To complete the proof we shall show that the projection $\pi$ of $Y$ onto $X$ is open (clearly $\pi$ is continuous). It suffices to prove that $\pi(Y \cap (G \times V)) = G$ for every open $G \subseteq X$ and for every open interval $V \neq 0$, $V \subseteq \mathbb{R}$.

Obviously $\pi(Y \cap (G \times V)) = G$. If $x \in G$, then, by (1), there is an $r \in H_x \cdot V$. Consequently $x \in \pi(Y \cap (G \times V))$, which yields $G \subseteq \pi(Y \cap (G \times V))$, q. e. d.

The problem whether every Hausdorff space with an enumerable basis is an interior image of a separable metric space is unsolved. Notice that (II) implies easily (H).}

References


On existential theorems in non-classical functional calculi

by

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Let $S^f$ be the Heyting propositional calculus, and let $S^f_\Sigma$ be the Heyting functional calculus. The individual variables of the system $S^f_\Sigma$ will be denoted by $x_1, x_2, \ldots$, the quantifiers — by $\Sigma$ and $\Pi$. The formulas from $S^f_\Sigma$ will be denoted by the letters $a, \beta$. If $a$ is a formula from $S^f_\Sigma$, then $\alpha [a]$ denotes the formula obtained from $a$ by replacing each free occurrence of $x_\alpha$ by $x_\beta$ (each bound occurrence of $x_\beta$ should be replaced earlier by $x_\beta$ which does not appear in $a, 1 \vdash \phi$).

Gödel\(^1\) formulated (without proof) the following theorem:

$$(\chi) \text{ Let } a, \varphi \text{ be two formulas from the Heyting propositional calculus } S^f. \text{ If the disjunction } a + \varphi \text{ is a theorem of } S^f_\Sigma, \text{ then either } a \text{ or } \varphi \text{ is a theorem of } S^f_\Sigma.\)$$

Theorem (\(\chi\)) was later proved by McKinsey and Tarski\(^2\) by an algebraical method. Another algebraical proof was given by Rieger\(^3\).

The purpose of this paper is to prove the following theorem (\(\chi\)) which is an extension of (\(\chi\)) over the Heyting functional calculus $S^f_\Sigma$. The second part of Theorem (\(\chi\)) shows that the Heyting functional calculus is the well formalization of Brouwer's ideas concerning existential theorems.

(\(\chi\)) If the formula $a + \beta$ is provable in $S^f_\Sigma$, then either $a$ or $\beta$ are provable in $S^f_\Sigma$. If the formula $S^f_\Sigma \alpha$ is provable in $S^f_\Sigma$, then there is a positive integer $g$ such that the formula $a [g]$ is provable in $S^f_\Sigma$.

Clearly if the sequence $x_1, \ldots, x_n$ contains all the free variables which appear in $a$, the integer $g$ can be chosen among the numbers $1, \ldots, n$. If $a$ contains no free variable, then $g$ is an arbitrary integer, $e. g. g = q.$

\(^1\) Presented at the Seminar on Foundations of Mathematics in the Mathematical Institute of the Polish Academy of Sciences in November 1955.
\(^2\) See K. Gödel [1]. See also A. Genzen [1].
\(^3\) See L. Rieger [1], p. 29.
We notice that
$(\gamma')$ Each formula from $S''_n$ without quantifiers is provable in $S''_n$ if and only if it is a substitution of a theorem of the Heyting sentential calculus $S''_n$.

Since the calculus $S''_n$ is decidable $^{10}$, we infer from $(\gamma)$ and $(\gamma')$ that
$(\gamma)$ Each formula $\beta$ from $S''_n$ of the form
$$\beta = \exists_1 \exists_2 \ldots \exists_n \phi \quad \exists_{\tau_1} \exists_{\tau_2} \ldots \exists_{\tau_n}$$
where $\phi$ contains no quantifier and $\exists_i$ is either the sign $\exists$ or $\bigcup \{i = 1, \ldots, n\}$, is decidable.

