About sets with strange isometrical properties (I)

by

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The problems considered in this paper were put forward by W. Sierpiński. They were suggested by the observation that certain plane sets are congruent with their subsets which are obtained by taking out of them a certain single point. Such are the set of points of the straight line with coordinates $1,2,3,\ldots$, and the set of points of the plane with complex coordinates $e^i, e^{2i}, e^{3i}, \ldots$

In this connection we have the following results and problems to deal with:

I. *A linear set E contains no more than one point p such that*

$$E-\{p\}\simeq E$$ [1])

(the symbol $\simeq$ denotes congruence of sets; $\{p\}$ denotes the set containing one point p).

II. *Does there exist a set E containing two different points p and q such that*

$$E-\{p\}\simeq E\simeq E-\{q\}?$$ [2])

III. *Does there exist a non-empty set E contained in the plane or in the 3-dimensional Euclidean space such that*

$$E-\{p\}\simeq E \quad \textit{for each} \quad p\in E?$$

(In the Hilbert space l^2 such a set exists) [3]).

Problem II and thus also III are negatively solved for linear sets and sets lying on the circumference of the circle (by theorem I). The answers to those problems are not known in the case of plane sets [4]).

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[1]) See Sierpiński [4], p. 1, or [6], p. 7. The same theorem remains true for sets lying on the circumference of the circle, which can be verified by an easy modification of the proof of Sierpiński.

[2]) The existence of such a plane set was affirmed in Sierpiński [4], p. 2, but the proof contains a mistake — see [5], p. 5, or [6], p. 117.

[3]) The problem and the example in l^2 are formulated in Sierpiński [4], p. 4, and in [6], p. 10.

[4]) See the Remark at the end of this paper.

The essential result of the present paper is a positive answer to problem III, and thus also II, in the case of the 3-dimensional Euclidean space (see theorems 2 and 3) [5]).

Many generalizations and simplifications, especially the constant use of algebraical methods in this article, have been suggested by J. Łoś, to whom the author wishes to express his sincere gratitude.

We adopt the following notation:

If S is a set of transformations of a space R, then for $E\subset R$ we put

$$S(E)=\sum_{\sigma\epsilon S}\sigma(E)$$

where $\sigma(E)$ denotes the transformation σ of the set E.

1 denotes the identity-transformation, and the unity of groups (all groups will be multiplicative).

Let G be a group. For each two sets $S,R\subset G$ and an element $\varphi\epsilon G$ we denote by φS and SR the sets of elements of G which have the forms $\varphi\sigma$ and $\sigma\varrho$ respectively where $\sigma\epsilon S$ and $\varrho\epsilon R$.

If M is a set of transformations, then we denote by $[M]$ the group generated by M (*i. e.* the smallest group of transformations containing M — the set of generators).

A group $[M]$ is said to be *free* if each of its elements $\varphi\neq 1$ can be represented only in one way in the form

$$\varphi_1^{k_1}\varphi_2^{k_2}\ldots\varphi_n^{k_n} \tag{1}$$

where n is a natural number, $k_1,k_2,\ldots,k_n$ are integers different from 0, $\varphi_1,\varphi_2,\ldots,\varphi_n\epsilon M$ and $\varphi_i\neq\varphi_{i+1}$ for $i=1,\ldots,n-1$.

The expression (1) is called the *canonical form* of the element φ.

In the proofs of our theorems we shall use the following theorem of Sierpiński:

(T_1) *There exists a free group of the rotations of the sphere* [6]) *with a set of generators of power* $2^{\aleph_0}$ [7]).

Our first theorem will be the following one:

THEOREM 1 [8]). *For each set P in the 3-dimensional Euclidean space which can be covered with a set L of straight lines of power $<2^{\aleph_0}$, having a point O in common and such that $O\notin P$, and for each family F of subsets of P which is of power $\leqslant 2^{\aleph_0}$ there exists a set E such that $P\subset E$, $\overline{\overline{E}}=\aleph_0\overline{\overline{P}}\,\overline{\overline{F}}$ and*

$$E-Q\simeq E \quad \textit{for each} \quad Q\epsilon F;$$

[5]) See also Mycielski [1] where these results are formulated.

