Closure homomorphisms and interior mappings

by

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This paper is a continuation of my paper Closure Algebras\(^1\) cited hereafter as CA.

A closure algebra \(A\) is a Boolean \(\sigma\)-algebra with a closure operation satisfying the well known axioms of Kuratowski:

\[ \begin{align*}
\text{I.} & \quad \overline{A + B} = \overline{A} + \overline{B}, \\
\text{II.} & \quad A \subseteq \overline{A}, \\
\text{III.} & \quad \overline{A' = A}, \\
\text{IV.} & \quad \overline{0} = 0.
\end{align*} \]

Every closure algebra is thus an “abstract algebra”\(^2\) with the following fundamental operations:

(a) Boolean enumerable addition \(\sum_{i=1}^{n} A_i\),

(b) Boolean complementation \(A'\),

(c) closure operation \(\overline{\cdot}\).

By a closure homomorphism we shall understand a homomorphism (in the sense of the Modern Algebra)\(^3\) with respect to the fundamental operations (a), (b), (c), i.e. a transformation \(h\) of a closure algebra \(A\) into another closure algebra \(B\) preserving all the operations (a), (b), (c):

\[ \begin{align*}
\text{(a') } & \quad h(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} h(A_i); \\
\text{(b') } & \quad h(A') = h(A); \\
\text{(c') } & \quad \overline{h(A)} = h(\overline{A}).
\end{align*} \]

The conditions (a’) and (b’) mean that \(h\) is a Boolean \(\sigma\)-homomorphism. Thus a closure homomorphism is a Boolean \(\sigma\)-homomorphism satisfying the condition (c’).

\section{The relation between closure homomorphisms and interior mappings}

By a topological space we shall mean a set \(X\) with a closure operation defined for all \(A \subseteq X\) such that I-IV hold (it is not assumed that \(\overline{X} = A\) if \(A\) is a one-point set). The closure algebra of all subsets of \(X\) will be denoted by \(\mathcal{E}(X)\). The closure algebra of all Borel subsets of \(X\) will be denoted by \(\mathcal{B}(X)\).

The letters \(X\) and \(Y\) will always denote topological spaces.

A mapping \(\varphi\) of \(X\) into \(Y\) is said to be open if \(\varphi(\overline{U})\) is open in \(Y\) for every open set \(U \subseteq X\) (in particular, \(\varphi(\mathcal{E}(X))\) is an open subset of \(\mathcal{E}(Y)\)). The mapping \(\varphi\) is said to be interior if it is open and continuous.

(1) A mapping \(\varphi\) of \(X\) into \(Y\) is open if and only if \(\varphi \circ \overline{\varphi^{-1}(Y)} = \overline{\varphi \circ \varphi^{-1}(Y)}\) for every set \(Y \subseteq Y\).

\section{Closure isomorphisms}

A closure isomorphism is a one-to-one closure homomorphism, i.e., a Boolean \(\sigma\)-isomorphism \(h\) satisfying the condition (c’).

Closure isomorphisms have been examined in CA under the name homeomorphisms\(^4\) since they are a generalization of the notion of homeomorphism from the Topology of Point Spaces.

Another class of homomorphisms examined in CA is the class of continuous homomorphisms\(^5\), i.e., Boolean \(\sigma\)-homomorphisms \(h\) of a closure algebra \(A\) into another \(B\), satisfying the condition

\[ h(\overline{A}) = \overline{h(A)} \]

for each \(A \subseteq A\).

Continuous homomorphisms are a natural generalization of the notion of a continuous point mapping from the Topology of Point Spaces. Clearly every closure homomorphism is continuous. The converse statement is not true.

The subject of the first part of this paper is the study of closure homomorphisms. It will be shown that closure homomorphisms are a generalization of the notion of interior mapping from the Topology of Point Spaces (see (ii)).

The second part contains some representation theorems for closure algebras with an enumerable basis. The representation problem for such closure algebras is not completely solved. It can be reduced to the question whether every Hausdorff space with an enumerable basis is an interior image of a separable metric space.

Incidentally I shall show that the dimension of a closure subalgebra \(S\) of a \(G\)-algebra\(^6\) \(A\) can be greater than the dimension of \(A\).

\section{Acknowledgments}

I wish to thank Professor S. T. I will not mention the names of the people who have helped me in the preparation of this paper.