Theorem $(\gamma)$ is a simple application of the results obtained in our paper on Algebraic Treatment of the Notion of Satisfiability $^{10}$ cited hereafter as [AT]. The main idea of the proof (the extension of the space $X$ to the space $X_{\alpha}$ — see p. 24) is essentially due to McKinsey and Tarski $^{4}$.

The knowledge of [AT] is assumed in the sequel. Terminology and notation are the same as in [AT], therefore they will not be explained here $^{4}$.

Theorems analogous to $(\gamma)$ hold also for the other non-classical functional calculi examined in [AT]. More exactly, we shall prove the following theorems where $\alpha$ and $\beta$ denote formulas from the functional calculus under consideration:

$(\alpha)$ If the formula $\alpha \rightarrow \beta$ is provable in the positive functional calculus $^{11}$ $S'_n$, then either $\alpha$ or $\beta$ is provable in $S'_n$. If the formula $\exists \gamma$ is provable in $S''_n$, then there is an integer $q$ such that $\exists \gamma_q$ is provable in $S''_n$.

$(\mu)$ If the formula $\alpha \rightarrow \beta$ is provable in the minimal functional calculus $^{11}$ $S'_n$, then either $\alpha$ or $\beta$ is provable in $S'_n$. If the formula $\exists \gamma$ is provable in $S''_n$, then there is an integer $q$ such that $\exists \gamma_q$ is provable in $S''_n$.

$(\upsilon)$ If the formula $\alpha \rightarrow \beta$ is provable in the functional calculus $^{11}$ $S'_n$, then either $\alpha$ or $\beta$ is provable in $S'_n$. If the formula $\exists \gamma$ is provable in $S''_n$, then there is an integer $q$ such that $\exists \gamma_q$ is provable in $S''_n$.

$(\lambda)$ If the formula $\alpha \rightarrow \beta$ is provable in the Lewis functional calculus $^{11}$ $S'_n$, then either $\alpha$ or $\beta$ is provable in $S'_n$. If the formula $\exists \gamma$ is provable in $S''_n$, then there is an integer $q$ such that $\exists \gamma_q$ is provable in $S''_n$.

Notice that the formulation of Theorem $(\lambda)$ is somewhat different from the formulation of Theorems $(\gamma)$, $(\mu)$, $(\upsilon)$, $(\gamma')$ can also be formulated in a purely algebraical way. The first part of these theorems asserts that the class $E$ of all non-provable formulas forms an ideal $I$ in the corresponding Lindenbaum algebra $^{13}$ $L_i (i = \gamma, \mu, \upsilon, \nu)$. The second part asserts that the ideal $I$ is enumerable additive in the following sense: if all components of an infinite sum corresponding to the quantifier $\exists \gamma$ $(i.e. \sum \gamma_q)$ are in $I$, then the sum also belongs to $I$. Clearly $I$ is the unique maximal ideal of $L_i$.

Clearly theorems analogous to $(\gamma)$ are also true for the positive propositional calculus $S''_n$, the minimal propositional calculus $S'_n$, the propositional calculus $S'_n$, and for the Lewis propositional calculus $S'_n$. In the case of $S''_n$, $\alpha \rightarrow \beta$ should be replaced by $\alpha \rightarrow \beta$.

Theorem $(\gamma)$ is a particular case of the following general theorem.

$(\alpha)$ Let $S$ be the propositional calculus described in [AT] $^{11}$, and let $S'$ be the functional calculus determined by $S$ ([AT] $^{11}$). Suppose that $S$ has the extension property $(E)$ $^{14}$. A formula $\alpha \in S'$ without quantifiers is provable in $S'$ if and only if $\alpha$ is a substitution of a theorem of the propositional calculus $S$.

In particular, Theorem $(\alpha)$ is true for all the systems examined in Part II of [AT]. The hypothesis that $S$ has the property $(E)$ seems to be inessential.