[6]) Or a group of rotations of the 3-dimensional Euclidean space around a point.

[7]) See Sierpiński [2], p. 238, lemma 1. The proof is effective.

[8]) The proof of this theorem will be effective only when $\overline{\overline{L}}\leqslant\aleph_0$.

the isometry transforming $E-Q$ in E being a rotation around the point O.

Yet if the set P is bounded (lies on a sphere S), then the corresponding set E is bounded (lies on the sphere S).

Let us observe that problem II is positively solved by this theorem in the case of the three dimensional space and moreover, that

1° Each set P of power $<2^{\aleph_0}$ fulfils the hypothesis of the theorem.

2° The family F can consist of all one-point subsets of P (since then evidently $\overline{\overline{F}}\leqslant 2^{\aleph_0}$).

The proof of theorem 1 will be preceded by two lemmas:

LEMMA 1. *A free group $[M]$ contains for each $N\subset M$ such a subset S_N that*

$$1\epsilon S_N, \tag{2}$$

$$\varphi S_N=S_N-\{1\} \quad \textit{for each} \quad \varphi\epsilon N, \tag{3}$$

$$\psi S_N=S_N \quad \textit{for each} \quad \psi\epsilon M-N. \tag{4}$$

Proof. Let S_N consist of 1 and of all elements of $[M]$ which have the canonical form

$$\varphi_1^{k_1}\varphi_2^{k_2}\ldots\varphi_n^{k_n} \quad \text{where} \quad k_n>0 \quad \text{if} \quad \varphi_n\epsilon N.$$

Evidently (2) is satisfied.

Let us prove (3). Let $\varphi\epsilon N$; then $\varphi^{-1}\notin S_N$ *i. e.* $1\notin\varphi S_N$. If $\tau\epsilon S_N$ and $\tau\neq 1$, then $\varphi^{-1}\tau\epsilon S_N$, *i. e.* $\tau\epsilon\varphi S_N$; which completes the proof because evidently $\varphi S_N\subset S_N$.

For proving (4) it is enough to observe that if $\psi\epsilon M-N$ and $\tau\epsilon S_N$ then $\psi\tau\epsilon S_N$ and $\psi^{-1}\tau\epsilon S_N$, because it means that $\psi S_N\subset S_N\subset\psi S_N$, q. e. d.

We shall say that a group G of transformations of a space R is *free on a set $P\subset R$* if for each $\sigma_1,\sigma_2\epsilon G$ and each $p_1,p_2\epsilon P$ the equality

$$\sigma_1(p_1)=\sigma_2(p_2)$$

implies the equality $\sigma_1=\sigma_2$ and therefore also $p_1=p_2$.

LEMMA 2. *If P and F fulfil the hypothesis of theorem 1 then there exists a free group $[M]$ of rotations around O which is free on P and in which $\overline{\overline{M}}=\overline{\overline{F}}$.*

Proof. Let M' be a set of rotations around O of power $2^{\aleph_0}$ generating a free group $[M']$ (theorem (T_1)). We shall prove the existence of such a set $M''\subset M'$ of power $2^{\aleph_0}$ that the group $[M'']$ is free on P. It will prove the lemma because we can take for M any subset of M'' which is of power $\overline{\overline{F}}$ (the existence of such a subset results effectively from the hypothesis $\overline{\overline{F}}\leqslant 2^{\aleph_0}$).

By hypothesis there exists a set L of straight lines with a common point $O\notin P$ which covers P and such that

$$\overline{\overline{L}}<2^{\aleph_0}. \tag{5}$$

For arbitrary straight lines $l_1, l_2 \in L$ let

$$A_{l_1 l_2}=\underset{\sigma}{E}\,(\sigma\in[M'],\ \sigma(l_1)=l_2),$$

and let $M_{l_1 l_2}$ be the set of those generators from M' which occur in the elements of $A_{l_1 l_2}$.

