1\leqslant r\leqslant n-1. \tag{12}$$

Since R is an n-dimensional pseudomanifold, it follows from (12) that

$$B^n(R)=\text{infinite cyclic}. \tag{13}$$

(12) and (13) give together

$$B^r(R)=B^r(S_n) \quad \text{for all} \quad r. \tag{14}$$

Consider now the fundamental group. If $\dim K>2$, hence if $\dim R>1$, then also $\pi_1(K)=\pi_1(K')$, $\pi_1(K)=0$. Hence

$$\pi_1(R)=0 \quad \text{if} \quad \dim R>1. \tag{15}$$

The lemma itself then follows from equalities (14), (15) and from the known theorem on the homotopy type of connected polyhedra (*e. g.* [6], 32.2).

Given a vertex a of a polytope K, the star $\mathrm{St}(a)$ of a in K is the subpolytope of K consisting of the simplexes which have a as a vertex and all their faces. Hence $\mathrm{St}(a)$ is a closed set. The set-theoretical boundary of $\mathrm{St}(a)$ will be called *boundary complex of a* and will be denoted by $\mathrm{Bd}(a)$. It is known that many topological properties of a polytope depend on the properties of boundary complexes. In particular, in terms of these complexes are defined homological manifolds which play an important part in modern topology, since duality theorems hold in them. We shall now investigate boundary complexes in r-polytopes.

THEOREM 6. *Let K be a connected r-polytope of dimension n. For every vertex a, $\mathrm{Bd}(a)$ has the homotopy type of the $(n-1)$-dimensional sphere.*

Proof. In view of theorem 3 we may assume that $\dim K>2$. By theorems 1 and 2, K is a closed pseudomanifold without local cut points. Hence $\mathrm{Bd}(a)$ is also a closed $(n-1)$-dimensional pseudomanifold. In consideration of lemma 9, in order to demonstrate our theorem it suffices to show that $\mathrm{Bd}(a)$ is an s-pseudomanifold. Let $S=\mathrm{St}(a)$ and $B=\mathrm{Bd}(a)$. Since S is a cone on B we have to show that if $p\in S-B$ then B is a retract of $S-(p)$; in virtue of the topological homogeneity of S along the rays from a, it suffices to show this for a point p arbitrarily near a. But this is obvious. For let $U\subset S$ be a canonical neighbourhood of a and let $p\in U$. Then $\mathrm{Fr}(U)$ is a retract of $U-(p)$, hence also $S-U$ is retract of $S-(p)$. Since $S-U$, does not contain the point a, B is a retract of $S-U$, hence also of $S-(p)$ and the theorem is proved.

COROLLARY. *A connected r-polytope is a homological manifold.*

We shall now establish a connection between the r-polytopes and spheroidal polytopes.

THEOREM 7. *Let K be an n-dimensional r-polytope. A necessary and sufficient condition in order that K be a spheroidal polytope is that K be of the homotopy type of the n-sphere.*

Proof. From theorems 9 and 12 of [5] and 32.2 of [6] it follows that the spheroidal space of dimension n has the homotopy type of the n-dimensional sphere. Hence the condition is necessary.

To prove the sufficiency let K be an n-dimensional r-polytope $(n>2)$ and let $B^r(K)=B^r(S_n)$, $\pi_1(S_n)=\pi_1(K)$, $r=0,1,\ldots$ Since every point of K may be considered as a vertex of a triangulation, it suffices to show that, given a vertex p of a triangulation of K, $K'=K-\mathrm{St}(p)$ is an absolute retract.

Now, let $q\in\Delta$, where Δ is an n-dimensional simplex contained in $\mathrm{St}(p)$. Let $K''=K-\Delta$. In the proof of theorem 6 we have established that $\mathrm{Bd}(p)$ is a retract of $\mathrm{St}(p)-(q)$. Since this retraction may obviously be extended to the retraction of the whole set $K-(q)$ to the set K', it follows from the relations $K'\subset K''\subset K-(q)$ that K' is a retract of K''. But by our hypothesis the fundamental group of K vanishes, hence so does also $\pi_1(K'')$. Therefore

$$\pi_1(K')=0. \tag{16}$$

It is easy to prove that $B^r(K)=B^r(K'')$ if $r\neq n$. Since K'' is an n-dimensional pseudomanifold with boundary, $B^n(K'')=0$. Hence

$$B^r(K')=0 \quad \text{for all} \quad r\geqslant 1. \tag{17}$$

By (16), (17) and the theorem of Hurewicz ([6], 14.1) K' is contractible to a point, and thus the proof is completed.

