Setting $\varphi_i=\bar{\varphi}_i\varphi$ for $i=1,2,\dots,n-1$, we obtain (6), q. e. d.

Theorem I follows immediately from Theorem II where $W=I$. More generally, we infer that the condition (α) in Theorem I can be replaced by any of the following conditions:

(α_1) *$f_i^{-1}(y)$ has a finite number of components for every $y\in I$ and $i=1,2,\dots,n$;*

(α_2) *the functions f_i ($i=1,2,\dots,n$) are of bounded variation and, for every $y\in I$, the set $f^{-1}(y)$ contains no interval;*

(α_3) *the functions f_i ($i=1,2,\dots,n$) are of bounded variation; if $f_j(x)=y_0=\mathrm{const}$ in an interval $x_1\leqslant x\leqslant x_2$, then the sets $f_i^{-1}(y_0)$ have a finite number of components.*

The case (α_1) follows from Theorem II where $W=I$.

If f_i is of bounded variation, then the set

$$Y_i=\mathop{E}_{y}\left(\overline{\overline{f_i^{-1}(y)}}\geqslant\aleph_0\right)$$

has measure zero. To obtain the case (α_2) it suffices to set in Theorem II $W=I-(Y_1+Y_2+\dots+Y_n)$. To obtain the case (α_3) it suffices to set in Theorem II $W=I-(Y_1+Y_2+\dots Y_n)+Y$ where Y is the (at most enumerable) set of all numbers y such that, for an integer $i=1,2,\dots,n$, the set $f_i^{-1}(y)$ contains an interval.

Notice that, if $f_1,f_2\in\mathfrak{F}_W$, the set A need not be connected or locally connected. In fact, let $x_n=3/4-1/4n$. Let

$$f_1(0)=0\,,\qquad f_1(x_{2k-1})=\frac{1}{2}+\frac{1}{2k},\qquad f_1(x_{2k})=\frac{1}{2}=f_1\left(\frac{3}{4}\right),\qquad f_1(1)=1,$$

and let f_1 be linear in all remaining intervals. Analogously let

$$f_2(0)=0\,,\qquad f_2(x_{2k-1})=\frac{1}{2}-\frac{1}{2k},\qquad f_2(x_{2k})=\frac{1}{2}=f_2\left(\frac{3}{4}\right),\qquad f_2(1)=1,$$

and let f_2 be linear in all remaining intervals. All the points (x_{2k},x_{2j}) are isolated points of A, and $f_1,f_2\in\mathfrak{F}_W$ where $W=I-(1/2)$.

The proof of Theorem II is simpler if we restrict more the class of functions under consideration. In particular, the proof is very simple in the case where the interval I can be divided into subintervals in each of which the functions are linear.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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On uniformization of functions (II)

by

R. Sikorski (Warszawa)

I shall give another proof of Theorems I and II from the paper of R. Sikorski and K. Zarankiewicz, *On uniformization of functions (I)* [1].

Both proofs are based on a connectedness property of the set

$$\mathop{E}_{(x_1,\dots,x_n)}\left(f_1(x_1)=f_2(x_2)=\dots=f_n(x_n)\right)$$

(see Lemmas (i) and (i′)). Theorem III (see p. 349), which seems to be interesting in itself, generalizes this property to the case of mappings $f_1,f_2,\dots,f_n$ of the k-dimensional cube Q_k into itself, the mappings being the identity on the boundary of Q_k.

The second proof makes no use of the principle of induction. It consists in the direct application of the method, used in the first part for the case of two functions, to the general case of n functions. However, this kind of proof requires more advanced topological means.

The second proof is based on the following lemmas which are generalizations of (i) and (ii) respectively.

(i′) *If $f_1,f_2,\dots,f_n\in\mathfrak{F}$, then the set*

$$A=\mathop{E}_{(x_1,\dots,x_n)}\left(f_1(x_1)=f_2(x_2)=\dots=f_n(x_n)\right) \tag{8}$$

is connected between the points $p_0=(0,0,\dots,0)$ and $p_1=(1,1,\dots,1)$ of the n-dimensional Euclidean space.

