On local disconnection of Euclidean spaces

by

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1. Introduction. Let S_n be the n-dimensional sphere defined in the $(n+1)$-dimensional Euclidean space E_{n+1} by the equation

$$x_1^2+x_2^2+\ldots+x_{n+1}^2=1.$$

We consider a closed set $F\subset S_{n+1}$. Let U_a^ε denote the ε-neighbourhood of a point $a\in F$ in S_{n+1}, *i. e.*

$$U_a^\varepsilon=\underset{x\in S_{n+1}}{E}[|x-a|<\varepsilon].$$

Let us denote by $b_0^{\varepsilon,\eta}$, for every two positive numbers ε and η, $\varepsilon>\eta$, the number of components of $U_a^\varepsilon-F$ which have a common point with U_a^η. If $\eta<\eta'$, then $b_0^{\varepsilon,\eta}\leqslant b_0^{\varepsilon,\eta'}$, and consequently there exists

$$\lim_{\eta\to 0} b_0^{\varepsilon,\eta}=b_0^\varepsilon. \tag{1}$$

Evidently $b_0^\varepsilon\geqslant b_0^{\varepsilon'}$ if $\varepsilon<\varepsilon'$. Consequently there exists a finite or infinite limit

$$\lim_{\varepsilon\to 0} b_0^\varepsilon=b_0(a,S_{n+1}-F). \tag{2}$$

The number $b_0(a,S_{n+1}-F)$ will be called the *number of components in which F decomposes the $n+1$ dimensional sphere S_{n+1} at the point a.*

In 1933 E. Čech [1]) proved, using the notion of local Betti numbers, that the number $b_0(a,S_{n+1}-F)$ is a topological invariant. The purpose of this paper is to give an elementary proof of this fact without using any notion of algebraic topology. The method of proof is based on the notion of Borsuk's cohomotopy groups [2]) and Borsuk's theorem [3]) on the structure of the n-th cohomotopy group of closed subset F of S_{n+1}.

[1]) E. Čech [1].
[2]) K. Borsuk [2].
[3]) K. Borsuk [3].

2. Definitions and notations. Throughout the paper by space we understand a metric space and by a mapping a continuous transformation.

$X\times Y$ will denote the Cartesian product of two spaces X and Y *i. e.* the set of all ordered pairs (x,y) with $x\in X$, $y\in Y$, metrized by the formula

$$|(x,y)-(x',y')|=\sqrt{|x-x'|^2+|y-y'|^2}.$$

If X_0 is a subset of X and f a mapping with the range X, then $f|X_0$ will denote the *partial mapping* of f defined in X_0 *i. e.* the mapping f_0 defined in X_0 by the formula $f_0(x)=f(x)$. We shall say that f constitutes an *extension* of f_0 on X; we then write $f_0\subset f$.

Y_n^X will denote the set of all mappings of X into a compact space Y_n. In the functional space Y_n^X we define a metric topology setting

$$|f-g|=\sup_{x\in X}|f(x)-g(x)| \quad \text{for every } f,g\in Y_n^X.$$

Two mappings $f,g\in Y_n^X$ are called *homotopic* (written $f\sim g$) if there exists a mapping $h\in Y_n^{X\times I}$ where I denotes the closed interval $0\leqslant t\leqslant 1$, such that

$$\begin{matrix} h(x,0)=f(x), \\ h(x,1)=g(x) \end{matrix} \quad \text{for every } x\in X.$$

The relation of homotopy, established in Y_n^X, is a relation of equivalence and thus the set of all mappings $f\in Y_n^X$ decomposes into disjoint classes of homotopic mappings. The class of all mappings homotopic with a mapping $f\in Y_n^X$ will be denoted by (f) and called the homotopy class of f. A mapping $f\in Y_n^X$ homotopic to a constant is said to be *unessential*; we then write $f\sim 1$.

If X_0 is a closed subset of a compact space X, then by $X\|X_0$ we denote the space obtained from X by identifying X_0 to a point q_{X_0}. It is known [4]) that for every space $X\|X_0$ there exists a natural mapping $\varphi\in(X\|X_0)^X$ which maps $X-X_0$ topologically onto $X\|X_0-q_{X_0}$.