Theorem 3 can be strengthened in the case of r-polytopes:

THEOREM 8. *Let K be a connected polytope of dimension <4. Then K is a manifold if and only if it is an r-space, and K is a spheroidal polytope if and only if it is an r-space and has the homotopy type of the sphere.*

Proof. First part: the necessity is obvious, the sufficiency follows at once from the corollary to theorem 6 and the fact that the at most 3-dimensional homological manifold is necessarily a manifold.

Second part: To demonstrate the necessity it is enough to observe that a spheroidal polytope of dimension <4 is a manifold and consequently also an r-space. Thus by theorem 7 it has the homotopy type of the sphere. The sufficiency follows from the same theorem.

§ 4. Some remarks and problems

1. The set of r-points of a space is somewhat similar to the set of points having neighbourhoods homeomorphic to the Euclidean space. But in any space this last set is open. Hence we have the following

PROBLEM 1. *Is the set of r-points of any space always open?*

Denote by A the set of r-points of a space K and denote by A_n the set of points of K having canonical neighbourhoods of diameter less than $1/n$. Then $A=\prod_n A_n$, and since the sets A_n are open, we infer that A is G_δ-set. The general problem reduces to showing that any point of a canonical neighbourhood in K is an r-point of K.

2. The canonical neighbourhood is an analogue to the notion of the open cell. But the open cell is contractible in itself. Hence

PROBLEM 2. *Are the canonical neighbourhoods contractible in themselves?*

Problem 2 remains open even in the case of r-polytopes, although obviously every neighbourhood of a point in an r-polytope contains a canonical neighbourhood contractible in itself. But in the general case even the following significant problem remains open:

PROBLEM 3. *Are the r-spaces locally contractible?*

3. Problems 2 and 3 are closely related to the following

PROBLEM 4. *Is the canonical neighbourhood compactified by the adjunction of a single point homeomorphic with a spheroidal space?*

Let us consider this relation more exactly. We shall say that a space is *locally spheroidal* if each point of it is contained in a neighbourhood homeomorphic with the spheroidal space from which one point has been removed.

It seems very probable that every ANR in which each neighbourhood of any point contains a canonical neighbourhood, contractible in itself, of that point is locally spheroidal. It is easy to see that such is the case if these canonical neighbourhoods can be retracted to a compact subset by a retraction differing as little as possible from the identity mapping. Obviously the star of a point in a polytope possesses this property. Since in an r-polytope the stars are canonical neighbourhoods contractible in themselves, we infer that any r-polytope is locally spheroidal. (This fact may serve as a basis for the theory developed in § 3.) Is the reciprocal theorem also true? This reduces to the following

PROBLEM 5. *Is any spheroidal polytope an r-polytope?*

Let us consider also

PROBLEM 6. *Is the boundary complex in an r-polytope also an r-polytope?*

If the answer to these two problems is positive, then on the basis of the two notions, that of the r-polytope and that of the spheroidal polytope, one can construct a theory which "substitutes" the usual theory of triangulable manifolds and spheres, but in which all the notions (except those of the polytope) have an internal topological meaning and in which the "Poincaré hypothesis" holds.

4. It can be shown that the at most 2-dimensional topological divisor of a locally contractible r-space is a manifold, hence an r-space. But the proof rests on theorem 2, and not on the definition of an r-point directly. Hence it would be interesting to investigate the following

PROBLEM 7. *Let $(a,b) \in A\times B$ be an r-point of a Cartesian product $A\times B$. Are then a and b r-points of A and B respectively?*

The positive answer to this problem would imply the positive answer to problem 6.

5. For the sake of completeness we state the fundamental

PROBLEM 8. *Does there exist an r-polytope which is not a manifold?*

Supplement

S. I. *Let K, K_1, $K\supset K_1$, be two compact spaces. Let $p\in K-K_1$ and let Z be a true cycle in a compact subset of K not containing p. Suppose that K_1 is a deformation retract of $K-(p)$ and denote by Z_f a cycle in K_1 assigned by this retraction to the cycle Z. Then there exists such a neighbourhood V of p that $Z\sim Z_f$ in $K-V$.*

Proof. Denote by C the carrier of Z and by $f(x,t)$ the deformation retracting K to K_1. Then by theorem 1 in [4] $Z\sim Z_f$ in the set $f(C,t)$, where $t\in\langle 0,1\rangle$. But this last set is compact and does not contain p, hence its complement in K is an open neighbourhood of p and may be considered as the desired neighbourhood V.

S. II. *Let $M_i = M_i' + F$ be a monotonic decreasing sequence of compact sets. Let Z be a true cycle in M_i and suppose that for each i $Z \underset{F}{\not\sim} 0$ in M_i. Then $Z\underset{F}{\not\sim} 0$ in $\prod_i M_i$.*

Proof. The proof is exactly the same as the proof of lemma 4 in [5].