Since each formula from the Lewis propositional calculus $S''_n$ is decidable $^{11}$, we infer from $(\lambda)$ and $(\alpha)$ that:

$^{10}$ See e.g. S. Jaskowski $^{1}$, G. Gentzen $^{1}$, M. Wajsberg $^{1}$, J. C. C. McKinsey and A. Tarski $^{3}$, L. Rieger $^{1}$.

$^{11}$ See References at the end of this paper. Theorem $(\gamma)$ can be also deduced from a fundamental theorem of G. Gentzen $^{1}$.

$^{12}$ The Heyting functional calculus is exactly described in [AT] $^{11}$.

$^{13}$ [AT] $^{11}$.

$^{14}$ [AT] $^{11}$.

$^{15}$ [AT] $^{11}$.

$^{16}$ See [AT], p. 69.

$^{17}$ See [AT], p. 81, 85, 88, 90, 92.

$^{18}$ In the case of $S''_n$, this theorem follows from the fact that a formula $\alpha \in S''_n$ is provable in $S''_n$ if and only if it is provable in $S''_n$ (see H. E. B. B. E. N. (1), p. 480). The proof of this theorem in the case of $S''_n$ and $S''_n$ is similar to that of $S''_n$. See McKinsey-Tarski $^{2}$ and $^{3}$. It is based on the algebraic interpretation mentioned in [AT] $^{11}$, 3, 12, 14. For the proof of this theorem in the case of $S''_n$, see McKinsey-Tarski $^{2}$.

$^{19}$ See [AT], p. 69.
(1') Each formula \( \beta \) from the Lewis functional calculus \( S^*_1 \), such that

\[
\beta = \bigvee_{\sigma} \bigwedge_{\tau} \ldots \bigwedge_{\xi} \sigma
\]

where \( \sigma \) contains no quantifier and \( \sigma \) is either the sign \( \Sigma 1 \) or \( \Pi 1 \) \((i = 1, \ldots, n)\), is decidable.

Since a formula \( \alpha \) from the functional calculus \( S^*_1 \) is provable in \( S^*_1 \) if and only if \( \alpha \) is provable in \( S^*_1 \), and since each formula from the sentential calculus \( S^*_1 \) is a theorem of \( S^*_1 \), if and only if it is a theorem in \( S^*_1 \), we infer that Theorem (1') remains true if we replace \( S^*_1 \) by \( S^*_1 \).

Proof of (2'). By [AT] 11.2 there is a topological space \( X \) such that the Heyting algebra \( H(X) \) of all open subsets of \( X \) is an \( S^*_1 \)-extension, i.e., there exists an \( S^*_1 \)-isomorphism \( h \) of \( L_\omega \) into \( H(X) \).

Let \( x_0 \) be a fixed element, \( x \neq X \), and let \( X_0 = X + (x_0) \). We shall treat the set \( X_0 \) as a topological space with the following definition of topology: open subsets of \( X_\omega \) are open subsets of \( X \) and the whole space \( X_\omega \).

There is exactly one open subset of \( X_\omega \) which contains the element \( x_0 \) via the whole space \( X_\omega \). Consequently

(2') If \( g_\alpha \in H(X_\omega) \) and \( X_\omega = \sum g_\alpha \), then there is a \( g \) such that \( G_\alpha = X_\omega \).

Since \( X_\omega \) is the open subset of \( X_\omega \), the formula

\[
g(A) = AX \quad \text{for} \quad A \in H(X_\omega)
\]

defines an \( S^*_1 \)-isomorphism \( g \) of the \( S^*_1 \)-algebra \( H(X_\omega) \) onto the \( S^*_1 \)-algebra \( H(X) \). The homomorphism \( g \) preserves all infinite sums and products.