(ii′) *If $f_1,f_2,\dots,f_n\in\mathfrak{F}_W$ (where W is a dense subset of I), then each component of the set A defined by (8) is locally connected.*

In fact, by (i′) the points p_0 and p_1 lie in a component A_0 of A. By (ii′) A_0 is locally connected. Therefore there exists an arc Γ

$$x_1=\varphi_1(t),\quad x_2=\varphi_2(t),\quad \dots,\quad x_n=\varphi_n(t),\qquad t\in I$$

[1] R. Sikorski and K. Zarankiewicz, *On the uniformization of functions (I)*, this volume, p. 339-344. See p. 339 and p. 342. The knowledge of this paper is here assumed.

contained in A_0 and joining the points p_0 and p_1. This means that $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathfrak{F}$ and

(9) $$f_1\varphi_1 = f_2\varphi_2 = \ldots = f_n\varphi_n,$$

i. e. that Theorem II (where (6) should be replaced by (9)) is true. Analogously to the first part we can now deduce Theorem I with conditions (α), (α_1), (α_2), (α_3).

Proof of (ii′) is the same as that of (ii). It is based on the following lemma (analogous to (iii)) where $L_{l,\xi}$ denotes the hyperplane

$$L_{l,\xi} = \underset{(x_1,\ldots,x_n)}{E} (x_l = \xi).$$

(iii′) *Let $V_1, V_2, \ldots, V_n$ be dense subsets of I and let X be a compactum contained in the unit cube Q_n. If, for $i = 1, 2, \ldots, n$ and for every $\xi \in V_i$, the set $XL_{i,\xi}$ has a finite number of components, then each component of X is locally connected.*

Lemma (i′) is a particular case ($k=1$) of Theorem III below. The direct proof of (i′) is somewhat simpler than that of Theorem III but the main idea is the same.

First we shall fix the notations. E_m is the m-dimensional Euclidean space. $Q_m = I \times I \times \ldots \times I$ (m-times) is the m-dimensional unit cube. S_{m-1} is the $(m-1)$-dimensional boundary of Q_m in E_m.

All cycles and chains under consideration will be taken mod p, where p is one of the integers $0, 2, 3, \ldots$ For simplicity we shall denote the set of all points of a chain (*i. e.* of all points of all their simplexes with coefficients $\neq 0$) by the same letter as the chain. The boundary of a chain K will be denoted by $K^{\cdot}$.

The Kronecker index of two chains (of suitable dimensions) K_1, K_2 in E_m or S_m will be denoted by $\chi(K_1, K_2)$. The linking coefficient of two cycles (of suitable dimensions) C_1, C_2 in E_m or S_m will be denoted by $\mathfrak{v}(C_1, C_2)$.

(iv) *Let k be a fixed integer, $0 < k < m$, let B be a compact subset of Q_m, and let g be a continuous mapping of Q_m into itself such that*

(β) $$C \sim g(C) \quad \textit{in the set} \quad Q_m - B$$

for every $(m-k-1)$-dimensional cycle $C \subset S_{m-1} - g^{-1}(B)$.
For every $(k-1)$-dimensional true cycle Z lying in $S_{m-1}B$,

if $Z \sim 0$ in B, then also $Z \sim 0$ in $g^{-1}(B)$ [2].

Notice that (β) implies the inclusion $S_{m-1}B \subset g^{-1}(B)$.

[2]) For the definition of homology and of true and convergent cycles — see P. Alexandroff, *Dimensiontheorie — Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Mathematische Annalen 106 (1932), p. 176-180. Notice that the homology mod 0 is taken relative to the group of rational numbers.

It will be convenient to assume in the proof that Q_m is a half-sphere of the geometrical m-dimensional unit sphere S_m. Consequently all simplexes under consideration are spherical.

Suppose that

(10) $$Z \sim 0 \quad \text{in} \quad B$$

and

(11) $$Z \not\sim 0 \quad \text{in} \quad g^{-1}(B).$$

It suffices to consider only the case where the true cycle

$$Z = (Z_1, Z_2, \ldots)$$

is convergent.

By (10) there is a sequence $\{D_j\}$ of k-dimensional chains such that
1° the diameters of the simplexes of D_j tend to zero if $j \to \infty$;
2° the vertices of the simplexes of D_j belong to B;
3° $D_j^{\cdot} = Z_j$ for all j.