\section{References}

1. See CA, p. 175.
2. A closure algebra \(A\) is called a \(G\)-algebra, if there is a sequence \((B_n)\) of open elements of \(A\) such that each open \(G \subseteq A\) is the sum of all \(X\), such that \(B_n \subseteq G\). See CA, p. 183. \(G\)-algebras are a generalization of separable metric space.
The proof is based on the following true statements:

(a) If \( X \subseteq E \subseteq Y \), then \( \varphi(X) \subseteq \varphi(Y) \).

(b) If \( U \) is open, then \( U \mathcal{C} U \).

Suppose \( \varphi \) is open. Let \( Y \subseteq Y \) and \( \Theta = \varphi^{-1}(Y) \). The set \( \varphi(\Theta) \) being open, we have by (a) and (b)

\[
\varphi(\varphi^{-1}(Y)) \subseteq \varphi(\Theta) \setminus \varphi(\Theta) \cdot Y = \varphi(\varphi^{-1}(Y)) \subseteq \varphi(\Theta) \setminus \varphi(\Theta) \cdot Y = 0 = 0.
\]

Hence \( \varphi^{-1}(Y) \subseteq \Theta = 0 \), i.e., \( \varphi^{-1}(Y) \subseteq \varphi^{-1}(Y) \).

Suppose now that \( \varphi^{-1}(Y) \subseteq \varphi^{-1}(Y) \) for each \( Y \subseteq Y \). Let \( \Theta \) be any open subset of \( X \). Put \( Y = \varphi^{-1}(Y) \). We have by (a) and (b)

\[
\varphi(Y) \subseteq \varphi(\Theta) \cdot Y = \varphi(\varphi^{-1}(Y)) \subseteq \varphi(\Theta) \cdot Y = \varphi(\varphi^{-1}(Y)) \subseteq \varphi(\Theta) \cdot Y = 0 = 0.
\]

Hence \( \varphi(Y) \subseteq \varphi(\Theta) \subseteq \varphi(Y) \), i.e., \( \varphi(Y) \) is open.

It follows immediately from (ii) that

\( Y \subseteq Y \) is a closure homomorphism.

(iii) Suppose \( Y \subseteq Y \) is less than the aleph inaccessible in the strict sense. Then \( Y \subseteq Y \) is an interior image of \( X \) (i.e., \( Y = \varphi(X) \)) if only if \( \varphi^{-1}(Y) \) is homeomorphic to a closure subalgebra of \( E \).

In fact, if \( Y = \varphi(X) \) and \( \varphi \) is interior, then \( k \) defined by (i) is a homeomorphism of \( S(Y) \) onto a closure subalgebra of \( E \). Conversely, if \( k \) is a homeomorphism of \( S(Y) \) onto a closure subalgebra of \( E \), then there is a mapping \( \varphi \) such that (i) holds. Then \( \varphi \) is an interior mapping and \( Y = \varphi(X) \).

Kolmogoroff, Kašdan and Anderson \(^{9}\) have given examples of compact metric spaces \( X \) and \( Y \) such that \( Y \) is an interior image of \( X \).

\(^{9}\) See [7], p. 17 and p. 25.

\(^{9}\) An aleph is said to be inaccessible in the strict sense if \( 1^\ast \mathcal{C} 2^\ast \) and \( \mathcal{C} Y_\alpha \) imply \( \mathcal{C} Y_\alpha \) if \( \mathcal{C} X \), then \( \mathcal{C} X \).

\(^{9}\) See [10], p. 12.

\(^{9}\) See [4], [5], [6].

\(^{9}\) An enumerable basis is a sequence \( (B_n) \) of open subsets such that each open subset \( G \) is the sum of all \( B_n \) such that \( B_n \subseteq \mathcal{C} G \). Moreover, an enumerable basis of a closure algebra \( A \) is a sequence \( (B_n) \) of open elements of \( A \) such that each open \( G \subseteq A \) is the sum of all \( B_n \) such that \( B_n \subseteq G \).

\(^{9}\) [4], p. 174. Eilenberg’s assumptions that \( X \) is compact and \( Y = \varphi(X) \) are superfluous.
Closure homomorphisms and interior mappings

(v) Let \( \varphi \) be a mapping of a metric space \( \mathcal{X} \) into another metric space \( \mathcal{Y} \). The following conditions are equivalent:

(a) \( \varphi \) is interior;

(b) \( \varphi^{-1}(\lim y_k) = \lim \varphi^{-1}(y_k) \) for every convergent sequence \( y_k \in \mathcal{Y} \);

(c) \( \varphi^{-1}(\lim y_k) = \lim_{x \in \mathcal{X}} \varphi^{-1}(x) \) for every convergent sequence \( y_k \in \mathcal{Y} \).

