additionaly, the principle of choice. But the method of Kelley permits us to demonstrate that the theorem of Tychonoff in the form (4,2) (for Hausdorff's spaces) implies the principle of choice for compact (Hausdorff's) spaces (4,3).

Let $\mathcal{M}=\{M_t\}_{t\in T}$ be a class of compact spaces, and let $p_t$ non $\in \sum_{t\in T} M_t$. We set $M_t^*=M_t+(p_t)$ and $\mathcal{M}^*=\{M_t^*\}_{t\in T}$. If we consider the point $p_t$ as isolated in $M_t^*$, then every $M_t^*$ is a compact space, and $M_t$ is closed in $M_t^*$.

Each set $F_{t_0}=\underset{X}{E}[\overline{\overline{X\cdot M_t^*}}=1$ for $t\in T$ and $X\cdot M_{t_0}^*\subset M_t]$ is a closed subset of the product space of $\mathcal{M}^*$ and $F_{t_1}\ldots F_{t_n}\neq 0$ for every finite number of $t_i\in T$. Therefore $\prod_{t\in T} F_t$ is non empty, but it is the product of $\mathcal{M}$. We have demonstrated that (4,2) is equivalent to each of the propositions (I)-(V).

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*Reçu par la Rédaction le 31. 12. 1952*

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# Intersections of prescribed power, type, or measure

by

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In 1914, Mazurkiewicz [5] showed that there exists a set of points in the plane, which intersects every straight line in the plane in precisely two points. Recently, Bagemihl [1] proved a general intersection theorem in the theory of sets, which, when applied to the plane, yields the following generalization of Mazurkiewicz's result: With every straight line $s$, associate a cardinal number $\mathfrak{q}_s\geqslant 2$ so that the sum of fewer than $2^{\aleph_0}$ of the numbers $\mathfrak{q}_s$ is always less than $2^{\aleph_0}$. Then there exists a set of points which intersects every straight line $s$ in exactly $\mathfrak{q}_s$ points.

In the present paper, after extending the general intersection theorem alluded to above, we obtain several theorems dealing with plane point sets which intersect every straight line in a set of prescribed power, order type, or measure. In particular, we show that the aforementioned $\mathfrak{q}_s$ may be chosen arbitrarily in the range $2\leqslant\mathfrak{q}_s\leqslant 2^{\aleph_0}$. Free use is made of the well-ordering theorem.

THEOREM 1. *Let $a$ be an arbitrary, fixed ordinal number, and $S$ be a set with*

$$\overline{\overline{S}}\leqslant\aleph_a. \tag{1}$$

*To every $s\in S$ let there correspond a set $L_s$ such that, for every $S'\subseteq S-\{s\}$ with $\overline{\overline{S'}}<\aleph_a$,*

$$\overline{\overline{L_s-\sum_{s'\in S'}L_{s'}}}\geqslant\aleph_a, \tag{2}$$

*and put $P=\sum_{s\in S}L_s$.*

*Suppose that for every $s\in S$ there exists a cardinal number $\mathfrak{l}_s$, with $1\leqslant\mathfrak{l}_s\leqslant\aleph_a$, such that the following holds: If $D\subset P$, $\overline{\overline{D}}<\aleph_a$, and $S_D$ is the set of elements $s'\in S$ for which $\mathfrak{l}_{s'}<\aleph_a$ and $\overline{\overline{L_{s'}D}}\geqslant\mathfrak{l}_{s'}$, then*

$$\overline{\overline{S_D}}<\aleph_a. \tag{3}$$

*With every $s\in S$ let there be associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ satisfying*

$$\mathfrak{l}_s\leqslant\mathfrak{q}_s\leqslant\aleph_a. \tag{4}$$

*Then there exists a set* $P^*\subseteq P$ *(with* $\overline{\overline{P^*}}\leqslant\aleph_\alpha$*) such that* $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$ *for every* $s\in S$.

We first prove

LEMMA 1. *Let* $D\subset P$, $\overline{\overline{D}}<\aleph_\alpha$, *and*

(5) $$\overline{\overline{L_sD}}\leqslant\mathfrak{q}_s \quad \textit{for every} \quad s\in S.$$

*Suppose that for some element* $e\in S$,

(6) $$\overline{\overline{L_eD}}<\mathfrak{q}_e.$$

*Denote by* $S_D^*$ *the set of elements* $s'\in S$ *for which* $\overline{\overline{L_{s'}D}}=\mathfrak{q}_{s'}<\aleph_\alpha$. *Then there exists an* $a\in L_e-(D+\sum_{s'\in S_D^*}L_{s'})$ *such that*

(7) $$\overline{\overline{L_s(D+\{a\})}}\leqslant\mathfrak{q}_s$$

*for every* $s\in S$. (Such an $a$ will be called an *admissible element* of $L_e$ relative to $D$).

