Effectiveness of the representation theory for Boolean algebras

by

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Stone calls Fundamental Existence Proposition of Ideal Arithmetic the lemma according to which:

(I) In every Boolean algebra there is a prime ideal.

This lemma plays a chief part in the demonstration that

(R) any Boolean algebra is isomorphic with a field of sets,

which is the most important result of Stone's representation theory.

According to Stone, (R) is effectively equivalent to (I). It was a long time ago noticed, that (I) holds only with the help of transfinite methods. All the known proofs of it (S. Ulam, A. Tarski, M. H. Stone) are based upon the principle of choice (or well-ordering theorem). A problem arises, whether the proposition (I) is really dependent on the principle of choice, and especially whether some particular cases of that principle are the consequences of the above-mentioned proposition.

A partial solution of this problem is given by W. Sierpiński. It is known that the result of (I), without the use of transfinite methods, is that in the field of all subsets of an arbitrary infinite set E, there exists a two-valued measure which vanishes for one-point sets. Sierpiński has proved that the existence of such a measure in the set of...
integers allows us to construct a non-measurable function in the sense of Lebesgue. It proves that (I) is at least of the same degree of inef- 
tivity, as the existence of such a function.

In this paper we give a full solution of the above mentioned problem. We 
demonstrate that (I) is effectively equivalent to other theorems, 
among others to the theorem of consistent choice. From this imme-
adiately results the principle of choice from finite sets, and even the or-
dering principle, therefore the problem is solved.

The problem whether the choice principle is independent of (I) is 
not discussed here; it remains open. We suppose that it shall have a 
positive solution.

Definitions and lemmas. A Boolean algebra is a set $\mathcal{A}$ of 
elements (denoted by $a, b, \ldots$) with three operations: addition ($a + b$), multi-
plication ($a \cdot b$) and complementation ($a'$), satisfying the well-known 
axioms. The relation $a + b = b, a \cdot b = b$ denotes $a \sqcup b, \lnot$ parti-
ly orders the set $\mathcal{A}$. This partial order has the least and the greatest 
element in $\mathcal{A}$; these elements are denoted by 0 and $|\mathcal{A}|$ respec-
tively. If $X$ is a subset of $\mathcal{A}$, then the smallest subalgebra of $\mathcal{A}$ 
containing $X$ exists; it is denoted by $[X]$. The fields of sets are examples of Boolean algebras. An $s$-ideal 
$[d$-ideal] of the algebra $\mathcal{A}$ is a proper subset $J$ of $\mathcal{A}$ satisfying 
the following condition:

$$a + b \in J \quad \text{if and only if} \quad a, b \in J,$$

$[a \cdot b \in J \quad \text{if and only if} \quad a, b \in J].$

An $s$-ideal $[d$-ideal] $J$ of $\mathcal{A}$ is called prime if, additionally, for every 
a $\in \mathcal{A}$ either $a \in J$, or $a' \in J$.

If $J$ is a prime $s$-ideal $[d$-ideal] of $\mathcal{A}$, then $\mathcal{A} - J$ is a prime $d$-ideal 
$[s$-ideal] of $\mathcal{A}$. The following lemma is obvious.*

**Lemma 1.** If $X \subseteq \mathcal{A}$ and $a_1 + a_2 + \ldots + a_n = |\mathcal{A}|$ for each 
a_1, a_2, \ldots, a_n, then there exist an $s$-ideal $[d$-ideal] $J$ of $\mathcal{A}$, which 
includes $X$.

By a measure in the Boolean algebra $\mathcal{A}$ we mean a non-negative 
real function $\mu$ on $\mathcal{A}$ which is additive (i.e., $\mu(a + b) = \mu(a) + \mu(b)$ 
for $a + b = 0, a, b \in \mathcal{A}$) and $\mu(|\mathcal{A}|) = 1$. If $X \subseteq \mathcal{A}$, $\mu$ is a 
measure in $\mathcal{A}$ and $f(a) = \mu(a)$ for $a \in X$, then $\mu$ is called exten-
sion of $f$ from $X$ to $\mathcal{A}$.

The measure $\mu$ in $\mathcal{A}$ is two-valued if $\mu(a) = 1$ or $\mu(a) = 0$ for $a \in \mathcal{A}$.