If \( \varphi(\mathcal{X}) = \mathcal{Y} \), the condition (b) means that \( \varphi^{-1}(y) \) is a continuous decomposition of \( \mathcal{X} \).

Suppose that \( \varphi \) is interior. Let \( y_0 = \lim y_k \). We have \( \{y_k\} = \bigcap_{k=1}^{\infty} \{y_k + \epsilon \} \).

Hence

\[
\varphi^{-1}(y_0) = \bigcap_{k=1}^{\infty} \varphi^{-1}(y_k) = \bigcap_{k=1}^{\infty} \varphi^{-1}(y_k + \epsilon) = \bigcap_{k=1}^{\infty} \varphi^{-1}(y_k + 2\epsilon) = \cdots
\]

that is,

\[
\varphi^{-1}(y_0) = L \varphi^{-1}(y_k).
\]

Since the last equation holds also for every subsequence \( \{y_k\} \) of \( \{y_n\} \), we obtain

\[
\varphi^{-1}(y_0) = \lim \varphi^{-1}(y_k),
\]

which proves the implication (a)\( \rightarrow \) (b).

The implication (b)\( \rightarrow \) (c) is trivial.

Suppose now that (c) is true. We shall show that \( \varphi \) is interior. By (iv) and by the remarks below the proof of (iv) it is sufficient to prove that

\[
\varphi^{-1}(\{y_1, y_2, \ldots\}) = \varphi^{-1}(\{y_1, y_4, y_7, \ldots\})
\]

for every convergent sequence \( y_k \in \mathcal{Y} \). Let \( y_0 = \lim y_k \). Clearly (γ) implies that \( \varphi^{-1}(y) \) is closed for every \( y \in \mathcal{Y} \).

We have

\[
\varphi^{-1}(y_1, y_2, \ldots) = \bigcap_{k=1}^{\infty} \{y_k\} = \bigcap_{k=1}^{\infty} \{y_k + \epsilon\} = \bigcap_{k=1}^{\infty} \{y_k + 2\epsilon\} = \cdots
\]

which completes the proof of (γ).

Let \( \mathcal{A} \) be a closure algebra and let \( \mathcal{E} \subset \mathcal{A} \). The symbol \( \mathcal{E} \mathcal{A} \) will denote the closure algebra \( \mathcal{A} \mathcal{E} \) of all elements \( \mathcal{A} \mathcal{E} \) with the following closure operation: \( \mathcal{A} \mathcal{E} = \mathcal{E} \mathcal{A} \mathcal{A} \).

The superior limit \( L \mathcal{X} \) of a sequence of sets \( \mathcal{X}_k \in \mathcal{X} \) is the set of all points \( x \in \mathcal{X} \) such that \( x = \lim \mathcal{X}_k, a \in \mathcal{X}_k, k < \infty \).... The inferior limit \( L \mathcal{X}_k \) of a sequence of sets \( \mathcal{X}_k \in \mathcal{X} \) is the set of all points \( a \in \mathcal{X} \) such that \( x = \lim \mathcal{X}_k, a \in \mathcal{X}_k, k > -1, \ldots, 0 \).

If \( L \mathcal{X} = L \mathcal{X}_k \) and \( L \mathcal{X}_k \) is the set of all points \( x \in \mathcal{X} \) such that \( x = \lim \mathcal{X}_k, a \in \mathcal{X}_k, k < -1, \ldots, 0 \).

If \( L \mathcal{X} = L \mathcal{X}_k \) and we write \( x = \lim \mathcal{X}_k \) and we say that the sequence \( (\mathcal{X}_k) \) converges to \( x \). See C. Kuratowski (7), p. 245, and p. 246.

(8) See C. Kuratowski (7), p. 246, IV, 8.

(9) See C. Kuratowski (7), p. 344, V, 1.

(10) See C. Kuratowski (7), p. 246, IV, 8.


(14) See C. Kuratowski (7), p. 245, IV, 8.


(18) See C. Kuratowski (7), p. 245, IV, 8.


(20) See C. Kuratowski (7), p. 245, IV, 8.


(22) See C. Kuratowski (7), p. 245, IV, 8.


is a closure homomorphism if and only if the ideal $\mathcal{I}$ is principal and generated by a closed element $F_0 \in \mathcal{A}$ (i.e., $B \in \mathcal{I}$ if and only if $B \subseteq \mathcal{P}_0$).

This remark follows immediately from (vii).