Let \( \varphi^m_n \in F^m [I_m, H(X_\omega)] \) \((m = 1, 2, \ldots)\) be the mapping defined as follows:

\[
\varphi^m_n(i_1, \ldots, i_m) = h\left[\varphi^m_n(x_{i_1}, \ldots, x_{i_m})\right] \quad \text{for} \quad i_1, \ldots, i_m \in I_m.
\]

Suppose that the formula \( \alpha + \beta \in S^*_1 \) is provable in \( S^*_1 \). By [AT] 5.1 and 5.3

\[
\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) + \left[ I_\beta, H(X_\omega) \right] \Phi_\beta(i), (\varphi^m_n) = \left[ I_\gamma, H(X_\omega) \right] \Phi_\gamma(i), (\varphi^m_n) = X_\omega = \text{the unit of} \ H(X_\omega).
\]

By (a) either

\[
\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) = X_\omega,
\]
or

\[
\left[ I_\beta, H(X_\omega) \right] \Phi_\beta(i), (\varphi^m_n) = X_\omega.
\]

Suppose that, for instance, the equation (a) is true. Since \( \varphi^m_n = \varphi^m \omega \), we have by [AT] 5.1 and 5.3

\[
X = g(X_\omega) = g\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) = \left[ I_\beta, H(X) \right] \Phi_\beta(i), (\varphi^m_n) = h(\alpha).
\]

Since \( h \) is an isomorphism, we infer that \( \alpha \) is the unit element of \( L_\omega \), i.e., that \( \alpha \) is provable in \( S^*_1 \) (see [AT] 4.2).

The first part of (2') is proved. The proof of the second part is similar.

Suppose that \( \beta = \sum \alpha \in S^*_1 \) is provable in \( S^*_1 \). By [AT] 5.1 and 5.3

\[
\sum \left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) = \left[ I_\beta, H(X_\omega) \right] \Phi_\beta(i), (\varphi^m_n) = X_\omega = \text{the unit of} \ H(X_\omega),
\]

where \((i')\) denotes the sequence \((i)\) where the \( p \)-th term is replaced by \( x_0 \).

By (a) there is an integer \( g \) such that the \( g \)-th component of the sum on the left side is equal to \( X_\omega \) i.e.,

\[
\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i'), (\varphi^m_n) = X_\omega
\]

where \( i_0 = i \) for \( i_0 \neq x_0 \) and \( x_0 = g \).

Let \( \delta = \varphi^m_n \). By [AT] 5.4

\[
\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) = \left[ I_\beta, H(X_\omega) \right] \Phi_\beta(i'), (\varphi^m_n) = X_\omega.
\]

Since \( \varphi^m_n = \varphi^m \omega \) we obtain from [AT] 5.1 and 5.2 that

\[
X = g(X_\omega) = g\left[ I_\alpha, H(X_\omega) \right] \Phi_\alpha(i), (\varphi^m_n) = \left[ I_\beta, H(X_\omega) \right] \Phi_\beta(i), (\varphi^m_n) = h(\delta).
\]

Since \( h \) is an isomorphism, we infer that \( \delta \) is the unit of \( L_\omega \), i.e., that

\[
\delta = \alpha \varphi^{m_0} \in S^*_1 \quad \text{is provable in} \ S^*_1 \quad \text{(see [AT] 4.2)}.
\]

Proof of (\( \sigma \)) is completely analogous to that of (\( \chi \)).

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\( ^{[1]} \) (\( i \)) is the sequence of all positive integers.
Proof of (a) is analogous to that of (h). Only the following supplement is needed.

Let $E$ be the $\mathcal{S}^r$-algebra $H(X)$. We define the operation $\sim$ in the $\mathcal{S}^r$-algebra $H(X)$ by the formula

$$\sim A = A \sim B \quad \text{for} \ A \in H(X),$$

where $\sim$ on the right side is taken in $H(X_0)$ (not in $H(X)$). Clearly $g$ is an $\mathcal{S}^r$-homomorphism of the $\mathcal{S}^r$-algebra $H(X_0)$ onto the $\mathcal{S}^r$-algebra $H(X)$.