The following lemmas give a connection between prime ideals and two-valued measures.

* [6], Theorem 2, p. 235-236.

**Lemma 2.** If $\mu$ is a two-valued measure in $\mathcal{A}$, then $E_{x} (\mu(x) = 0)$ is a prime $s$-ideal and $E_{x} (\mu(x) = 1)$ a prime $d$-ideal of $\mathcal{A}$.

**Lemma 3.** If $J_{\mu}$ is a prime $s$-ideal, then the function 
$$\mu(x) = \begin{cases} 1 & \text{for } x \in \mathcal{A} - J_{\mu} \\ 0 & \text{for } x \in J_{\mu} \end{cases}$$
is a measure in $\mathcal{A}$.

These lemmas allow us to translate some propositions from the lan-
guage of Boolean algebra to the language of the measure theory.

In this paper by a topological space we always mean a Hausdorff 
space (i.e., a space in which for all pairs of different points $p_{1}, p_{2}$ 
two exclusive neighbourhoods $G_{1}$ and $G_{2}$ exist, so that $p_{1} \not\in G_{1}$ 
and $p_{2} \not\in G_{2}$). The notions of compact (= bicompact) space and product space always 
have the usual meaning.**

If $\mathcal{M} = (M_{t})_{t \in T}$ is a family of topological spaces, then the relation 
c($p_{1}, p_{2}$) defined for $p_{1}, p_{2} \in \bigcup_{T} M_{t}$ is called a relation of 
consistency for the class $\mathcal{M}$ if it is symmetrical (i.e., $c(p_{1}, p_{2}) = c(p_{2}, p_{1})$) and the set 
$E_{c(p_{1}, p_{2})} [c(p_{1}, p_{2}), p_{1} \in M_{1}, p_{2} \in M_{2}]$ is closed in every product space of 
different spaces $M_{1}, M_{2} \in \mathcal{M}$.

A set $X \subseteq \bigcup_{T} M_{t}$ is called a partial choice-set from sets of $\mathcal{M}$, if $X - M_{t}$ is closed for every $t \in T$.

1. Method. The method of this paper is non-axiomatic. It may 
only be seen that all proofs could be formalized in every sufficiently 
large system of axiomatic set theory [e.g., in the system $\Sigma$ of Bernays-
Gödel**], without the axiom of choice.

2. Results. We deal here with six propositions, the Fundamental 
Existence Proposition of Ideal Arithmetic (I) and the following five:

(II) If $J$ is an $s$-ideal $[d$-ideal] of a field of sets $\mathcal{A}$, there is a prime 
$s$-ideal $[d$-ideal] $J_{\mu}$ of $\mathcal{A}$, which contains $J$.

(II*) If $\mathcal{A}$ is a sub-field of a field of sets $\mathcal{A}$, and $\mu_{1}$ a two-valued 
measure in $\mathcal{A}$, there exists a two-valued measure $\mu$ in $\mathcal{A}$, which is an 
extension of $\mu_{1}$.

(III) The product space of non-empty compact spaces is non-empty and 
compact.

* [6], Theorem 2, p. 235-236.

** Cf. [1], Definition 1, p. 59 and Definition 1, p. 42, 43.

*** Those notions are introduced in [6], see (3.1), (3.2), (3.3).

††† Cf. [2].
(IV) If \( \mathbf{M} = (M_i)_{i \in \tau} \) is a family of compact spaces, \( x \) a relation of consistency for the class \( \mathbf{M} \), and if, moreover, for every finite set \( T \subseteq \tau \) there exists an \( \omega \)-consistent choice from the class \( (M_i)_{i \in \tau} \), then there exists an \( \omega \)-consistent choice from the whole class \( \mathbf{M} \).

(V) If \( \mathbf{A} \) is a Boolean algebra, \( A \) a subset of \( \mathbf{A} \), \( f \) a real valued function defined on \( A \) and \( W \) a closed subset of the \([0,1]\) interval, and if, moreover, for any finite \( X \subseteq \mathbf{A} \) there exists a measure \( v \) on \([0,1]_b\) such that \( \mu(a) \in W \), for \( a \in X \), and \( \mu(a) = f(a) \), for \( a \in \mathbf{A} \cdot X \), then there exists a measure \( \mu \) in the whole algebra \( \mathbf{A} \), which is an extension of \( f \), and \( \mu(a) \in W \) for every \( a \in \mathbf{A} \).