§ 2. The representation problem for closure algebras with an enumerable basis

(ix) Suppose that the closure algebra $\mathcal{A}$ is of the form $\mathcal{A} = B[I]$ where $B$ is a Boolean $\sigma$-algebra and $I$ is a $\sigma$-ideal of $B$. If $\mathcal{A}$ has an enumerable basis, it is possible to define a closure operation in $B$ in such a way that

(a) $B$ is a closure algebra with an enumerable basis;

(b) the closure algebra $\mathcal{A}$ is identical with the closure algebra which we obtain by the division of the closure algebra $B$ by the ideal $I$ using the method described in CA 9 (p. 186).

The proof of this theorem is analogous to the proof of CA 14.1. It is even simpler in the above case.

If a closure algebra is a $\sigma$-field of sets, it is called a closure field.

(x) Every closure field is weakly homeomorphic to a $T_0$-space. Every closure field with an enumerable basis is weakly homeomorphic to a $T_0$-space with an enumerable basis.

The proof is similar to that of CA 13.1.

Since every Boolean $\sigma$-algebra is isomorphic to a quotient algebra $\mathcal{X}/I$ where $\mathcal{X}$ is a $\sigma$-field of sets and $I$ is a $\sigma$-ideal of $\mathcal{X}$, we find from (ix) that

(xi) Every closure algebra $\mathcal{A}$ with an enumerable basis is isomorphic to a quotient closure algebra $\mathcal{X}/I$ where $\mathcal{X}$ is a closure field with an enumerable basis, and $I$ is a $\sigma$-ideal of $\mathcal{X}$.

Combining (ix) and (x) we find that

(xii) Every closure algebra $\mathcal{A}$ with an enumerable basis is weakly homeomorphic to a closure quotient algebra $\mathcal{Z}(\mathcal{X})/I$ where $\mathcal{X}$ is a $T_0$-space with an enumerable basis (i.e., $\mathcal{Z}(\mathcal{A})$ is homeomorphic to $\mathcal{Z}(\mathcal{X})/I$, $L = \mathcal{A}/\mathcal{B}(\mathcal{X})$).

Theorems (ix), (x), (xi), (xii) are analogous to Theorems CA 14.1, 13.1, 14.2 and 15.1 proved for $C$-algebras. The question arises whether

a theorem analogous to CA 15.2 is true for closure algebras with an enumerable basis. I hope that the following non-proved statement is true:

\begin{enumerate}
\item[(H)] For every closure algebra $\mathcal{A}$ with an enumerable basis there is a closure subalgebra $\mathcal{S}$ of the closure algebra $\mathcal{Z}(\mathcal{X})$ of all subsets of the Hilbert cube $\mathcal{K}$, and a $\sigma$-ideal $\mathcal{I}$ of $\mathcal{S}$ such that $\mathcal{A}$ is weakly homeomorphic to $\mathcal{S}/\mathcal{I}$.
\end{enumerate}

The difficulty of the above problem lies in the fact that we know no characterization of closure subalgebras of closure algebras $\mathcal{Z}(\mathcal{X})$ where $\mathcal{X}$ is a separable metric space. By CA 4.3 each such subalgebra has an enumerable basis and therefore is weakly homeomorphic to a $T_0$-space.

A simple example given in CA (p. 184, footnote 21) shows that a complete four-element closure subalgebra of $\mathcal{Z}(\mathcal{R})$, where $\mathcal{R}$ is the set of all real numbers, need not be weakly homeomorphic to a $T_0$-space. It seems probable that

\begin{enumerate}
\item[(H)] If $\mathcal{X}$ is a $T_0$-space with an enumerable basis, then $\mathcal{Z}(\mathcal{X})$ is homeomorphic to a complete closure subalgebra of $\mathcal{Z}(\mathcal{Y})$ where $\mathcal{Y}$ is a separable metric space.
\end{enumerate}

On account of (ii), hypothesis (H) may be formulated in the following equivalent form:

\begin{enumerate}
\item[(H')] Each $T_0$-space with an enumerable basis is an interior image of a separable metric space.
\end{enumerate}

We can only prove that

\begin{enumerate}
\item[(xiii)] Each $T_0$-space $\mathcal{X}$ with an enumerable basis is an interior image of a totally disconnected Hausdorff space with an enumerable basis.
\end{enumerate}

Let $\mathcal{R}$ denote the set of all real numbers. Consider the Cartesian product $\mathcal{X} \times \mathcal{R}$ with the usual topology. $\mathcal{X} \times \mathcal{R}$ has an enumerable basis.