Proof: It is easy to see from (4) and (3) that $\overline{\overline{S_D^*}}\leqslant\overline{\overline{S_D}}<\aleph_\alpha$. Therefore, by (2), $\overline{\overline{L_e-\sum_{s'\in S_D^*}L_{s'}}}\geqslant\aleph_\alpha$, so that $L_e-(D+\sum_{s'\in S_D^*}L_{s'})$ contains at least one element — call it $a$.

Now let $s\in S$. Then $s$ satisfies at least one of the following conditions: 1. $s\in S_D^*$, 2. $\overline{\overline{L_sD}}<\mathfrak{q}_s$, 3. $\mathfrak{q}_s=\aleph_\alpha$. If 1, then (7) follows from the fact that $a$ non-$\in\sum_{s'\in S_D^*}L_{s'}$; if 2, then (7) is obvious; if 3, then (7) follows from (5). This completes the proof of the lemma.

Proof of Theorem 1. The case $\overline{\overline{S}}=0$ is trivial. We may therefore assume that $\overline{\overline{S}}>0$. Note (1) and (4), consider $\mathfrak{q}_s$ replicas of every $L_s (s\in S)$, well-order the resulting complex of $\sum_{s\in S}\mathfrak{q}_s\leqslant\aleph_\alpha^2=\aleph_\alpha$ sets to form a sequence

(8) $$L_0, L_1, \ldots, L_\xi, \ldots \qquad (\xi<\varrho),$$

where $1\leqslant\varrho\leqslant\omega_\alpha$, and denote by $\mathfrak{q}_\xi$ $(\xi<\varrho)$ the respective cardinal numbers associated with the sets (8) according to (4).

We define sets $A_\xi\subset P$ $(\xi<\varrho)$, by induction on $\xi$, as follows: Let $A_0$ consist of a single element of $L_0$. Suppose that $0<\xi<\varrho$, and that for every $\mu<\xi$ a set $A_\mu\subset P$, with $\overline{\overline{A_\mu}}\leqslant 1$, has been defined so that $\overline{\overline{L_s\sum_{\mu<\xi}A_\mu}}\leqslant\mathfrak{q}_s$ for every $s\in S$. Evidently $\overline{\overline{\sum_{\mu<\xi}A_\mu}}\leqslant\overline{\overline{\xi}}<\aleph_\alpha$. If $\overline{\overline{L_\xi\sum_{\mu<\xi}A_\mu}}=\mathfrak{q}_\xi$, let $A_\xi=0$. If, however, $\overline{\overline{L_\xi\sum_{\mu<\xi}A_\mu}}<\mathfrak{q}_\xi$, let $A_\xi$ consist of a single admissible element of $L_\xi$



relative to $\sum_{\mu<\xi}A_\mu$; the existence of such an element is guaranteed by Lemma 1. The sets $A_\xi$ $(\xi<\varrho)$ thus defined are obviously mutually exclusive.

Put $P^*=\sum_{\xi<\varrho}A_\xi$. Then $P^*\subseteq P$ and $\overline{\overline{P^*}}\leqslant\overline{\varrho}\leqslant\aleph_\alpha$.

Now let $s\in S$. It is clear from the definition of the sets $A_\xi$ $(\xi<\varrho)$ that $\overline{\overline{L_s\sum_{\mu\leqslant\xi}A_\mu}}\leqslant\mathfrak{q}_s$ for every $\xi<\varrho$. If $\overline{\overline{L_s\sum_{\mu\leqslant\xi}A_\mu}}<\mathfrak{q}_s$ for every $\xi<\varrho$, then $\overline{\overline{L_s\sum_{\mu<\varrho}A_\mu}}\leqslant\mathfrak{q}_s$. There are $\mathfrak{q}_s$ values of $\xi<\varrho$ for which $L_s=L_\xi$, and for every such $\xi$, $\overline{\overline{L_sA_\xi}}=1$, so that $\overline{\overline{L_s\sum_{\mu<\varrho}A_\mu}}\geqslant\mathfrak{q}_s$. Hence $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$. If, however, for some $\xi'<\varrho$, $\overline{\overline{L_s\sum_{\mu\leqslant\xi'}A_\mu}}=\mathfrak{q}_s$, then $\mathfrak{q}_s\leqslant\overline{\overline{\xi'+1}}<\aleph_\alpha$, and, from the definition of an admissible element, it follows that $\overline{\overline{L_s\sum_{\xi'<\xi<\varrho}A_\xi}}=0$, so that in this case too $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$. This completes the proof of Theorem 1.