Evidently (IV) is a translation of (I) from the language of Boolean algebras into the language of the measure theory so as for instance the proposition

(I') in every Boolean algebra, there is a two-valued measure

is a similar translation of (I). Therefore (I), (II), and (IV), (I') are respectively equivalent.

The proposition (III) is the well known theorem of Tychonoff (18) with the addition of the condition of non-emptiness of the product space. This condition is equivalent to the choice principle from compact spaces, which implies the choice principle from finite sets.

The proposition (IV) is the principle of consistent choice, it was discussed in our previous paper (20).

Finally (V) is a theorem of extension of measure. It follows from the known theorems of A. Horn and A. Tarski (19) and from a theorem of E. Marczewski (21).

The main result of this paper is that all propositions (I)-(V) are effectively equivalent.

3. The implication (I) \( \Rightarrow \) (II). This implication is known. Let \( \mathbf{A} \) be a field of sets and \( J \) an \( \{x, y\} \) ideal of \( \mathbf{A} \). It results from (I) that in the quotient algebra \( \mathbf{A} / J \) there exists a prime ideal \( J \); by setting \( J = \set{E \mid X, Y \subseteq \mathbf{A} \} \) we obtain a prime ideal \( J \) of \( \mathbf{A} \).

4. Proof of the implication (II) \( \Rightarrow \) (III). One part of this implication is due to N. Bourbaki (22). He has remarked that (II) has the following theorem as its consequence:

(4.1) Any topological space \( \mathbf{M} \) is compact if and only if for every two-valued measure \( \mu \) defined for all subsets of \( \mathbf{M} \), there exists precisely one point \( p \in \mathbf{M} \) such that \( \mu(G) = 1 \) for every neighborhood \( G \) of \( p \), and that from (4.1) and (II) it follows that

(4.2) the product space of compact spaces is compact.

Therefore it is sufficient to show that from (II) it results that:

(4.3) The product of non-empty compact spaces is non-empty.

Let \( \mathbf{M} = (M_i)_{i \in \tau} \) be a family of compact spaces; we set

\[ X = \prod_{i \in \tau} X_i \quad \text{and} \quad \prod_{i \in \tau} M_i = 1 \text{ for every } \tau \in T. \]

\[ C_i = \prod_{i \in \tau} \mathbf{M}_i = 1. \]

\( \mathbf{X} \) is evidently the family of partial choice-sets from the sets of \( \mathbf{M}_i \) and \( C \), the family of such partial choice-sets which have one element in common with \( M_i \). Clearly we have

\[ \prod_{i \in \tau} C_i = 0 \quad \text{for every } \tau \text{ and } t_1, t_2, \ldots, t_n \in T \]

and in view of lemma 1 there exists an \( \alpha \)-ideal \( J \) of the field \( \mathbf{X} \) of all subsets of \( \mathbf{X} \), such that \( C_i \in J \) for all \( i \in T \). But in consequence of (II) we conclude that there exists a prime ideal \( J \), which includes \( J \). The function

\[ \mu(E) = \begin{cases} 0 & \text{if } E \in \mathbf{M} \setminus J, \\ 1 & \text{if } E \in J, \end{cases} \]

is a two-valued measure in \( \mathbf{X} \) such that

\[ \mu(C_i) = 1 \quad \text{for every } i \in T. \]

Because \( C_i \in J \subseteq J \), for all \( i \in T \).

Denoting by \( \mathcal{O}(p) \) the set \( \set{E \in \mathbf{M} \mid p \in \mathbf{M} \} \), we find that

\[ \mu(p, q \in \mathbf{M} \mid p \neq q, \text{ then } \mathcal{O}(p) \cdot \mathcal{O}(q) = 0, \]

\[ \sum_{p \in \mathbf{M}} \mathcal{O}(p) = C_i. \]

It follows therefore that the function

\[ m_i(E) = \mu(\sum_{p \in \mathbf{M}} \mathcal{O}(p)) \]

(4.3) See Lemma 3, p. 51.
defined for all $EC_{M_i}$ is a two-valued measure. But in view of (4.1) every such measure distinguishes in $M_i$ one and only one point whose every neighbourhood has the measure $1$.