We shall prove that

$$\overline{\overline{M}}_{l_1 l_2}<\aleph_0. \tag{6}$$

Let $\varphi,\psi,\chi\in A_{l_1 l_2}$ and $\varphi\neq\psi$; then

$$\varphi(l_1)=l_2,\qquad \psi(l_1)=l_2,\qquad \chi(l_1)=l_2,$$

and therefore

$$\varphi^{-1}\chi(l_1)=l_1,\qquad \psi^{-1}\chi(l_1)=l_1.$$

By these equalities the rotations $\varphi^{-1}\chi, \psi^{-1}\chi$ have a common axis l_1, because, constituting elements of a free group, they cannot be rotations with the angle π. This implies that their product is commutative

$$(\varphi^{-1}\chi)(\psi^{-1}\chi)=(\psi^{-1}\chi)(\varphi^{-1}\chi),\qquad i.\,e.\qquad \psi\varphi^{-1}\chi\psi^{-1}\varphi=\chi.$$

The hypothesis that $\varphi\neq\psi$, and $[M']$ is free implies that this equality is possible only if each generator of χ occurs in φ or ψ (because the canonical forms of the right and left side terms must be equal).

Then $M_{l_1 l_2}$ consists only of the generators occuring in φ or ψ, which proves (6).

Let

$$M''=M'-\sum_{l_1,l_2\in L} M_{l_1 l_2}. \tag{7}$$

By (5) and (6) we have $\overline{\overline{M}}''=2^{\aleph_0}$.

Let us prove that $[M'']$ is free on P. Let $p_1,p_2\in P$; then there exist such straight lines l_1,l_2, that $p_1\in l_1\in L$ and $p_2\in l_2\in L$.

Let $\sigma_1,\sigma_2\in[M'']$ and let us suppose that

$$\sigma_1(p_1)=\sigma_2(p_2). \tag{8}$$

Then, as σ_1,σ_2 are rotations around O, we have also

$$\sigma_1(l_1)=\sigma_2(l_2),$$

which implies $\sigma_2^{-1}\sigma_1\in A_{l_1 l_2}$. But by (7) the only common element of $[M'']$ and $A_{l_1 l_2}$ is 1 (only if $l_1=l_2$); therefore $\sigma_2^{-1}\sigma_1=1$, *i. e.*

$$\sigma_1=\sigma_2\quad\text{and thus}\quad p_1=p_2. \tag{9}$$

We have proved that (8) implies (9), *i. e.* that $[M'']$ is free on P. Thus we have proved that the set M' contains a set M'' with the properties we need; q. e. d.

Proof of theorem 1. P and F fulfil the hypothesis of the theorem. Let $[M]$ be the group fulfilling lemma 2 and S_N the set from lemma 1. As $\overline{\overline{M}}=\overline{\overline{F}}$, we can suppose that $M=\{\varphi_Q\}_{Q\in F}$, where $\varphi_{Q'}\neq\varphi_{Q''}$ if $Q'\neq Q''$.

Let

$$N_p=\underset{\sigma}{E}\,(\sigma=\varphi_Q,\ p\in Q\in F), \tag{10}$$

$$E=\sum_{p\in P} S_{N_p}(p). \tag{11}$$

We shall prove that E fulfils theorem 1.

Evidently $P\subset E$ and $\overline{\overline{E}}=\aleph_0\overline{\overline{P}}\,\overline{\overline{F}}$.

If P lies on a sphere S then $\overline{\overline{P}}<2^{\aleph_0}$ (because each straight line cuts S in two points at most and if $\overline{\overline{P}}\geqslant 2^{\aleph_0}$, we cannot have $\overline{\overline{L}}<2^{\aleph_0}$ [9])). Then we can choose as the point O the centre of S, and (11) implies $E\subset S$. It is also obvious that if P is bounded, then E is bounded.

It remains to prove that

$$\varphi_Q(E)=E-Q\quad\text{for each}\quad Q\in F. \tag{12}$$

In fact, by (10) if $p\in Q$ then $\varphi_Q\in N_p$ and if $p\in P-Q$ then $\varphi_Q\in M-N_p$. Then by (11) and lemma 1

$$\varphi_Q(E)=\sum_{p\in Q}\varphi_Q S_{N_p}(p)+\sum_{p\in P-Q}\varphi_Q S_{N_p}(p)=\sum_{p\in Q}(S_{N_p}-\{1\})(p)+\sum_{p\in P-Q}S_{N_p}(p).$$

By lemma 2 and (11) this proves (12), q. e. d.