The following theorem was presented by the authors (see [2]) to the American Mathematical Society, May 28, 1952:

A complex $\mathfrak{C}$ of cardinal numbers is said to be *strongly less than* $\aleph_\alpha$, if every sum of fewer than $\aleph_\alpha$ terms belonging to $\mathfrak{C}$ is less than $\aleph_\alpha$. *Let* $a$ *be an arbitrary, fixed ordinal number,* $S$ *and* $P$ *be sets, and to every* $s\in S$ *let there correspond a subset,* $L_s$, *of* $P$. *Suppose that the following conditions are satisfied:*

(I) $\overline{\overline{S}}\leqslant\aleph_\alpha$.

(II) $\overline{\overline{L_s}}\geqslant\aleph_\alpha$ *for every* $s\in S$.

(III) *If* $s\in S$, *there is a cardinal number* $\mathfrak{n}_s\geqslant 1$ *such that, if* $s'\in S$ *and* $s'\neq s$, *then* $\overline{\overline{L_sL_{s'}}}\leqslant\mathfrak{n}_s$, *and the complex of cardinal numbers* $\mathfrak{n}_s$ $(s\in S)$ *is strongly less than* $\aleph_\alpha$.

(IV) *There is a cardinal number* $\mathfrak{m}\geqslant 1$ *with the following properties:* (a) $\mathfrak{d}^{\mathfrak{m}}<\aleph_\alpha$ *for every* $\mathfrak{d}<\aleph_\alpha$; (b) *if* $P'\subseteq P$, $\overline{\overline{P'}}=\mathfrak{m}$, *and* $\mathfrak{m}_{P'}$ *is the number of elements* $s\in S$ *for which* $P'$ *is a subset of* $L_s$, *then the complex of cardinal numbers* $\mathfrak{m}_{P'}$ *obtained by letting* $P'$ *run through all the subsets of* $P$ *having* $\mathfrak{m}$ *elements, is strongly less than* $\aleph_\alpha$.

*Suppose that with every* $s\in S$ *there is associated in an arbitrary manner a cardinal number* $\mathfrak{q}_s$ *such that*

(V) $\mathfrak{m}+\mathfrak{n}_s-1\leqslant\mathfrak{q}_s\leqslant\aleph_\alpha$.

*Then there exists a subset* $P^*$ *of* $P$ *(with* $\overline{\overline{P^*}}\leqslant\aleph_\alpha$*) such that* $\overline{\overline{L_sP^*}}=\mathfrak{q}_s$ *for every* $s\in S$.

We shall now show that this theorem, which is a generalization of one due to Bagemihl [1], is contained in Theorem 1, by proving that (I)-(V) imply (1)-(4) (that the converse is not true will be evident from the proof).

Condition (1) follows trivially from (I).

To prove (2), suppose that $S'\subseteq S-\{s\}$ and $\overline{\overline{S'}}<\aleph_\alpha$. If $s'\in S'$, then, according to (III), $\overline{\overline{L_{s'}L_s}}\leqslant\mathfrak{n}_{s'}$, and (III) also implies that

$$\overline{\overline{L_s\sum_{s'\in S'}L_{s'}}}\leqslant\sum_{s'\in S'}\overline{\overline{L_{s'}L_s}}\leqslant\sum_{s'\in S'}\mathfrak{n}_{s'}<\aleph_\alpha.$$

According to (II), then,

$$\overline{\overline{L_s - L_s \sum_{s' \in S'} L_{s'}}} = \overline{\overline{L_s - \sum_{s' \in S'} L_{s'}}} \geqslant \aleph_\alpha,$$

which is (2).

Now to prove (3), take every $\mathfrak{l}_s$ in Theorem 1 to be the $\mathfrak{m}$ in (IV). Let $D \subset P$, $\overline{\overline{D}} < \aleph_\alpha$, and $S_D$ be the set of elements $s' \in S$ for which $\overline{\overline{L_{s'}D}} \geqslant \mathfrak{l}_{s'}$ $(=\mathfrak{m})$. There are not more than $\overline{\overline{D}}^{\mathfrak{m}}$ subsets of $D$ of $\mathfrak{m}$ elements. By (a) of (IV), $\overline{\overline{D}}^{\mathfrak{m}} < \aleph_\alpha$, and by (b) of (IV), $\overline{\overline{S_D}} < \aleph_\alpha$, which proves (3).

Finally, according to (V), (4) is certainly satisfied if $\mathfrak{l}_s = \mathfrak{m}$.

