A characterization of $\Sigma\Delta$-rings of subsets

by

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1. A family F of subsets of a fixed set S is called a *ring* if F is closed for finite unions and intersections; F is called a *$\Sigma\Delta$-ring* if it is closed for arbitrary unions and intersections [1]). It is clear that a ring is a distributive lattice when partially ordered by set-inclusion. Further it can be shown that every $\Sigma\Delta$-ring is a completely distributive lattice with respect to set inclusion [2]). But whereas it is true that every distributive lattice is isomorphic (for definition, see below) with a ring of subsets, it does not hold in general that every completely distributive lattice is isomorphic with a $\Sigma\Delta$-ring of subsets (see Remark below).

In this note we obtain a characterization of a complete lattice which is isomorphic with a $\Sigma\Delta$-ring of subsets, in terms of a new order notion which we have called supercompactness: this is closely connected with the notion of completely prime dual ideal of Birkhoff [3]) and is stronger than the concept of compactness introduced by Nachbin [4]). We also deduce as a corollary a new characterization for the Boolean set-algebra B_R of all subsets of a set R.

2. Definitions. For basic concepts and results in lattice theory we refer the reader to [1] and [3] (Chpt. III). We shall define here only a few relevant terms. An element s of a lattice L is called *supercompact*, if $\Sigma a_i \geqslant s$ always implies some $a_i \geqslant s$, where $\geqslant$ is the p. o. relation in L, and Σa_i denotes any existing lattice sum in L. We remark that s is super-

[1]) See R. Vaidyanathaswamy [4], p. 12.

[2]) By a *completely distributive lattice* is meant a complete lattice L satisfying the generalised distributive law stated on p. 10 of [4] and its dual: viz., $\sum_{i\in Z}\prod_{k\in F_i} a_k = \prod_{g\in G}\sum_{k\in g} a_k$ and dually; where a_k (k ranging in F) is any family of elements of L, F_i (i ranging in Z) a disjoint partition of F, G the family of all subsets g of F having just one member in common with each F_i. Every $\Sigma\Delta$-ring satisfies these generalised distributive laws (see p. 10 of [4]), and is therefore a completely distributive lattice.

[3]) See G. Birkhoff [2], p. 12.

[4]) See L. Nachbin [3], p. 137.

compact, if and only if, the dual ideal $\{s\}$ of all elements $x \geqslant s$ is completely prime (a dual ideal A is called *completely prime* whenever $\Sigma a_i \in A$ implies some $a_i \in A$) [5]).

A lattice L is said to be *isomorphic* with a lattice L' if there is a one-to-one reversible mapping f: $a \leftrightarrow a'$ of L onto L' preserving order both ways, *i. e.* $a \leqslant b$, if and only if, $f(a) \leqslant f(b)$. It is clear that elements corresponding to one another under an isomorphism f have identical order properties; thus in particular, $f(a)$ is supercompact, if and only if this is true of a.

3. We prove now the

THEOREM A. *For a lattice L to be isomorphic with a $\Sigma\Delta$-ring L' of subsets of a set S, it is necessary and sufficient that L be*

(1) *complete,*

(2) *have an additive basis of supercompact elements (i. e. each element $a \in L$ is expressible as a lattice sum of supercompact elements).*

Proof. To prove the necessity of the conditions we need only to show that every $\Sigma\Delta$-ring L' satisfies them. That L' is a complete lattice follows immediately from the definition of a $\Sigma\Delta$-ring, and thus it only remains to demonstrate that L' fulfils condition (2).

We may clearly assume that the set S is the union of all elements in L'. Now to each point p in S we can associate an element a'_p in L' defined thus: a'_p is the intersection $\cap a'$ of all elements a' in L' containing (p) (a'_p exists since L' is closed for arbitrary intersections, and a'_p is evidently the smallest element of L' containing (p)). We now assert that a'_p is a supercompact element of L'. For, if $\Sigma a'_i \geqslant a'_p$, then $\Sigma a'_i = \cup a'_i \supseteq a'_p \supseteq (p)$ so that, for some a'_i, $a'_i \supseteq (p)$ and for that a'_i we have $a'_i \geqslant a'_p$, since a'_p is the *smallest* element of L' containing p. Thus a'_p is a supercompact element of L'. Further for any element a' in L we have obviously $a' = \Sigma a'_p$ (p ranging in the set a'). This completes the proof of the necessity part.