Now we shall prove the theorems solving problem III.

THEOREM 2. *There exists a set E of power $2^{\aleph_0}$ lying on the sphere in the 3-dimensional Euclidean space such that for each at most enumerable set D*

$$E-D\simeq E\ \text{[10])}.$$

To prove this theorem we used two lemmas:

LEMMA 3. *Let $[M]$ be a free group, a set $N\subset M$ be given,*

$$S\subset[N]; \tag{i}$$

and for a sequence $\{Q_\xi\}_{\xi<\alpha}$ of subsets of S there exists such a sequence of generators $\{\varphi_\xi\}_{\xi<\alpha}\subset N$ that

$$\varphi_\xi S=S-Q_\xi\quad\textit{for each}\quad \xi<\alpha. \tag{ii}$$

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[9]) See Sierpiński [3] p. 153.

[10]) Here and subsequently the symbol $\simeq$ denotes congruence of sets by rotation; reflexion and translation are excluded. A generalization of this theorem will be given in part II of this paper.

Then for each set $Q \subset S$ *and for each generator* $\varphi \in M - N$ *there exists such a set* $R = R(N, S, Q, \varphi)$, *that*

(iii) $$\overline{\overline{R}} = \aleph_0 \overline{\overline{S}}\,\overline{\overline{N}},$$

(iv) $$S \subset R \subset [N + \{\varphi\}],$$

(v) $$\varphi_\xi R = R - Q_\xi \quad \textit{for each} \quad \xi < \alpha,$$

(vi) $$\varphi R = R - Q.$$

Proof. Let

$$\varphi_1^{k_1} \varphi_2^{k_2} \ldots \varphi_n^{k_n}$$

be the canonical form of an element $\tau \in [N + \{\varphi\}] - \{1\}$.

(13) I_φ will be the set of all those τ in which $\varphi_n = \varphi$.

(14) I_N will be the set of all those τ in which $\varphi_n \in N$.

We verify (as in the proof of lemma 1) that

(15) $$\varphi_\xi I_\varphi = I_\varphi, \qquad \varphi_\xi(I_N + \{1\}) = I_N + \{1\} \quad \text{for each} \quad \xi < \alpha$$

and that

(16) $$\varphi I_N = I_N, \qquad \varphi(I_\varphi + \{1\}) = I_\varphi + \{1\}.$$

Let us observe that

(17) the sets Q, $\sum_{k=1}^{\infty} \varphi^k Q$, $I_N\left(\sum_{k=1}^{\infty} \varphi^k Q\right)$, $S - Q$, $I_\varphi(S - Q)$ are disjoint

with one another, which follows from the fact that by (i), (13) and (14) the canonical forms of elements belonging to different of those sets are different.

We put

(18) $$R(N, S, Q, \varphi) = S + I_\varphi(S - Q) + (I_N + \{1\})\left(\sum_{k=1}^{\infty} \varphi^k Q\right)$$

(it is the sum of the sets occuring in (17)). Then by (17) we have also

(19) $$R(N, S, Q, \varphi) = \sum_{k=0}^{\infty} \varphi^k Q + I_N\left(\sum_{k=1}^{\infty} \varphi^k Q\right) + (I_\varphi + \{1\})(S - Q).$$

Now we shall verify that R fulfils (iii), (iv), (v) and (vi). (iii) is evidently true by (13), (14) and (18). (iv) results from (i) and (18). (v) results from (ii), (15), (17) and (18). (vi) results from (16), (17) and (19).

Thus lemma 3 is proved.

LEMMA 4. *A free group* $[M]$, *where* $\overline{\overline{M}} = 2^{\aleph_0}$ *contains such a set* U *of power* $2^{\aleph_0}$ *that for each non-empty at most enumerable set* $Q \subset U$ *there exists such a generator* $\varphi_Q \in M$ *that*

$$\varphi_Q U = U - Q.$$

Proof. Let ω_λ denote the smallest ordinal number of power $2^{\aleph_0}$ [11]), and let $M = \{\varphi_\xi\}_{\xi < \omega_\lambda}$ [12]).