The following examples show that if one of the hypotheses (1)-(4) of Theorem 1 is not satisfied, then the conclusion of this theorem may no longer be true:

$\overline{(1)}$, (2), (3), (4): Let $S = \{\mu\}_{\mu < \omega_{\alpha+1}+\omega_\alpha}$, $P = \{(\xi, \eta)\}_{\xi < \omega_{\alpha+1}, \eta < \omega_\alpha}$, $L_\xi = \{(\xi, \eta)\}_{\eta < \omega_\alpha}$ for every $\xi < \omega_{\alpha+1}$, $L_{\omega_{\alpha+1}+\eta} = \{(\xi, \eta)\}_{\xi < \omega_{\alpha+1}}$ for every $\eta < \omega_\alpha$. Then $\mathfrak{l}_\mu = 1$, and we take $\mathfrak{q}_\mu = 1$, for every $\mu \in S$.

If $\overline{\overline{L_\xi P^*}} = 1$ for every $\xi < \omega_{\alpha+1}$, then $\overline{\overline{L_{\omega_{\alpha+1}+\eta} P^*}} = \aleph_{\alpha+1}$ for some $\eta < \omega_\alpha$, contradicting $\mathfrak{q}_{\omega_{\alpha+1}+\eta} = 1$.

(1), $\overline{(2)}$, (3), (4): Let $S = \{\mu\}_{\mu < \omega_{\alpha+1}}$, $P = \{\xi\}_{\xi < \omega_\alpha}$, $L_\xi = \{\xi\}$ for every $\xi < \omega_\alpha$, $L_{\omega_\alpha+1} = \{\xi\}_{\xi < \omega_\alpha}$. Then $\mathfrak{l}_\mu = 1$, and we take $\mathfrak{q}_\mu = 1$, for every $\mu \in S$.

We have $\overline{\overline{L_{\omega_\alpha+1} P^*}} = \aleph_\alpha \neq \mathfrak{q}_{\omega_\alpha+1}$.

(1), (2), $\overline{(3)}$, (4): Let $P$ be the set of points of a projective plane in which every line contains $\aleph_\alpha$ points, and let the sets $L_s$ be the lines in this plane. Take every $\mathfrak{l}_s = 1$; then (3) is not satisfied. Take also every $\mathfrak{q}_s = 1$.

If we consider any two points of $P^*$, these two points determine a line $L_s$, and thus $\overline{\overline{L_s P^*}} \geqslant 2 > \mathfrak{q}_s$.

(1), (2), (3), $\overline{(4)}$: In the preceding example, take every $\mathfrak{l}_s = 2$. If we take every $\mathfrak{q}_s = 1$ or $> \aleph_\alpha$, then it is obvious that $P^*$ does not exist.

Theorem 1 can be applied, *e. g.*, to the points and straight lines of a Euclidean plane. In this case we interpret $S$ as a set of indices for the set of straight lines in the plane and $L_s$ as the set of points which constitute the straight line $s \in S$, $P$ as the set of points in the plane, and $\aleph_\alpha = 2^{\aleph_0}$. Conditions (1) and (2) are evidently satisfied, and (3) clearly holds if we take every $\mathfrak{l}_s = 2$. Thus we obtain

COROLLARY 1. *With every straight line $s$ in a Euclidean plane associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a set of points which intersects every straight line $s$ in precisely $\mathfrak{q}_s$ points.*



This result was obtained by Mazurkiewicz [5] for the case $\mathfrak{q}_s = 2$ for every $s$; by Bagemihl [1] for the case $\mathfrak{q}_s \geqslant 2$ for every $s$, the complex of cardinal numbers $\mathfrak{q}_s$ being strongly less than $2^{\aleph_0}$; and in the above form, independently and practically simultaneously (early in 1952), by Sierpiński [8] and by the present authors.

As is easily seen, condition (3) is also satisfied under the weaker assumptions of

COROLLARY 2. *With every point $p$ of the set, $P$, of points of a Euclidean plane associate a set, $S_p$, of $\mathfrak{k}_p$ straight lines in this plane, each of which contains the point $p$, and let the complex of cardinal numbers $\mathfrak{k}_p$ $(p \in P)$ be strongly less than $2^{\aleph_0}$. Let $S' = \sum_{p \in P} S_p$. With every $s \in S'$ associate a cardinal number $\mathfrak{q}_s$ such that $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, and with every $s$ non-$\in S'$ associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a set of points which intersects every straight line $s$ in the plane in precisely $\mathfrak{q}_s$ points.*

Let the word *curve* mean any set of points $(x, y)$ satisfying an equation of the form $y = f(x)$ where $f$ is a single-valued function of a real variable (*cf.* p. 11 of [9]). Take $S'$ in Corollary 2 to be the set of all straight lines parallel to the $y$-axis, and let $\mathfrak{q}_s = 1$ for every $s \in S'$, $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$ for every $s$ non-$\in S'$. Then $\mathfrak{k}_p = 1$ for every $p \in P$, and Corollary 2 yields