The set of all those points is evidently a choice-set from the class $\mathcal{M}$.

5. The Implication (III) $\rightarrow$ (IV). The proof of this implication was given in our previous paper $^{32}$. As the proposition (IV) may be interesting, we also give other formulations of it:

(IV) If $\mathcal{F} = \{ (a_i, b_i) \}_{i \in \mathbb{I}}$ is a family of pairs and $\pi(x_1, x_2, a_0)$ is a totally symmetrical relation (i.e., such that $\pi(x_1, x_2, a_0)$ implies $\pi(x_0, x_1, a_0)$ for every permutation $x_1, x_2, a_0$ of 1, 2, 0) defined for $x_1, x_2, a_0 \in X$, so that for every finite set $S \subset C S$ there is a choice-set $X$ from $\{ (a_i, b_i) \}_{i \in \mathbb{I}}$ such that $\pi(x_1, x_2, a_0)$ for all $x_1, x_2, a_0 \in X$, then there exists a choice-set from the whole $\mathcal{F}$, with the same property.

The proposition (I) follows from (IV) by substituting $\mathcal{F} = \{ (x, x') \}_{i \in \mathbb{I}}$ (where $\mathbb{I}$ is a Boolean algebra) and $\pi(x_1, x_2, a_0) = (x_1, x_2, a_0 \neq 0)$.

Since every finite set can be looked upon as a bicomponent, therefore (IV) holds true if every set $M_i$ of $\mathcal{M}$ is finite. That particular case of (IV) in which every set $M_i$ is precisely of the power $m$ (where $m$ is a finite cardinal number) we denote by (IV$^m$).

We obtain (IV$^+$) from (IV$^m$) by substituting $\mathbb{I} = S^m$ (the set of all ordered pairs $\langle a_1, b_1 \rangle$, where $a_1, b_1 \in \mathbb{I}$),

$$M_{(a_1, b_1)} = \{ (a_1, b_1), (b_1, a_1) \}$$

and $\pi(x_1, x_2, a_0)$ if and only if $\pi(x_0, x_1, a_0)$ for $k \leq l \leq n \leq 4$.

It is easy to see that the implication (IV$^m$) $\rightarrow$ (IV$^m$) holds true if $m \geq k$, and so the propositions (I), (IV) (IV$^+$) and (IV$^m$), $m = 4, 5, \ldots$ are effectively equivalent.

6. Proof of the Implication (IV) $\rightarrow$ (V). Let $A, A, f$ and $W$ be respectively: a Boolean algebra, a subset of $A$, a function on $A$ and a closed subset of the $[0, 1]$ interval. Suppose that $A, A, f$ and $W$ fulfill the conditions of (V). Let $\mathcal{E}$ be the family of all finite subsets of $A$ and let $M$ for $X \times \mathcal{E}$ be the set of all real functions on $X$ with values in $W$ which are extensions of $f$ from $X \times \mathbb{I}$ to $X$. Evidently $\mathcal{M} = (M, \mathcal{E})$ is a family of compact spaces. For $X, X \times \mathcal{E}$ and $\phi_1 \in M_X$, $\phi_2 \in M_X$, we denote that the functions $\phi_1$ and $\phi_2$ are additive respectively in $X, X \times \mathcal{E}$ and $\phi_1(x) = \phi_2(x)$ for $x \in X$. $\sigma$ appears to be a relation of consistency for the class $\mathcal{M}$ and, in consequence of (IV), there exists a $\sigma$-consistent choice-set $\Phi$ from $\mathcal{M}$. From $\sigma$-consistency of $\Phi$ it follows that for every $a \in A$ and $\phi_1, \phi_2 \in \Phi$, $\phi_1(a) = \phi_2(a)$ if only $\phi_1$ and $\phi_2$ are defined for $a$ (i.e. if $\phi_1 \phi_2 \Phi \cdot M_X$, $\phi_2 \Phi \cdot M_X$ and $a \in X, X \times \mathcal{E}$). Therefore we can set $\mu(a) = \phi(a)$, where $\phi \in \Phi_{M_X}$. $\mu$ is a measure in $\mathcal{M}$ with the required properties.