To prove the sufficiency part we let L satisfy conditions (1) and (2), and denote by $S = \{s\}$ the totality of non-zero [6]) supercompact elements s in L and by S_a, the subset of S comprising those s's with $s \leqslant a$, where a is an arbitrary element of L. It follows immediately from the definition of the S_a's that $S_{\Pi a_i} = \cap S_{a_i}$; $S_0 = \emptyset$. Also we have $S_{\Sigma a_i} = \cup S_{a_i}$ (since $s \leqslant$ any a_i obviously implies $s \leqslant \Sigma a_i$, while $s \leqslant \Sigma a_i$ implies, on ac-

[5]) A more detailed study of the properties of supercompact elements will be made in a forthcoming paper.

[6]) This restriction is not quite essential but is made here for the convenience of making the $\Sigma\Delta$-ring L' (which we are going to construct) include the null-set.

count of supercompactness of s, $s \leqslant$ some a_i). Thus the S_a's constitute a $\Sigma\Delta$-ring L' of subsets.

Consider now the mapping f: $a \to S_a$ of L onto [7] L'. This gives an isomorphism of L with L'. For in the first place it is one-to-one reversible, ($f(a)=f(b) \to a=b$, since s is an additive basis of L). Secondly $a \leqslant b$ implies evidently $S_a \subseteq S_b$, while $S_a \subseteq S_b$ implies $a \leqslant b$, since a and b are the lattice sum of elements in S_a and S_b respectively. This completes the proof.

Now let L be a lattice satisfying besides conditions (1) and (2) of Theorem A also the condition:

(3) *To each element $a \neq 1$ there corresponds an element $b \neq 0$ with $ab=0$.*

Denote by S the totality of non-zero supercompact elements s of L and by L' the $\Sigma\Delta$-ring constructed from S as in the proof of the second half of Theorem A. Then in view of the isomorphism f of L with L', L' also satisfies condition (3). For a given s in S, let S_s be the image of s in L' under f ($s \in S_s$), and $(S_s)^*$ be the union of all elements in L' not containing s (as a point); then $s \notin (S_s)^*$ so that $(S_s)^*$ is the largest element of L' not containing s. Also $S_s \cup (S_s)^* = S$. For $S_s \cup (S_s)^* \subseteq S$, and $S_s \cup (S_s)^* \subset S$ would imply by condition (3), the existence of an element $S_b (\neq \emptyset)$ in L' with $S_b \cap \big(S_s \cup (S_s)^*\big) = \emptyset$. Hence also $S_b \cap S_s = \emptyset$, whence $s \notin S_b$, so that $S_b \subseteq (S_s)^* \subseteq S_s \cup (S_s)^*$, which leads to the contradiction $S_b = S_b \cap (S_s \cup (S_s)^*) = \emptyset$. Thus $S_s \cup (S_s)^* = S$.

Now let $s_0 \in S_s$ so that $s_0 \leqslant s$. If $s_0 \neq s$ then $s \notin S_{s_0}$, and since $s \in S = S_{s_0} \cup (S_{s_0})^*$, it follows that $s \in (S_{s_0})^*$. But then $s_0 \in (S_{s_0})^*$ (since $s_0 \leqslant s$ and $s \in (S_{s_0})^*$ which contradicts the definition $(S_{s_0})^*$). Consequently we have $S_s = (s)$. It follows that L' includes all one-pointic subsets of S, and being itself a $\Sigma\Delta$-ring, L' comprises all subsets of S, whence L' coincides with the Boolean algebra B_S of all subsets of S.

On the other hand in any Boolean set-algebra B_R condition (3) holds, since for any element $X \neq R$ in B_R we have the complement $X' \neq \emptyset$ and $X \cap X' = \emptyset$.

We have thus proved the

THEOREM B. *A lattice L is isomorphic with a Boolean set algebra B_R, if and only if, L satisfies conditions* (1)-(3).

4. Remarks. The closed interval $I=[0,1]$ considered as a chain has no supercompact elements except 0, since each element $a \neq 0$ in I can be expressed as the lattice sum Σx of all elements $x<a$. However, it is easy to verify that it is a completely distributive lattice. Thus we have in I a completely distributive lattice which is not a $\Sigma\Delta$-ring of subsets.

[7]) That f maps L onto L results immediately from the definition of L'.

Note added in proof. Since sending this paper to the editor (in November 1952) it came to our notice that G. N. Raney has given in his paper *Completely distributive lattices*, Proc. Amer. Math. Soc. 3 (1952), p. 677-680, a result (Theorem 2) which is identical with Theorem A above.

References

[1] G. Birkhoff, *Lattice theory*, second edition, New York 1948.

[2] — *Representations of lattices by sets*, Trans. Amer. Math. Soc. 64 (1948), p. 299-315.

[3] L. Nachbin, *On a characterization of the lattice of all ideals in a Boolean ring*, Fund. Mathematicae 36 (1949), p. 137-142.

[4] R. Vaidyanathaswamy, *Treatise on set topology*, 1947.

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