COROLLARY 3. *With every straight line $s$ (in a Euclidean plane) which is not parallel to the $y$-axis associate a cardinal number $\mathfrak{q}_s$ such that $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$. Then there exists a curve which intersects every straight line $s$ which is not parallel to the $y$-axis in precisely $\mathfrak{q}_s$ points.*

Corollary 2 suggests the following problems dealing with the Euclidean plane:

What is a necessary and sufficient condition on a set, $S^*$, of straight lines so that, if with every $s \in S^*$ there is associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ in the range $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, and if with every $s$ non-$\in S^*$ there is associated in an arbitrary manner a cardinal nurmber $\mathfrak{q}_t$ in the range $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, there exists a set of points which intersects every straight line $s$ in precisely $\mathfrak{q}_s$ points?

What is the answer if $\mathfrak{q}_s = 1$ $(s \in S^*)$ instead of $\mathfrak{q}_s$ being chosen arbitrarily in the range $1 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$?

We have been able to solve the following problem:

What is a necessary and sufficient condition on a set, $S^*$, of straight lines so that, if with every $s \in S^*$ there is associated in an arbitrary manner a cardinal number $\mathfrak{q}_s$ in the range $2 \leqslant \mathfrak{q}_s \leqslant 2^{\aleph_0}$, there exists a set of points, $P^*$, which intersects every straight line $s \in S^*$ in precisely $\mathfrak{q}_s$ points and every straight line $s$ non-$\in S^*$ in less than 2 points?

The answer is: *$S^*$ is the set of straight lines joining all pairs of points of a point set $M$ having the property that if a straight line contains at least 2 points of $M$, it contains $2^{\aleph_0}$ points of $M$.*

To see that the condition is necessary, take $\mathfrak{q}_s=2^{\aleph_0}$ $(s\in S^*)$; then $P^*$ is such a set $M$. To show that the condition is sufficient, in Theorem 1 take $\aleph_a=2^{\aleph_0}$, $S=S^*$, $P=M$, $L_s=$ the subset of $M$ which is contained in the straight line $s\in S^*$, $\mathfrak{l}_s=2$ $(s\in S^*)$; the set $P^*$ of Theorem 1 is then obviously one that has the properties required of the set $P^*$ in the problem.

In what follows we still deal with the Euclidean plane, but when we speak of a straight line we shall tacitly assume that one of the two possible orientations has been assigned to it, and when we speak of a subset of a straight line we shall regard the subset as ordered by the orientation of the line, so that the (order) type of such a subset is well-defined.

THEOREM 2. *With every straight line $s$ associate in an arbitrary manner a finite or an enumerable order type $\tau_s\neq 0, 1$. Then there exists a set of points whose intersection with every straight line $s$ is a set of type $\tau_s$.*

Proof: Well-order the set of straight lines to form a sequence

$$s_0, s_1, \ldots, s_\xi, \ldots \qquad (\xi<\omega_\gamma),$$

where $\omega_\gamma$ is the initial number of $Z(2^{\aleph_0})$, and let the sequence of associated types be

$$\tau_0, \tau_1, \ldots, \tau_\xi, \ldots \qquad (\xi<\omega_\gamma).$$

Let $T_0$ be a set of points on $s_0$ such that $\overline{T}_0=\tau_0$. Let $0<\xi<\omega_\gamma$, and suppose that, for every $\mu<\xi$, a set, $T_\mu$, of points on $s_\mu$ has been defined, such that $\overline{T}_\mu=\tau_\mu$, and so that at most 2 points of $s_\xi$ belong to $\sum_{\mu<\xi}T_\mu=T'_\xi$. We have

$$\overline{\overline{T'_\xi}}=\overline{\overline{\sum_{\mu<\xi}T_\mu}}\leqslant\sum_{\mu<\xi}\overline{\overline{T_\mu}}\leqslant\overline{\overline{\xi}}\,\aleph_0<2^{\aleph_0}.$$

Consequently, there are fewer than $2^{\aleph_0}$ straight lines such that each contains at least 2 points of the set $T'_\xi$. Therefore the set, $V_\xi$, of points on $s_\xi$ which are not on any of these lines different from $s_\xi$ is everywhere dense on $s_\xi$, so that every interval of $s_\xi$ contains a subset of $V_\xi$ of any given finite or enumerable type.