3. Cohomotopy groups. In this section we give the definition and some properties of Borsuk's cohomotopy groups needed in the sequel.

By a *product* of the mappings $f,g\in S_n^X$ we understand the mapping $f\times g\,(S_n\times S_n)^X$ defined by the formula

$$(f\times g)(x)=\big(f(x),g(x)\big) \quad \text{for every } x\in X.$$

[4]) See for instance C. Kuratowski [4], p. 42.

It is known [5]) that:

(3) *If X is a compactum and* $\dim X<2n$, *then for every* $f\times g\in(S_n\times S_n)^X$ *there exists a mapping* $h\in(S_n\times S_n)^{X\times I}$ *satisfying the conditions*

$$h(x,0)=(f\times g)(x)\quad\text{for every } x\in X,$$
$$h(X,1)\subset b_0\times S_n+S_n\times b_0,$$

where b_0 is an arbitrary point of S_n.

In this case we define the *sum* of the homotopy classes $(f),(g)\subset S_n^X$ in the following manner: Setting

$$\omega(x)=h(x,1)\quad\text{for } x\in X,$$
$$\vartheta_n(y,b_0)=y\quad\text{for } (y,b_0)\in S_n\times b_0,$$
$$\vartheta_n(b_0,y)=y\quad\text{for } (b_0,y)\in b_0\times S_n,$$
$$S_n\wedge S_n=b_0\times S_n+S_n\times b_0,$$

we have $\omega\in(S_n\wedge S_n)^X$, $\vartheta_n\in S_n^{S_n\wedge S_n}$ and $\vartheta_n\omega\in S_n^X$. We define the *sum* $(f)+(g)$ of the homotopy classes (f), $(g)\subset S_n^X$ by setting

$$(f)+(g)=(\vartheta_n\omega).$$

It is known [6]) that:

(4) *If X is a compactum and* $\dim X<2n-1$, *then the homotopy classes* $(f)\subset S_n^X$ *constitute an Abelian group with the operation defined as addition of homotopy classes.*

This group is called the n-th *Borsuk group* or n-th *cohomotopy group* of X and is denoted in the sequel by $B_n(X)$; the order of this group will be denoted by $b_n(X)$.

The *zero* element of $B_n(X)$ is the homotopy class which contains unessential mappings $f\in S_n^X$. An *inverse* element to $(f)\in B_n(X)$ is obtained in the following manner: Setting

$$\varrho_n(x_1,x_2,\ldots,x_n,x_{n+1})=(x_1,x_2,\ldots,x_n,-x_{n+1}),$$

for every $(x_1,x_2,\ldots,x_{n+1})\in S_n$, we define the inverse element $-(f)$ as the homotopy class containing $\varrho_n f\in S_n^X$.

If X_0 is a closed subset of a compactum X (with $\dim X<2n-1$), then

(5)
$$(f)+(g)=(h)\quad\text{implies}\quad(f|X_0)+(g|X_0)=(h|X_0),$$
$$(f_1)=-(f_2)\quad\text{implies}\quad(f_1|X_0)=-(f_2|X_0).$$

Let f be a mapping of X into Y. If $g\in S_n^Y$; then $gf\in S_n^X$, if $g_1\sim g_2$, then $g_1f\sim g_2f$. Let us set, for every homotopy class $(g)\subset S_n^Y$, $\hat{f}[(g)]=(gf)\subset S_n^X$.

[5]) See K. Borsuk [2] and E. Spanier [5].

[6]) See K. Borsuk [2] and E. Spanier [5], p. 211.

It is known [7]) that:

(6) *If X and Y are compacta and* $\dim X<2n-1$, $\dim Y<2n-1$, *then the mapping $\hat{f}$ (induced by f) is a homomorphism of $B_n(Y)$ into $B_n(X)$.*

Let X_0 be a closed subset of X. By $H_n(X,X_0)$ we denote the set of homotopy classes $(f)\subset S_n^X$ such that $f|X_0\sim 1$. Then:

If X is a compactum and $\dim X<2n-1$, *then the set $H_n(X,X_0)$ is a subgroup of $B_n(X)$.*

Proof. Let $(f),(g)\in H_n(X,X_0)$ and $(f)+(g)=(h)$. By (5) it is $(h|X_0)=(f|X_0)+(g|X_0)$. But $f|X_0\sim 1$ and $g|X_0\sim 1$, hence $h|X_0\sim 1$ and $(h)\in H_n(X,X_0)$. If $(f_1)\in H_n(X,X_0)$ and $(f_2)=-(f_1)$, then, by (5), $(f_2|X_0)=-(f_1|X_0)$. But $f_1|X_0\sim 1$, hence $f_2|X_0\sim 1$ and $(f_2)\in H_n(X,X_0)$.