We shall define by transfinite induction three sequences of subsets of $[M]$

(vii) $$\{S_\xi\}_{\xi < \omega_\lambda}, \qquad \{Q_\mu^\eta\}_{\eta, \mu < \omega_\lambda}, \qquad \{Q_\xi\}_{\xi < \omega_\lambda}$$

such that

(20) $$1 \leqslant \overline{\overline{Q}}_\xi \leqslant \aleph_0, \quad \overline{\overline{S}}_\xi = \aleph_0 \overline{\xi} \quad \text{if} \quad \xi < \omega_\lambda;$$

(21) $$Q_\xi \subset S_\xi \subset [\{\varphi_\chi\}_{\chi \leqslant \zeta}] \quad \text{if} \quad \xi \leqslant \zeta < \omega_\lambda;$$

(22) $$\varphi_\xi S_\zeta = S_\zeta - Q_\xi \quad \text{if} \quad \xi \leqslant \zeta < \omega_\lambda;$$

(23) $$S_\xi \subset S_\zeta, \quad S_\zeta \neq S_\xi \quad \text{if} \quad \xi < \zeta < \omega_\lambda;$$

(24) for each $\eta < \omega_\lambda$ the sequence $\{Q_\mu^\eta\}_{\mu < \omega_\lambda}$ consists of all non-empty at most enumerable subsets of S_η;

(25) for each pair μ, η, where $\mu, \eta < \omega_\lambda$, there exists such a $\xi < \omega_\lambda$ that $Q_\mu^\eta = Q_\xi$.

This proves lemma 4, because it is enough to put

$$U = \sum_{\xi < \omega_\lambda} S_\xi.$$

In fact (20) and (23) imply $\overline{\overline{U}} = 2^{\aleph_0}$. Further it is obvious, that if $Q \subset U$ and $1 \leqslant \overline{\overline{Q}} \leqslant \aleph_0$, then there exists such an $\alpha < \omega_\lambda$, that $Q \subset \sum_{\xi < \alpha} S_\xi$. Then by (23) we have $Q \subset S_\alpha$, and by (24) we have $Q \in \{Q_\mu^\alpha\}_{\mu < \omega_\lambda}$; thus by (25) there exists such a $\beta < \omega_\lambda$ that $Q = Q_\beta$. It follows from (22) and (23) that $\varphi_\beta U = U - Q$, which proves that U fulfils lemma 4.

Then let us give the inductive definition mentioned above of the sequences (vii) fulfilling the conditions (20)-(25).

We put

(viii) $Q_0 = \{\varphi_0\}$; $S_0 = \{\varphi_0, \varphi_0^2, \varphi_0^3, \ldots\}$; $\{Q_\mu^0\}_{\mu < \omega_\lambda}$ is the sequence of all non-empty at most enumerable subsets of S_0 [12]).

It is obvious that the conditions (20)-(24) are then fulfilled.

Let us suppose that for a certain ordinal number $\alpha < \omega_\lambda$ the sequences $\{S_\xi\}_{\xi < \alpha}$, $\{Q_\mu^\eta\}_{\mu < \omega_\lambda}^{\eta < \alpha}$, $\{Q_\xi\}_{\xi < \alpha}$ are already defined and fulfil (20)-(24). Then it follows that

(26) There exist such pairs of ordinal numbers $(\mu'_\alpha, \eta'_\alpha)$, where $\mu'_\alpha < \omega_\lambda$ and $\eta'_\alpha < \alpha$, that $Q_{\mu'_\alpha}^{\eta'_\alpha} \notin \{Q_\xi\}_{\xi < \alpha}$ but for each pair (μ, η), where $\mu < \omega_\lambda$, $\eta < \alpha$ and $\mu + \eta < \mu'_\alpha + \eta'_\alpha$, we have $Q_\mu^\eta \in \{Q_\xi\}_{\xi < \alpha}$.

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[11]) We do not assume the continuum hypothesis, which implies $\omega_\lambda = \omega_1$.

[12]) The axiom of choice is used here.

Let (μ_a, η_a) be such a pair in which μ_a is the smallest of the ordinal numbers μ'_a; η_a is then also uniquely defined and $<a$.