Proof of (v) is analogous to that of (h). Only the following supplement is needed.

We define the operation $\sim$ in $H(X)$ as follows: if $A \in H(X_0)$, then $\sim A$ in the $\mathcal{S}^r$-algebra $H(X_0)$ (where $\sim$ is taken in the $\mathcal{S}^r$-algebra $H(X)$). Clearly $g$ is an $\mathcal{S}^r$-homomorphism of the $\mathcal{S}^r$-algebra $H(X_0)$ onto $H(X)$.

Proof of (i) is also analogous to that of (h). Instead of $H(X)$ and $H(X)$ we should write everywhere $C(X)$ and $C(X)$ respectively. The mapping $g$ is an $\mathcal{S}^r$-homomorphism of $C(X_0)$ onto $C(X)$ since $X$ is open in $X_0$.\footnote{\cite{Sikorski}}

Remarks. Theorem (n) can be directly deduced from Theorem (h) and \cite{AT} 15.5. Theorem (h) follows directly from (h) and \cite{AT} 15.2.

In the proof of (h) we can replace $H(X)$ by any other $\mathcal{S}^r$-extension $H$ of $L_0$, which need not be the Heyting algebra of all open subsets of a topological space. The Heyting algebra corresponding to $H(X)$ must then be somewhat differently defined \footnote{\cite{Sikorski}}.

Proof of (a). Let $L$ be Lindenbaum algebra constructed for the system $\mathcal{S}^r$. The exact definition of $L$ is completely analogous to the Lindenbaum algebra $L$ described in \cite{AT} 4 (where $\hat{R} = \emptyset$).

Elements of $L$ are classes $[c]$ ($c \in S$) of equivalent formulas, i.e. $\tau \in [c]$ if and only if the formula $\tau = \phi$ is a theorem of $S$ ($\tau, \phi \in S$). The definition of algebraic operations is the same as in \cite{AT} 4.

Clearly $L$ is an $\mathcal{S}^r$-algebra, and $[c]$ is the unit element of $L$ if and only if $c$ is a theorem of $S$.

Let $L^*$ be an $\mathcal{S}^r$-algebra containing $L$ as a subalgebra \footnote{\cite{McKinsey-Tarski}}.

Let $\beta$ be a one-to-one transformation of the set of all primitive formulas $P^r_{\alpha}(\alpha_1, ..., \alpha_n)$ onto the set $(\alpha_1, \alpha_2, ..., \alpha_n)$ of all sentential variables (see \cite{AT} §§ 1-2). If $\alpha \in \mathcal{S}^r$ contains no quantifier, we shall denote by $\alpha_0$ the formula obtained from $\alpha$ by replacing formulas $P^r_{\alpha}(\alpha_1, ..., \alpha_n)$ by the sentential variables $P^r_{\alpha}(\beta_1, ..., \beta_n)$ respectively. Clearly $\alpha_0 \in S$ and $\alpha$ is a substitution of $\alpha_0$.

Now let $\phi^r_{\alpha}(x_1, x_2, ..., x_n)$ be defined as follows:

$$\phi^r_{\alpha}(x_1, x_2, ..., x_n) = \phi^r_{\alpha}(P^r_{\alpha}(\alpha_1, ..., \alpha_n))$$

for $x_1, x_2, ..., x_n \in I^r$.

It is easy to prove by induction with respect to the length of $\alpha$ that

$$\phi^r_{\alpha}((x_1, x_2, ..., x_n)) = [\phi^r_{\alpha}(\alpha_1, ..., \alpha_n)].$$

Consequently, if $\alpha$ is provable in $\mathcal{S}^r$, then $[\alpha]$ is the unit of $L$ \cite{AT} 5.3, i.e. $[\alpha]$ is a theorem of $\mathcal{S}^r$.