The order of the group $H_n(X,X_0)$ will be denoted by $h_n(X,X_0)$.

4. Some lemmas. Let F be a proper closed subset of S_{n+1}. Let $G_0,G_1,\ldots,G_i,\ldots$ be a finite or infinite sequence of all components of $S_{n+1}-F$. In every component G_i we choose an arbitrary point p_i and a spherical $n+1$ dimensional element Q_i with centre p_i and boundary S_{ni}.

It is known [8]) that:

(7) *There exists a one-one correspondence between the set of all homotopy classes* $(f)\subset S_n^F$ *and the set of all sequences* $\{(f_i)\}$, *where* $(f_i)\subset S_n^{S_{ni}}$, $f_i=f|S_{ni}$, $i=1,2,\ldots$ *and f_i is unessential for almost all i. This correspondence is an isomorphism between the cohomotopy group $B_n(F)$ and the direct sum* [9]) $\sum_{i=1}^{\infty}B_n(S_{ni})$.

Now let $r^*_{p_0p_i}$ be for every $i\geqslant 1$ a mapping of S_{ni} onto S_n, homotopic to a homeomorphism of S_{ni} onto S_n. Let $r_{p_0p_i}$ be an extension of the mapping $r^*_{p_0p_i}\in S_n^{S_{ni}}$ on $S_{n+1}-(p_0)-(p_i)$. Then, for $i\neq j$, the mapping $r_{p_0p_i}|S_{nj}$ is unessential (because $r_{p_0p_i}|S_{nj}\subset r_{p_0p_i}|Q_{nj}\in S_n^{Q_{nj}}$), and for every $i=1,2,\ldots$ the homotopy class $(r_{p_0p_i}|S_{ni})\subset S_n^{S_{ni}}$ is a generator of the free cyclic group $B_n(S_{ni})$. From this, applying (7), we obtain the following

[7]) See E. Spanier [5], p. 214.

[8]) See K. Borsuk [3], p. 227 and 240.

[9]) By the direct sum $\sum_i A_i$ of Abelian groups A_i, $i=1,2,\ldots$ we understand the Abelian group A constituted by all sequences $\{a_i\}$ with $a_i\in A_i$, where $a_i=0$ for almost all indices i and where the group operation is defined by the formula $\{a_i\}+\{a_i'\}=\{a_i+a_i'\}$. It is clear that if a_i is a generator of the free cyclic group A_i and $\delta_j^i=0$ for $i\neq j$, $\delta_j^i=1$ for $i=j$, then the sequence $\{\delta_i^1\cdot a_1\},\{\delta_i^2\cdot a_2\},\ldots,\{\delta_i^j\cdot a_j\},\ldots$ constitutes the basis of the group $A=\sum_i A_i$.

Lemma 1. *The sequence of the homotopy classes*

$$(8)\qquad (r_{p_0p_1}|F),(r_{p_0p_2}|F),\ldots,(r_{p_0p_i}|F),\ldots\subset S_n^F$$

constitutes the basis of the n-th cohomotopy group of F.