By (11) there exists [3]) an $(m-k)$-dimensional cycle K such that

(12) $$K \cdot g^{-1}(B) = 0,$$

and K is linked with Z, *i. e.*

(13) $$\mathfrak{v}(K, Z_j) \neq 0 \quad \text{for almost all [4]) } j.$$

We can suppose that K is in general position with every chain D_j. By (13) and 3°

(14) $$\chi(K, D_j) \neq 0 \quad \text{for almost all } j.$$

We can also assume that the interior of each simplex of K is contained either in Q_m or in $S_m - Q_m$. Let L be the chain formed of all simplexes of K (with the same coefficients) which are contained in Q_m, and let $\mathit{Ł}$ be the chain formed of all the remaining simplexes of K (also with the same coefficients). Since $D_j \subset Q_m$, we have

$$\chi(\mathit{Ł}, D_j) = 0 \quad \text{for almost all } j.$$

Since $K = L + \mathit{Ł}$, we have

$$\chi(L, D_j) = \chi(L, D_j) + \chi(\mathit{Ł}, D_j) = \chi(K, D_j).$$

Hence, by (14)

(15) $$\chi(L, D_j) \neq 0 \quad \text{for almost all } j.$$

Since the vertices of Z_j belong to S_{m-1} (= the $(m-1)$-dimensional „equator" of S_m), there exist chains D_j^* in S_m such that

[3]) See P. Alexandroff, loc. cit., p. 184.
[4]) That is, for all $j > j_0$ where j_0 is a sufficiently great integer.

1* the diameters of the simplexes D_j^* tend to zero if $j \to \infty$;

2* the simplexes of D_j^* are disjoint with the interior of Q_m; if a simplex Δ of D_j^* lies in S_{m-1}, then Δ is also a simplex of D_j;

3* $D_j^{*\cdot} = -Z_j$.

The conditions 1*, 2*, 3* hold, for instance, if D_j^* is symmetrical to D_j with respect to the equator hyperplane.

Let $\mathcal{Z}_j = D_j + D_j^*$ and $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots)$. By 1°, 1*, 3°, 3* $\mathcal{Z}$ is a k-dimensional true cycle in S_m. By 2*

$$\chi(L, \mathcal{Z}_j) = \chi(L, D_j).$$

Hence, by (15),

(16) $$\chi(L, \mathcal{Z}_j) \neq 0 \quad \text{for almost all} \quad j.$$

Let $C = L^\cdot$. C is an $(m-k-1)$-dimensional cycle and, by (12),

(17) $$C \subset S_{m-1} - g^{-1}(B).$$

By (16)

(18) $$\mathfrak{v}(C, \mathcal{Z}_j) \neq 0 \quad \text{for almost all} \quad j.$$

By (β), (17), (18), 1°, 2°, 2*

(19) $$\mathfrak{v}\big(g(C), \mathcal{Z}_j\big) \neq 0 \quad \text{for almost all} \quad j,$$

i. e. the continuous cycle $g(C)$ and the true cycle $\mathcal{Z}$ are linked.

On the other hand, $C = L^\cdot$ and $Lg^{-1}(B) = 0$ by (12). Hence $g(C) = g(L)^\cdot$ and $g(L) \subset Q_m - B$. Consequently by 1°, 1*, 2°, 2*

$$\mathfrak{v}\big(g(C), \mathcal{Z}_j\big) = 0 \quad \text{for almost all} \quad j,$$

i. e. $g(C)$ and $\mathcal{Z}$ are not linked — in contradiction to (19). Lemma (iv) is proved.

Now let k and n be two fixed positive integers, $n > 1$, and let $\mathfrak{F}_k$ be the set of all continuous mappings of Q_k into itself such that

(20) $$f(x) = x \quad \text{for} \quad x \in S_{k-1},$$

i. e. f is the identity on the boundary of Q_k.