S. III. *Let M, M_0, M_1, F be compact sets satisfying $M = M_0+M_1$, $F\subset M_0\cdot M_1$. Suppose that there exists in $M_0\cdot M_1$ an r-dimensional true cycle Z^r which is not F-homologous to zero in $M_0\cdot M_1$ but $Z^r\underset{F}{\not\sim} 0$ in M_i, $i=0,1$. Then there exists in M a true $(r+1)$-dimensional F-cycle Z^{r+1} not F-homologous to zero in M. If Z^r is a power cycle so is also Z^{r+1}.*

Proof. (Cf. the proof of lemma 2 in [5]). Since $Z^r\underset{F}{\not\sim} 0$ in M_i there exist in M_i chains Q_i^{r+1} such that $\dot{Q}_i^{r+1} = Z^r+\varphi_i$ where φ_i is a chain in F.

Let $Z^{r+1} = Q_0^{r+1} - Q_1^{r+1}$. Thus $\dot{Z}^{r+1} = \varphi_0-\varphi_1$ and it follows that Z^{r+1} is an F-cycle in M. We shall show that it is the desired cycle. Suppose, on the contrary, that there exists in M a true chain $Q^{r+2} = \{q_k\}$ such that $\dot{Q}^{r+2} = Z^{r+1}+\varphi$, φ in F. By an infinitesimal displacement we may obtain from Q^{r+2} a chain $Q^{r+2*} = \{q_k^*\}$ such that

a) if a vertex a of a chain q_k belongs to M_i, then the corresponding vertex a^* also belongs to M_i and $a\in M_0\cdot M_1$ implies $a^*=a$;

b) every simplex of q_k^* is in one of the sets M_0, M_1 (see lemma 12 in [3]). Hence

$$(Q^{r+2*})^{\cdot} = (\dot{Q}^{r+2})^* = (Z^{r+1}+\varphi)^* = Q_0^{r+1*} - Q_1^{r+1*} + \varphi. \tag{18}$$

Let $Q^{r+2*} = P_0^{r+2} - P_1^{r+2}$ where $P_0^{r+2} = \{p_k\}$ and p_k consists of all those simplexes of q_k^* which are in M_0. Setting this equality into (18) we have

$$Q_0^{r+1*} - \dot{P}_0^{r+2} = Q_1^{r+1*} - \dot{P}_1^{r+2} - \varphi.$$

The chain on the left side is in M_0, that on the right is in M_1 — hence both are in $M_0\cdot M_1$. Let $P^{r+1} = Q_0^{r+1*} - \dot{P}_0^{r+2}$. We have

$$\dot{P}^{r+1} = (Q_0^{r+1*})^{\cdot} = (\dot{Q}_0^{r+1})^* = (Z^r+\varphi_0)^* = Z^r+\varphi_0$$

since $Z^r+\varphi_0$ is in $M_0\cdot M_1$.

Thus we infer that Z^r is F-homologous to zero in $M_0\cdot M_1$ in contradiction to our assumption.

S. IV. *Let M, M_0, M_1, F be as in S. III. Suppose that*

(a) *each n-dimensional true F-cycle in M_i is F-homologous to zero in M_i;*

(b) *each $(n-1)$-dimensional true F-cycle in $M_0\cdot M_1$ is F-homologous to zero in $M_0\cdot M_1$.*

Then every n-dimensional true F-cycle in M is in M F-homologous to zero. If the assumptions (a) *and* (b) *are restricted to power cycles, then he theorem holds, but also with regard to power cycles only. (In all but one applications of this lemma the situation is that $M_0\cdot M_1 = A+F$ with A and F compact and disjoint. In that case we can replace* (b) *by requiring every $(n-1)$-dimensional true cycle in A to bound in A.)*

Proof. (Cf. the proof of lemma 14 in [3].) Let Z^n be a true n-dimensional F-cycle in M. We have to show that $Z^n\underset{F}{\sim} 0$ in M. As in the proof of S. III we may assume (if need by applying an infinitesimaly small displacement) that

$$Z^n = Z_0^n - Z_1^n \tag{19}$$

where Z_i^n is a chain in M_i. Let $Z_i^n = \{z_{ki}\}$, $k=1,2,\ldots$, $i=0,1$. Denote by z_{ki}^{n-1} the chain consisting of all those simplexes of z_{ki} which have at least one vertex not in F. Let $Z_i^{n-1} = \{z_{ki}^{n-1}\}$. Thus

1^0 from equality $\dot{Z}_i^n = Z_i^{n-1} - \varphi_i$, where φ_i is a chain in F, we infer that Z_i^{n-1} is an F-cycle in M_i;

2^0 since $\dot{Z}^n = (Z^{n-1} - Z_1^{n-1}) - (\varphi_0 - \varphi_1)$ is a chain in F, we infer that $Z_0^{n-1} = Z_1^{n-1}$.