Lemma 2. *Let F be a closed subset of S_{n+1} ($F\neq S_{n+1}$) and G an open connected subset of S_{n+1} ($G\neq S_{n+1}$). If $G_0,G_1,\ldots,G_k$ are all components of $S_{n+1}-F$ such that $G\cdot G_i\neq 0$ for $i=0,1,\ldots,k$, then $k=h_n(F,F-G)$.*

Proof. Let us order all components of $S_{n+1}-F$ in a finite or infinite sequence $G_0,G_1,\ldots,G_k,G_{k+1},\ldots$ and choose in every component G_i a point p_i in such a manner that $p_0,p_1,\ldots,p_k\in G$. We infer by lemma 1 that the homotopy classes

$$(9)\qquad (r_{p_0p_1}|F),\ (r_{p_0p_2}|F),\ \ldots,\ (r_{p_0p_k}|F)\in B_n(F)$$

are linearly independent. Since $F-G$ does not disconnect S_{n+1} between any pair of points $p_0,p_1,\ldots,p_k$, then $r_{p_0p_i}|F-G\sim 1$ for every $i=1,2,\ldots,k$. Thus we have shown that there exist at least k linearly independent elements of the group $H_n(F,F-G)$.

Now let us have any mapping $f\in S_n^F$ such that $f|F-G\sim 1$. We shall prove that the homotopy class (f) is a linear combination of the classes (9). By lemma 1 the homotopy class (f) is a linear combination of a finite number of elements of the sequence (8):

$$(10)\qquad (f)=c_1(r_{p_0p_{i_1}}|F)+c_2(r_{p_0p_{i_2}}|F)+\ldots+c_m(r_{p_0p_{i_m}}|F).$$

Since the set $F-G$ disconnects S_{n+1} between every pair of the points of the sequence $p_0,p_{k+1},p_{k+2},\ldots$, we have

$$(11)\qquad (r_{p_0p_j}|F-G)\neq(\mathrm{const}\,|F-G)\quad\text{for every } j>k.$$

From this and from $f|F-G\sim 1$ we infer that the linear combination (10) cannot contain any class $(r_{p_0p_j}|F)$ for $j>k$. Consequently the homotopy class (f) is a linear combination of the homotopy classes (9) and the proof of lemma 2 is completed.

Lemma 3. *If H_1, H_2 and G are three open neighbourhoods of a point $a\in F$ ($\dim F<2n-1$) such that $H_1\subset H_2\subset G$, then*

$$(12)\qquad h_n[F\|F-G,(F\|F-G)-H_1]\leqslant h_n[F\|F-G,(F\|F-G)-H_2].$$

Proof. Let us set $F^*=F\|F-G$. For every $f\in S_n^{F^*}$ the relation $f|F^*-H_1\sim 1$ implies $f|F^*-H_2\sim 1$. It follows that $H_n[F^*,F^*-H_1]\subset H_n[F^*,F^*-H_2]$ and consequently also (12).

Lemma 4. *If H, G_1 and G_2 are three open neighbourhoods of a point $a\in F$ ($\dim F<2n-1$) such that $H\subset G_1\subset G_2$, then*

$$(13)\qquad h_n(F^*,F^*-H)\leqslant h_n[F^*\|F^*-G_1,(F^*\|F^*-G_1)-H],$$

where F^* denotes the set $F\|F-G_2$.

Proof. Let φ be a natural mapping of F^* onto $F^{**}=F^*\|F^*-G_1$ and $\hat{\varphi}$ the induced homomorphism of $B_n(F^{**})$ into $B_n(F^*)$. If $f\in S_n^{F^{**}}$ and $f|F^{**}-H\sim 1$, then $f|F^*-H\sim 1$. From this we infer that

$$\hat{\varphi}\{H_n(F^{**},F^{**}-H)\}\subset H_n(F^*,F^*-H).$$

Now let $(g)\subset S_n^{F^*}$ and $g|F^*-H\sim 1$. Without loss of generality we can suppose that $g(F^*-H)=p_0\in S_n$. It follows that $g(F^*-G_1)=p_0$. We define the mapping h of F^{**} into S_n as follows:

$$h(x)=\begin{cases} g[\varphi^{-1}(x)] & \text{for } x\in F^*\|F^*-G_1-(q_{F^*-G_1}),\\ p_0 & \text{if } x=q_{F^*-G_1}.\end{cases}$$

Evidently $h\in S_n^{F^{**}}$, $h|F^{**}-H\sim 1$ and $g(x)=h[\varphi(x)]$ for every $x\in F^*$. It follows that $\hat{\varphi}\{(h)\}=(g)$ and $\hat{\varphi}\{H_n[F^{**}\|F^{**}-H]\}=H_n(F^*,F^*-H)$. From this we infer the inequality (13).