If $s_\xi$ contains no point of $T'_\xi$, let $T_\xi$ be any subset of $V_\xi$ such that $\overline{T}_\xi=\tau_\xi$. If $s_\xi$ has precisely one point, $p$, in common with $T'_\xi$, write $\tau_\xi=\sigma_\xi+1+\varrho_\xi$, and let $T_\xi$ consist of the points of a subset, of type $\sigma_\xi$, of $V_\xi$ preceding $p$, $p$ itself, and the points of a subset, of type $\varrho_\xi$, of $V_\xi$ succeeding $p$. Finally, if $s_\xi$ has two points, $p, p'$, in common with $T'_\xi$, write $\tau_\xi=\sigma_\xi+1+\varrho_\xi+1+\zeta_\xi$, and define $T_\xi$ in the obvious manner. Denote by $T$ the union of the sets $T_\xi$ $(\xi<\omega_\gamma)$ thus defined. Evidently $T$ intersects every $s$ in a set of type $\tau_s$, and the proof of the theorem is complete.



Call an order type $\tau$ a *subtype of the continuum*, if the linear continuum contains an ordered subset of type $\tau$. We shall prove

THEOREM 3. *With every straight line $s$ associate in an arbitrary manner a subtype, $\tau_s\neq 0, 1$, of the continuum. Then, if $2^{\aleph_0}=\aleph_1$, there exists a set of points whose intersection with every straight line $s$ is a set of type $\tau_s$ and measure* 0.

Let us term a linear perfect set *sparse*, if it is a dyadic discontinuum (cf. [3], p. 134), $D$, whose dyadic schema at the $n$-th stage consists of $2^n$ mutually exclusive closed intervals, the length of the largest of which is $o(4^{-n})$.

Suppose that $s_1, s_2, s_3$ are three straight lines and $E_1, E_2$ are sets of points on $s_1, s_2$, respectively. Consider the set of all straight lines each of which is different from $s_3$ and is determined by a point of $E_1$ and a point of $E_2$. This set of lines intersects $s_3$ in a set of points which we shall call the *mutual projection of $E_1$ and $E_2$ on $s_3$* and denote by $(E_1, E_2; s_3)$.

LEMMA 2. *Let $s_1, s_2, s_3$ be three straight lines and $D_1, D_2$ be sparse perfect sets on $s_1$, $s_2$, respectively. Then*

$$\operatorname{meas}(D_1, D_2; s_3)=0.$$

Proof: Denote by $p_{13}, p_{23}$ the points of intersection (if nonexistent, to be disregarded in what follows) of $s_1$ and $s_3$, $s_2$ and $s_3$, respectively. Let $\varepsilon>0$, $p$ be an arbitrary, but fixed, point, $C$ be a circle with radius $\varepsilon^{-1}$ and center $p$, and $C_{13}$ and $C_{23}$ be circles of radius $\varepsilon$ and with the respective centers $p_{13}, p_{23}$. Choose $\varepsilon$ so small that $C_{13}$ and $C_{23}$ lie in the interior of $C$, and hold $\varepsilon$ fixed for the time being. Denote by $R_\varepsilon$ the region inside $C$ and outside $C_{13}$ and $C_{23}$, and let $D_1^\varepsilon=D_1R_\varepsilon$, $D_2^\varepsilon=D_2R_\varepsilon$. Suppose that $I_1^n, I_2^n$ signify the parts in $R_\varepsilon$ of any two intervals of the $n$-th stage of the dyadic schemata representing $D_1, D_2$, respectively. Then it is not difficult to see that

$$\operatorname{meas}\big(R_\varepsilon\cdot(I_1^n, I_2^n; s_3)\big)<c_\varepsilon\cdot(\operatorname{meas}I_1^n+\operatorname{meas}I_2^n)=c_\varepsilon\cdot o(4^{-n}),$$

where $c_\varepsilon$ is a constant depending only on $\varepsilon$. Since at most $4^n$ pairs of intervals of the $n$-th stage come into question,

$$\operatorname{meas}\big(R_\varepsilon\cdot(D_1^\varepsilon, D_2^\varepsilon; s_3)\big)<4^nc_\varepsilon\cdot o(4^{-n}),$$

and letting $n\to\infty$ we see that

$$\operatorname{meas}\big(R_\varepsilon\cdot(D_1^\varepsilon, D_2^\varepsilon; s_3)\big)=0.$$

Now let $\{\varepsilon_k\}$ be a sequence of sufficiently small positive numbers tending monotonically to 0. Then

$$0=\sum_{k=1}^{\infty}\operatorname{meas}\left(R_{\varepsilon_k}\cdot(D_1^{\varepsilon k},D_2^{\varepsilon k};s_3)\right)\geqslant\operatorname{meas}\sum_{k=1}^{\infty}\left(R_{\varepsilon_k}\cdot(D_1^{\varepsilon k},D_2^{\varepsilon k};s_3)\right)$$
$$=\operatorname{meas}(D_1,D_2;s_3),\qquad\text{q. e. d.}$$