Since $\mathcal{X} \times \mathcal{R}$, we can associate with every $x \in \mathcal{X}$ a set $E_x$ of irrational numbers such that

\begin{enumerate}
\item[(1)] $E_x$ is dense in $\mathcal{R}$ for every $x \in \mathcal{X}$;
\item[(2)] if $x_1 \neq x_2$, then $E_{x_1} \cap E_{x_2} = \emptyset$.
\end{enumerate}

The space $\mathcal{Y} = \bigcup_{x \in \mathcal{X}} E_x \times \mathcal{R} \times \mathcal{X}$ (with the topology induced by $\mathcal{X} \times \mathcal{R}$) has an enumerable basis. $\mathcal{Y}$ is a totally disconnected Hausdorff space. In fact, let $(x_1, r_1) = (x_2, r_2)$ be two points in $\mathcal{Y}$. If $x_1 = x_2$, then $x_1 = x_2$. If $x_1 \neq x_2$, then $r_1 = r_2$ also since $r_1 \in E_{x_1}$, $r_2 \in E_{x_2}$ and $E_{x_1} \cap E_{x_2} = \emptyset$. Suppose, for instance, that $r_1 < r_2$. Let $r_3$ be a rational number such that $r_1 < r_3 < r_2$. The sets $U_1 = \mathcal{Y} \cap \{r < r_1\}$, $U_2 = \mathcal{Y} \cap \{r > r_2\}$ are disjoint neighbourhoods of $(x_1, r_1)$ and $(x_2, r_2)$ respectively, and $\mathcal{Y} = U_1 + U_2$.


A space $\mathcal{X}$ is totally disconnected if for every pair $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$, there are open sets $U_1$ and $U_2$ such that $x_1 \in U_1, x_2 \in U_2, \mathcal{X} = U_1 + U_2, U_1, U_2 = 0$. 2*
On existential theorems in non-classical functional calculi

by

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Let $S_i$ be the Heyting propositional calculus, and let $S_{\Sigma}$ be the Heyting functional calculus. The individual variables of the system $S_{\Sigma}$ will be denoted by $x_1, x_2, \ldots$, the quantifiers — by $\Sigma$ and $\Pi$. The formulas from $S_{\Sigma}$ will be denoted by the letters $\alpha, \beta$. If $\alpha$ is a formula from $S_{\Sigma}$, then $\alpha[x_i/\beta]$ denotes the formula obtained from $\alpha$ by replacing each free occurrence of $x_i$ by $x_i$ (each bound occurrence of $x_i$ should be replaced earlier by $x_i$ which does not appear in $\alpha$, $1 \neq q$).

Gödel formulated (without proof) the following theorem:

\[(\chi)\quad \text{Let } \alpha, \beta \text{ be two formulas from the Heyting propositional calculus } S_i. \text{ If the disjunction } \alpha + \beta \text{ is a theorem of } S_{\Sigma}, \text{ then either } \alpha \text{ or } \beta \text{ is a theorem of } S_{\Sigma}.\]

Theorem (\(\chi\)) was later proved by McKinsey and Tarski [2] by an algebraical method. Another algebraical proof was given by Rieger [9].

The purpose of this paper is to prove the following theorem (\(\tau\)) which is an extension of (\(\chi\)) over the Heyting functional calculus $S_{\Pi}$. The second part of Theorem (\(\chi\)) shows that the Heyting functional calculus is the well formalization of Brouwer's ideas concerning existential theorems.

\[(\tau)\quad \text{If the formula } \alpha + \beta \text{ is provable in } S_{\Pi}, \text{ then either } \alpha \text{ or } \beta \text{ are provable in } S_{\Pi}. \text{ If the formula } \Sigma \alpha \text{ is provable in } S_{\Pi}, \text{ then there is a positive integer } q \text{ such that the formula } \alpha[x_i/\beta] \text{ is provable in } S_{\Pi}.\]

Clearly, if the sequence $x_1, x_2, \ldots$ contains all the free variables which appear in $\alpha$, the integer $q$ can be chosen among the numbers $1, 2, \ldots, q$. If $\alpha$ contains no free variable, then $q$ is an arbitrary integer, e. g. $q=1$.

\[1\] Presented at the Seminar on Foundations of Mathematics in the Mathematical Institute of the Polish Academy of Sciences in November 1958.

\[2\] See E. G. Godel [1]. See also G. Gentzen [1].

\[3\] See L. Rieger [1], p. 29.