7. Proof of the implication (V) $\rightarrow$ (I*). If in (V) $A$ is the empty set, $f$ the empty function and $W$ the set which consists of two numbers 0 and 1, then (V) may be expressed in the following form

(V*) If in every finite subalgebra $A_i$ of the Boolean algebra $A$ there exists a two-valued measure, then also in the whole algebra $A$ there exists such a measure.

The antecedent of (V*) holds for every algebra $A$ and therefore so does the succeeds, which is (I*).

8. Table of results. We have proved that all the propositions (I)-(V) are effectively equivalent. Let us denote by (VI) the theorem of extending partial order to order $^{33}$, by (VII) the ordering principle, by (VIII) the principle of choice from finite sets and finally by (0) the axiom of choice.

The propositions (V)-(VII) $\rightarrow$ (VI) $^{34}$, therefore we obtain the following table

\[
\begin{array}{cccc}
0 & I & II & (V) \\
& III & (VI) & (VII) \\
IV & V & (VIII)
\end{array}
\]

Nevertheless we do not know, whether any implication outside those resulting from this table is valid for any pair of the propositions (0)-(VIII).

9. Remark. In a previous volume of this Journal J. L. Kelley $^{35}$ showed that the theorem of Tychonoff for Kuratowski's closure spaces (i.e. the spaces in which the closure of every set is defined, fulfilling the known axioms of Kuratowski; those spaces need not be Hausdorff's spaces) implies the axiom of choice (0).

This result is not astonishing, because the theorem (1.1) of Bourbaki is valid only for Hausdorff's spaces, and this compels us, when proving the theorem of Tychonoff for Kuratowski's spaces, to use once more more.

$^{32}$ See e.g. [6], Theorem 2.1, p. 234.

$^{33}$ [6], it is easy to see that in the proof of (2.1) only the theorem of Tychonoff is used, besides, of course, that of Theorem 1 of [6], which is effective.

$^{34}$ Kelley [4].
Intersections of prescribed power, type, or measure

by

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In 1914, Mazurkiewicz [5] showed that there exists a set of points in the plane, which intersects every straight line in the plane in precisely two points. Recently, Bagemihl [1] proved a general intersection theorem in the theory of sets, which, when applied to the plane, yields the following generalization of Mazurkiewicz's result: With every straight line \(s\) associate a cardinal number \(q_s \geq 2\) so that the sum of fewer than \(2^q\) of the numbers \(q_s\) is always less than \(2^q\). Then there exists a set of points which intersects every straight line \(s\) in exactly \(q_s\) points.

In the present paper, after extending the general intersection theorem alluded to above, we obtain several theorems dealing with plane point sets which intersect every straight line in a set of prescribed power, order type, or measure. In particular, we show that the aforementioned \(q_s\) may be chosen arbitrarily in the range \(2 \leq q_s \leq 2^q\). Free use is made of the well-ordering theorem.

**Theorem 1.** Let \(a\) be an arbitrary, fixed ordinal number, and \(S\) be a set with \(S \leq \kappa_a\). Let every \(s \in S\) let there correspond a set \(L_s\) such that, for every \(S' \subseteq S - \{s\}\) with \(S' \leq \kappa_a\),

\[
\sum_{s \notin S'} L_s \geq \kappa_a,
\]

and put \(P = \sum_{s \in S} L_s\).

Suppose that for every \(s \in S\) there exists a cardinal number \(L_s\), with \(1 \leq L_s \leq \kappa_a\), such that the following holds: If \(D \subseteq P\), \(D < \kappa_a\), and \(S_D\) is the set of elements \(s \in S\) for which \(L_s < \kappa_a\) and \(L_s D \geq L_s\), then

\[
S_D < \kappa_a.
\]

With every \(s \in S\) let there be associated in an arbitrary manner a cardinal number \(p_s\), satisfying

\[
L_s \leq p_s \leq \kappa_a.
\]