Proof of Theorem 3. Well-order the set of straight lines to form a sequence

$$s_0,s_1,\ldots,s_\xi,\ldots\qquad(\xi<\omega_1),$$

and let the sequence of associated types be

$$\tau_0,\tau_1,\ldots,\tau_\xi,\ldots\qquad(\xi<\omega_1).$$

Let $D_0$ be a sparse perfect set on $s_0$. Since $D_0$ is perfect, it contains a subset $T_0$ of type $\tau_0$, and since $D_0$ is of measure 0, so is $T_0$. Let $0<\xi<\omega_1$, and suppose that, for every $\mu<\xi$, a sparse perfect set $D_\mu$, and a set $T_\mu\subset D_\mu$, with $\overline{T}_\mu=\tau_\mu$, have been defined on $s_\mu$ in such a manner that at most 2 points of $s_\xi$ belong to $\sum_{\mu<\xi}T_\mu=T'_\xi$. According to Lemma 2,

$$0=\operatorname{meas}\sum_{\mu<\nu<\xi}(D_\mu,D_\nu;s_\xi)=\operatorname{meas}\sum_{\mu<\nu<\xi}(T_\mu,T_\nu;s_\xi),$$

because the sum contains at most $\aleph_0$ terms (this is where we make use of the assumption $2^{\aleph_0}=\aleph_1$). Hence, the point set

$$V_\xi=s_\xi-\sum_{\mu<\nu<\xi}(T_\mu,T_\nu;s_\xi)$$

is of positive measure in every interval of $s_\xi$. Consequently, every interval of $s_\xi$ contains a perfect subset of $V_\xi$, and therefore also contains a sparse perfect subset of $V_\xi$, which in turn contains a subset having as its type any given subtype of the continuum. The rest of the proof is verbally identical with the last paragraph of the proof of Theorem 2. It is merely necessary to note that, if $D,D',D''$ are sparse perfect sets on some straight line, and every point of $D$ precedes every point of $D'$, and every point of $D'$ precedes every point of $D''$, then the union $D+D'+D''$ is also a sparse perfect set.

It would be interesting to know whether or not the assumption $2^{\aleph_0}=\aleph_1$ is necessary in Theorem 3.

THEOREM 4. *With every straight line $s$ associate in an arbitrary manner a number $m_s$ such that $0\leqslant m_s\leqslant\infty$. Then, if $2^{\aleph_0}=\aleph_1$, there exists a set of points whose intersection with every straight line $s$ is a set of measure $m_s$.*

Proof: Well-order the set of straight lines to form a sequence

$$s_0,s_1,\ldots,s_\xi,\ldots\qquad(\xi<\omega_1),$$

and let the sequence of associated measures be

$$m_0,m_1,\ldots,m_\xi,\ldots\qquad(\xi<\omega_1).$$



Let $M_0$ be a point set, of measure $m_0$, on $s_0$. Let $0<\xi<\omega_1$, and suppose that, for every $\mu<\xi$, $M_\mu$ is a point set, of measure $m_\mu$, on $s_\mu$. Since $\overline{\xi}\leqslant\aleph_0$, the intersection of $\sum_{\mu<\xi}M_\mu$ with $s_\xi$ is an at most enumerable set of points, and is therefore of measure 0, so that it is possible to define $M_\xi$ as a subset, of measure $m_\xi$, of $s_\xi-\sum_{\mu<\xi}M_\mu$. Evidently the set $M=\sum_{\xi<\omega_1}M_\xi$ intersects each $s$ in a set of measure $m_s$, q. e. d.

Instead of assuming that $2^{\aleph_0}=\aleph_1$, it is sufficient to assume that every linear set of power less than $2^{\aleph_0}$ has measure 0.

*Added during printing.* Following a kind suggestion of K. Gödel's, we are able to show that the assumption $2^{\aleph_0}=\aleph_1$ is unnecessary in Theorem 3. Specifically, we shall prove

THEOREM 5. *With every straight line $s$ in the plane, associate in an arbitrary manner a subtype, $\tau_s\neq 0,1$, of the continuum. Then there exists a set of points whose intersection with every straight line $s$ (assumed, for simplicity, to be oriented in the positive sense relative to a set of Cartesian coordinate axes) is a set of type $\tau_s$.*

Proof. Well-order the set of straight lines to form a sequence

$$s_0,s_1,\ldots,s_\xi,\ldots\qquad(\xi<\omega_\gamma),$$

where $\omega_\gamma$ is the initial number of $Z(2^{\aleph_0})$, and let the sequence of associated types be

$$\tau_0,\tau_1,\ldots,\tau_\xi,\ldots\qquad(\xi<\omega_\gamma).$$

Every $s_\xi$ has an equation either of the form

$$\text{(i)}\quad y=a_\xi x+b_\xi\qquad\text{or}\qquad\text{(ii)}\quad x=c_\xi,$$

where $a_\xi$, $b_\xi$, $c_\xi$ are real numbers, the "constants belonging to $s_\xi$".