(27) If we put $N=\{\varphi_\xi\}_{\xi<a}$, $S=\sum_{\xi<a} S_\xi$, $Q=Q^{\eta_a}_{\mu_a}$, $\varphi=\varphi_a$, then the hypotheses of lemma 3 are fulfilled.

Therefore we can put, further,

(ix) $Q_a=Q^{\eta_a}_{\mu_a}$; $S_a=R(\{\varphi_\xi\}_{\xi<a}, \sum_{\xi<a} S_\xi, Q_a, \varphi_a)$ (lemma 3); $\{Q^a_\mu\}_{\mu<\omega_\lambda}$ is the sequence of all non-empty at most enumerable subsets of S_a [12]).

It is obvious that the conditions (20)-(24) are fulfilled because lemma 3 by remark (27) gives a suitable definition of S_a.

Therefore (viii) and (ix) are an inductive definition of the sequences (vii), which fulfil (20)-(24).

It remains to prove that the sequences (vii) defined in this way fulfil condition (25). By the first of the definitions (ix) it is sufficient to prove that the sequence $\{(\mu_\xi, \eta_\xi)\}_{\xi<\omega_\lambda}$ contains each pair (μ, η) occurring in (25). It follows by means of (26), because for each such pair (μ, η) the power of the set of all pairs (μ', η') for which $\mu'+\eta' \leqslant \mu+\eta$ is $<2^{\aleph_0}$, but $\overline{\overline{\omega}}_\lambda=2^{\aleph_0}$.

Thus by the previous remarks lemma 4 is proved.

Proof of theorem 2. Let $[M]$, where $\overline{\overline{M}}=2^{\aleph_0}$, be a free group of rotations of a sphere S around its centre (theorem (T_1)) and p such a point on S that for each $\sigma, \tau \in [M]$

(28) $$\sigma(p)=\tau(p) \quad \text{implies} \quad \sigma=\tau$$

(if such a point does not exist it is sufficient to take one generator out of M and to choose the point p on its axis; such a point will already have property (28)).

Let U be a subset of $[M]$ fulfilling lemma 4.

We put

$$E=U(p).$$

Evidently $\overline{\overline{E}}=2^{\aleph_0}$ and E lies on S. By lemma 4 and property (28) it is obvious that if $D \subset E$ and $1 \leqslant \overline{\overline{D}} \leqslant \aleph_0$, then there exists such a rotation $\varphi \in M$ that

$$\varphi(E)=E-D,$$

which proves theorem 2.

THEOREM 3. *For each infinite cardinal number $\mathfrak{m} \leqslant 2^{\aleph_0}$, there exists on the sphere in the 3-dimensional Euclidean space a set E of power $\mathfrak{m}$ such that*

$$E-F \simeq E$$

for each finite set F.

In the case of $\mathfrak{m}=\aleph_0$ this theorem can be proved effectively. Such an effective proof is similar to the proof of theorem 2; obvious modifications must be introduced only in lemma 4 where $2^{\aleph_0}$ must be replaced by $\aleph_0$ and the sets Q may be non-empty and finite. We observe further that to obtain theorem 2 the axiom of choice was used only in the proof of lemma 4 when we arranged the set M in an ω_λ-sequence, and in the definitions (viii) and (ix). In the present case we need only the existence of an ω_0-sequence of rotations of the sphere which generates a free group; and this follows effectively from (T_1). In the modifications of the definitions (viii) and (ix) which must be introduced we need only to arrange in ω_0-sequences the sets of all finite subsets of sets ordered in ω_0-sequences, which can also be done effectively.

In the case of $\mathfrak{m}=2^{\aleph_0}$ theorem 3 is evidently true by theorem 2.

Proof of theorem 3. We shall use theorem 2. Let H be a set fulfilling that theorem. Then we can assign to each point $p \in H$ such a rotation φ_p that

(29) $$\varphi_p(H)=H-\{p\}.$$

It is evidently sufficient for proving our theorem to construct such a set E of power $\mathfrak{m}$ that

(30) $$E-\{p\} \simeq E \quad \text{for each} \quad p \in E,$$

because for such a set E the congruence $E-F \simeq E$ is also valid for any finite set F.