5. The local cohomotopy numbers. Let a be an arbitrary point of a compactum F with $\dim F<2n-1$. Let $U_a^\varepsilon(F)$ denote the ε-neighbourhood of a in F, *i. e.*

$$U_a^\varepsilon(F)=\underset{x\in F}{E}[|x-a|<\varepsilon].$$

Let us set

$$(14)\qquad b_n^{\varepsilon,\eta}(a,F)=h_n\big[F\|F-U_a^\varepsilon(F),\big(F\|F-U_a^\varepsilon(F)\big)-U_a^\eta(F)\big]\quad\text{for}\quad 0<\varepsilon<\eta.$$

By lemma 3, if $\eta<\eta'$, then $b_n^{\varepsilon,\eta}\leqslant b_n^{\varepsilon,\eta'}$. Consequently there exists

$$(15)\qquad \lim_{\eta\to 0} b_n^{\varepsilon,\eta}(a,F)=b_n^\varepsilon(a,F).$$

By lemma 4, if $\varepsilon<\varepsilon'$, then $b_n^\varepsilon(a,F)\geqslant b_n^{\varepsilon'}(a,F)$. Consequently there exists a finite or infinite limit

$$(16)\qquad \lim_{\varepsilon\to 0} b_n^\varepsilon(a,F)=b_n(a,F).$$

The number $b_n(a,F)$ will be called the *local cohomotopy number of F at the point $a\in F$.*

From the definition of $b_0^{\varepsilon,\eta}$ and $b_n^{\varepsilon,\eta}(a,F)$ and by lemma 2, we infer that in the case of $F \subset S_{n+1}$

$$b_0^{\varepsilon,\eta} = b_n^{\varepsilon,\eta}(a,F)+1. \tag{17}$$

From (17), (1), (2), (15) and (16) we obtain the following

THEOREM. *If $a \in F = \overline{F} \subset S_{n+1}$, then the number of components $b_0(a, S_{n+1}-F)$ in which the set F decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point a is determined by the local cohomotopy number $b_n(a,F)$ of F at the point a by the formula*

$$b_0(a, S_{n+1}-F) = b_n(a,F)+1. \tag{18}$$

Since the number $b_n(a,F)$ is topologically invariant, we obtain the following

COROLLARY. *The number of components $b_0(a, S_{n+1}-F)$ in which a closed set $F \subset S_{n+1}$ decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point $a \in F$ is topologically invariant.*

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Effectiveness of the representation theory for Boolean algebras [1])

by

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Stone calls Fundamental Existence Proposition of Ideal Arithmetic the lemma according to which:

(I) *In every Boolean algebra there is a prime ideal.*

This lemma plays a chief part in the demonstration that

(R) *any Boolean algebra is isomorphic with a field of sets*,

which is the most important result of Stone's representation theory.

According to Stone, (R) is effectively equivalent [2]) to (I). It was a long time ago noticed [3]), that (I) holds only with the help of transfinite methods. All the known proofs of it (S. Ulam [4]), A. Tarski [5]), M. H. Stone [6])) are based upon the principle of choice (or well-ordering theorem). A problem arises, whether the proposition (I) is really dependent on the principle of choice, and especially whether some particular cases of that principle [7]) are the consequences of the above-mentioned proposition.

A partial solution of this problem is given by W. Sierpiński [8]). It is known that the result of (I), without the use of transfinite methods, is that in the field of all subsets of an arbitrary infinite set E, there exists a two-valued measure which vanishes for one-point sets. Sierpiński has proved that the existence of such a measure in the set of

[1]) Presented to the Polish Mathematical Society, Warsaw Section, on May 12, 1950.

[2]) Cf. [8], Fund. Exist. Prop., p. 78; Theorem 67, p. 106 and Theorem 70, p. 110.

[3]) Cf. [9], p. 812.

[4]) [11].

[5]) [10], Lemma 1, p. 43.

[6]) [8], Theorem 63, p. 100.

[7]) By particular cases of the principle of choice we mean those forms of this principle in which the family of sets $\{M_t\}_{t \in T}$ is subject to some restrictions *e. g.* that every M_t is finite, or that it is a bicompact space, etc.

[8]) [7].