Let $M$ be the system of algebraically independent real numbers constructed by J. von Neumann [6]. The set $M$ contains a perfect subset (cf. S. Ruziewicz and W. Sierpiński [7], p. 18). Every perfect set contains $2^{\aleph_0}$ mutually exclusive perfect subsets (see, *e. g.*, C. Kuratowski and W. Sierpiński [4], p. 195).

Let $I_0,I_1,\ldots,I_n,\ldots$ $(n<\omega)$ be the set of all nonempty open intervals with rational endpoints, and let $B_0,B_1,\ldots,B_n,\ldots$ $(n<\omega)$ be $\aleph_0$ mutually exclusive, bounded, perfect subsets of $M$. For every $n<\omega$, there exists a one-to-one transformation $t'=r_nt+r'_n$, where $r_n$, $r'_n$ are rational numbers and $r_n\neq 0$, under which the image of $B_n$ is a perfect subset, $C_n$, of $I_n$. The sets $C_n$ $(n<\omega)$ are mutually exclusive, and $\sum_{n<\omega}C_n$ is an algebraically independent system of real numbers (cf. J. von Neumann [6], p. 140). Each $C_n$ contains $2^{\aleph_0}$ mutually exclusive perfect subsets $P_n^\xi$ $(\xi<\omega_\gamma)$. For every $\xi<\omega_\gamma$, we define $P_\xi=\sum_{n<\omega} P_n^\xi$; then every nonempty open interval of real numbers contains a perfect subset of $P_\xi$.

Denote by $R_0$ a subset, of type $\tau_0$, of $P_0$, and let $T_0$ be the set of points on $s_0$ whose abscissas or ordinates form the set $R_0$ according as $s_0$ is of the form (i) or (ii). Now suppose that $0<\xi<\omega_\gamma$, and that we have defined, for every $\mu<\xi$, the set $R_\mu \subseteq P_\mu$ and the set $T_\mu$ on $s_\mu$ in such a manner that the orthogonal projection of $T_\mu$ onto the $x$- or the $y$-axis is $R_\mu$ according as $s_\mu$ is of the form (i) or (ii), that $\overline{T}_\mu=\tau_\mu$, and that at most 2 points of $s_\xi$ belong to $\sum_{\mu<\xi} T_\mu=T'_\xi$. Denote by $K_\xi$ the set of all constants belonging to at least one $s_\mu$ $(\mu\leqslant\xi)$; obviously $\overline{\overline{K_\xi}}<2^{\aleph_0}$. Consequently, the cardinal number of the set, $D_\xi$, of elements of $P_\xi$ which are algebraically dependent on the system of real numbers $K_\xi+\sum_{\mu<\xi} R_\mu$ is less than $2^{\aleph_0}$. Hence, every nonempty open interval of real numbers contains a perfect subset of $Q_\xi=P_\xi-D_\xi$. Let $V_\xi$ be the set of points on $s_\xi$ whose abscissas or ordinates form the set $Q_\xi$ according as $s_\xi$ is of the form (i) or (ii). No straight line different from $s_\xi$ and determined by two points of $T'_\xi$ can intersect $s_\xi$ in a point of $V_\xi$, for otherwise the system of real numbers $K_\xi+\sum_{\mu<\xi} R_\mu+Q_\xi$ would not be algebraically independent, contradicting the definition of $Q_\xi$. The proof of our theorem can now be completed in accordance with the last paragraph of the proof of Theorem 2 above (if we define $R_\xi$ as the orthogonal projection of $T_\xi$ onto the appropriate coordinate axis).

*Added in proof.* We are indebted to A. Rosenthal for calling our attention to his papers *Über Gebilde mit einzigem Ordnungsindex,* Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, 1922, p. 221-240, and *Über die Nichtexistenz von Kontinuen in gewissen Mengen mit einziger Ordnungszahl,* Sitzungsberichte der Heidelberger Akademie der Wissenschaften, mathematisch-natur-wissenschaftliche Klasse, 1934, p. 49-56, the first of which contains, in addition to other results, a special case of our Theorem 1. A paper by J. Moneta, *Application du théorème du continu,* Cahiers Rhodaniens 4 (1952), p. 29-42 in which an unnecessary appeal is made to the (unproved) relation $2^{\aleph_0}=\aleph_1$, contains a particular case of our Corollary 2.

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*Reçu par la Rédaction le 17. 3. 1953*