For each set $K \subset H$, let

$$G_K=[\underset{\varphi}{E}(\varphi=\varphi_p, p \in K)].$$

Let E_1 be an arbitrary subset of H of power $\mathfrak{m}$, and $E_{n+1}=G_{E_n}(E_n)$ for $n=1,2,\ldots$

We put

(31) $$E=H \frown \sum_{n=1}^{\infty} E_n \text{ }^{13}).$$

Thus we have $\overline{\overline{E}}=\aleph_0 \mathfrak{m}$, because $E_1 \subset E$.

For proving (30) it is sufficient to show that

(32) $$\varphi_p(E)=E-\{p\} \quad \text{for each} \quad p \in E.$$

It is obvious that

(33) $$\varphi_p\Big(\sum_{n=1}^{\infty} E_n\Big)=\sum_{n=1}^{\infty} E_n \quad \text{for each} \quad p \in E.$$

The equalities (29), (33) and (31) obviously imply (32), q. e. d.

[13]) $\frown$ is the intersection sign.

Remark. None of the constructions on the sphere which we have described above can be done in this way on the plane, because there exists no free group of isometries of the plane with more then one generator. It follows that for each pair a, b of similarities of the plane the following relation holds:

$$(\mathrm{x}) \qquad a^2b^2a^{-2}b^{-4}a^2b^2a^{-2}b^2a^2b^{-4}a^{-2}b^2=1\ ^{14}).$$

We shall prove a certain generalization of the relation (x).

Let us take the notation $(a,b)=aba^{-1}b^{-1}$ — it is the so called commutator of the elements a and b. We have the following assertion:

(T_2) *For each four similarities of the plane $\varphi, \psi, \chi, \eta$ the following relation holds*:

$$\big((\varphi^2,\psi^2),(\chi^2,\eta^2)\big)=1.$$

(For example the relation (x) follows by the substitution $\varphi=a$, $\psi=b$, $\chi=b^{-1}$, $\eta=a$).

Proof. Let ζ be a similarity of the plane with a complex coordinate z. Then ζ^2 is a similarity without reflexion (preserving orientation), *i. e.*

$$(34) \qquad \zeta^2(z)=a_{\zeta^2}z+b_{\zeta^2},$$

where a_{ζ^2} and b_{ζ^2} are complex numbers uniquely defined by ζ (so $a_{\zeta^2}\neq 0$). Thus we have also

$$(35) \qquad a_{\zeta^{-2}}=\frac{1}{a_{\zeta^2}}.$$

From (34) and (35) it follows that for each two similarities σ and τ the similarity (σ^2,τ^2) is of the form $z+b$ (where b is a complex number), *i. e.* it is a translation. The product of translations is commutative; this proves (T_2), because the commutator of commutative elements vanishes.

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[14]) This follows from a similar relation given by Sierpiński in his paper [5], p. 1.

Continuous functions in the logarithmic-power classification according to Hölder's conditions

by

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Introduction

We shall denote by $\omega(h)$ functions defined and never assuming zero for $h>0$, monotonic, non-decreasing and tending to zero for $h\to 0$. In addition we shall suppose that

$$(1) \qquad \lim_{h\to+0}\Lambda(h)<\infty \quad \text{where} \quad \Lambda(h)=\sup_{0<t\leqslant h}\frac{t}{\omega(t)}.$$

As regards functions denoted in the sequel by $f(x)$ we shall always suppose that they are continuous, defined and bounded in the interval $(-\infty,+\infty)$.

Let H_ω denote the class of functions which for every x and every h [1]) *satisfy the generalized condition of Hölder*

$$(2) \qquad |f(x+h)-f(x)|\leqslant M\omega(|h|),$$

where M denotes a constant dependent only on $f(x)$. We shall suppose that $\omega(h)$ satisfies the condition (1) [2]).

[1]) If condition (2) is satisfied for every h where $|h|<a$ for a certain positive constant a, then $f(x)$ will belong to class H_ω.

[2]) In the case of $\lim \Lambda(h)=\infty$ only constant functions would belong to class H_ω.