Proof of (h'). The expression $\phi^r_{\alpha}(x_1, x_2, ..., x_n)$ described in \cite{AT} § 4 (p. 69-70) was not uniquely determined. However, it will be uniquely determined if, for instance, we require the integer $l$ (see the definition in \cite{AT}, p. 70) to be the least positive integer such that $x = p$ and $\alpha$ contains neither $l$ nor $\sum_{i=1}^n \alpha_i$. In the sequel we assume the definition of $\phi^r_{\alpha}(x_1, x_2, ..., x_n)$ from \cite{AT} § 4 with the above correction. Hence $\phi^r_{\alpha}(x_1, x_2, ..., x_n)$ is uniquely determined.

If $\beta = \sum_{i \in I^r} \alpha \in S^*, \alpha \in S^*$, then we shall denote by $Z(\beta)$ the set of all formulas $\phi^r_{\alpha} \in \mathcal{S}^r$ where either $q = p$ or $\alpha$ contains at least one occurrence of $\alpha_0$.

If $\beta = \sum_{i \in I^r} \alpha \in S^*$, then $Z(\beta)$ is the set containing only one element: the formula $\alpha$.

More generally, if $\mathcal{R} \in S^r$ is a set of formulas $\beta$ of the form

$$\beta = \sum_{i \in I^r} \alpha \quad \text{or} \quad \beta = \prod_{i \in I^r} \alpha,$$

then $Z(\mathcal{R})$ is the union of all sets $Z(\beta)$ where $\beta \in R$. Clearly $Z(\mathcal{R})$ is finite if $\mathcal{R}$ is finite, and the number of elements of $Z(\mathcal{R})$ can easily be estimated from the above.

Suppose now that $\beta \in S^*$ is a formula of the form $(\sum_i)$ or $(\prod_i)$. Let $R_\beta = Z(\mathcal{R}_\beta)$ and, by induction, $R_{\beta \cdot R} = Z(\mathcal{R}_{\beta \cdot R})$ ($k = 2, ..., n$). It follows from (h') that $\beta$ is provable if and only if $R_\beta$ contains at least one provable formula. By an easy induction with respect to $k$, we find that $\beta$ is provable if and only if $R_\beta$ contains at least one provable formula. Consequently $\beta$ is provable if and only if $R_\beta$ contains at least one provable formula. However, all formulas in $R_\beta$ contain no quantifier. Hence, by (h'), we can decide by a finite method whether there is a provable formulas in $R_\beta$.

Proof of (h') is completely analogous to that of (h').
Sur le phénomène de convergence de M. Sierpiński

par

J. Popruženko (Lodz)

M. Sierpiński a démontré en 1928 le Théorème suivant:

(T) Si $x_1 - 2^n$, il existe une suite convergente de fonctions $f(x)$ ($n = 1, 2, ...$), définies pour $0 \leq x \leq 1$, qui converge non uniformément sur tout ensemble non dénombrable 1.

Nous appelons phénomène de convergence de Sierpiński la singularité qui se présente dans la thèse du Théorème (T), à savoir la convergence non uniforme dans tout ensemble indénombrable d'un certain champ de convergence.

Si l'on abandonne l'hypothèse du continu, la question de l'existence des singularités de ce genre reste ouverte; la méthode de M. Sierpiński, essentiellement liée à l'hypothèse du continu, ne donne aucun renseignement sur ce sujet.

L'étude approfondie du Théorème (T) et des problèmes qui s'y rattachent m'a conduit à mettre en évidence une singularité connexe dont l'existence dans les espaces d'une certaine puissance indénombrable peut être démontrée sans prémisses hypothétiques.

Dans la Note présente, je m'occupe de cette démonstration, puis je donne certaines applications du résultat acquis à la théorie de la mesure abstraite.

1. Prélèvements

Étant données deux suites infinies d'entiers positifs $a = (a_1, a_2, ...)$ et $b = (b_1, b_2, ...)$, convenons d'écrire

1) W. Sierpiński [1] et [2], p. 32, Proposition C.