Let $Z^{n-1} = Z_0^{n-1} = Z_1^{n-1}$. By 1^0 and 2^0 Z^{n-1} is an F-cycle in $M_0 \cdot M_1$. Hence (b) yields an n-dimensional true chain P^n in $M_0 \cdot M_1$ such that $\dot{P}^n = Z^{n-1} - \psi$, ψ in F. Putting

(20) $$R_i^n = Z_i^n - P^n$$

we infer from equality $\dot{R}_i^n = \psi - \varphi_i$ that R_i^n is an n-dimensional true F-cycle in M_i. Thus (a) yields an $(n+1)$-dimensional true chain Q_i^{n+1} in M_i such that

(21) $$\dot{Q}_i^{n+1} = R_i^n - \chi_i, \qquad \chi_i \text{ in } F.$$

From (19), (20), (21) we have

$$\dot{Q}_0^{n+1} - \dot{Q}_1^{n+1} = (R_0^n - R_1^n) - (\chi_0 - \chi_1)$$
$$= (Z_0^n - Z_1^n) - (P^n - P^n) - (\chi_0 - \chi_1) = Z^n - (\chi_0 - \chi_1),$$

hence $Z^n \underset{F}{\sim} 0$ in M and the proof is completed.

References

[1] P. Alexandroff, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 161-238.

[2] P. Alexandroff und H. Hopf, *Topologie*, Berlin 1935.

[3] K. Borsuk, *O zagadnieniu topologicznego scharakteryzowania sfer euklidesowych*, Wiadomości Matematyczne 38 (1934), p. 1-30.

[4] — *Zur kombinatorischen Eigenschaften der Retrakte*, Fundamenta Mathematicae 21 (1935), p. 91-98.

[5] — *Über sphäroidale und H-sphäroidale Räume*, Récueil Math. Mosc. 1 (43) (1936), p. 643-660.

[6] S. Eilenberg, *Extension and classification of continuous mappings. Lectures in Topology*, Ann Arbor 1941, p. 57-99.

[7] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1948.

[8] E. R. van Kampen, *On some characterizations of 2-dimensional manifolds*, Duke Math. Journ. 1 (1935), p. 74-93.

[9] C. Kuratowski, *Topologie II*, Warszawa-Wroclaw 1950.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 4. 6. 1954

A formula with no recursively enumerable model

by

A. Mostowski (Warszawa)

G. Kreisel [4] was the first to construct a first-order formula which has no recursive model [1]). A formula with the same property was also constructed independently by the present author in a paper read before the VIII Congress of the Polish Mathematicians in the autumn of 1953 (see Mostowski [6]). Both formulas were obtained by suitable modifications of the axioms of the set-theory proposed by Bernays [1].

The present paper contains another example of a formula which has no recursive model. This example seems to be simpler than the former ones in so far, as it makes no reference to the axiomatic set-theory and uses exclusively tools known from the theory of recursive functions.

The formula to be given below was found in the course of unsuccessful attempts to construct a formula no model of which would belong to the smallest field of sets generated by the classes $\boldsymbol{P}_1^{(n)}$ and $\boldsymbol{Q}_1^{(n)}$ [2]). It is published in the hope that it might suggest a solution of this problem.

It has been justly observed that many recent papers in the field of symbolic logic do not supply full proofs of the statements they contain. While it would certainly not be reasonable to require from all papers to give exhaustive proofs it is certainly necessary to publish full proofs from time to time. This line is followed in the present paper.

1. Post's theory of recursively enumerable sets [7]. Let G be a free semigroup (with cancellation) generated by the free generators a, b, c. Thus the elements of G are finite strings $x_1 x_2 \ldots x$ where each x_j is either a or b or c and the multiplication of strings is performed simply by juxtaposing them. The void string is not admitted in G. Elements of G will be denoted by lower case Greek letters. The length $l(\alpha)$ of a string α is defined as the number of letters it contains.

A string α is said to be (a) a *segment* of β; (b) a *part* of β; (c) a *rest* of β if either $\alpha = \beta$ or (a) there is a γ such that $\beta = \alpha\gamma$; (b) there are γ, δ

[1]) Kreisel's paper contains even a slightly stronger result; cf. the theorem on p. 47 of his paper.

[2]) For terminology see my paper [5]. The problem was formulated by Kreisel [4]; cf. p. 47.