Up to the end of the proof of Theorem III the letter x with indices will exclusively denote points of the k-dimensional space E_k. If $x_1, x_2, \ldots, x_n \in E_k$, *i. e.* $x_i = (\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,k})$ for $i = 1, 2, \ldots, n$, then $(x_1, x_2, \ldots, x_n)$ denotes the point

$$(\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,k}, \xi_{2,1}, \xi_{2,2}, \ldots, \xi_{2,k}, \ldots, \xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,k})$$

of the nk-dimensional space E_{nk}. Obviously $(x_1, x_2, \ldots, x_n) \in Q_{nk}$ if and only if $x_1, x_2, \ldots, x_n \in Q_k$. Similarly $(x_1, x_2, \ldots, x_n) \in S_{nk-1}$ $(x_1, x_2, \ldots, x_n \in Q_k)$ if and only if one of the points $x_1, x_2, \ldots, x_n$ lies in S_{k-1}.

Let B be the set of all points $(x, x, \ldots, x) \in Q_{nk}$ (where $x \in Q_k$), and let $Z = BS_{nk-1}$, *i. e.* Z is the set of all points $(x, x, \ldots, x)$ where $x \in S_{k-1}$.

B is a k-dimensional complex homeomorphic to Q_k, and Z is a $(k-1)$-dimensional complex homeomorphic to S_{k-1}. The homeomorphism is realized by the projection π

$$\pi(x, x, \ldots, x) = x.$$

We shall consider the $(k-1)$-dimensional sphere Z as a cycle formed of consistently oriented simplexes with coefficients $=1$. Obviously $Z \sim 0$ in B since Z is the boundary of B.

THEOREM III. *Let $f_1, f_2, \ldots, f_n \in \mathfrak{F}_k$ and let*

$$A = \underset{(x_1, \ldots, x_n)}{E} \big(f_1(x_1) = f_2(x_2) = \ldots = f_n(x_n)\big).$$

Then $Z \subset A \subset Q_{nk}$ and $Z \sim 0$ in A.

The inclusion $A \subset Q_{nk}$ is obvious. The inclusion $Z \subset A$ follows from (20).

In order to prove that $Z \sim 0$ in A it suffices to apply Lemma (iv) where $m = kn$, B and Z have the meaning defined above, and

$$g(x_1, x_2, \ldots, x_n) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big).$$

In fact, g is a mapping of Q_{nk} into itself and $A = g^{-1}(B)$. It follows from (20) that g maps each $(nk-1)$-dimensional face of Q_{nk} into itself. Hence

$$g(S_{nk-1}) \subset S_{nk-1}.$$

Consequently g transforms the set

$$R = S_{nk-1} - g^{-1}(B) = S_{nk-1} - A$$

into the set $S_{nk-1} - Z = S_{nk-1} - B$.

The condition (β) is fulfilled since the mapping $g|R$ (*i. e.* the mapping g restricted to the set R) is homotopic with the identity mapping (of R into itself) in the set $S_{nk-1} - Z$. The homotopy is realized by the function $h(y,t)$ $(y \in R,\ t \in I)$ where $h(y,t)$ is the point which divides the segment $\overline{y, g(y)}$ in the relation t. This follows from the following property: if $y = (x_1, x_2, \ldots, x_n) \in S_{nk-1} - Z$ and $g(y) \in S_{nk-1} - B$, then the segment $\overline{y, g(y)}$ is contained in $S_{nk-1} - Z$. In fact, the points y and $g(y)$ lie on the same $(kn-1)$-dimensional face of Q_{nk}. Thus the segment $\overline{y, g(y)}$ is contained in S_{nk-1}. Suppose there is a point $z = ty + (1-t)g(y) \in Z$ $(0 \leqslant t \leqslant 1)$, *i. e.* $z = (x, x, \ldots, x)$, where $x \in S_{k-1}$. Since $x = tx_j + (1-t)f_j(x_j)$ for $j = 1, 2, \ldots, n$, $x_j \in Q_k$, $f_j(x_j) \in Q_k$, and $x \in S_{k-1}$, we infer that $x_j \in S_{k-1}$. Consequently $f_j(x_j) = x_j$ by (20) and $x = tx_j + (1-t)f_j(x_j) = x_j$ for $j = 1, 2, \ldots, n$. Hence $y = z \in Z$ in contradiction to the hypothesis $y \notin Z$.

Notice that Lemma (i′) implies immediately the following

Corollary I. *For arbitrary functions* $f_1, f_2, \dots, f_n \in \mathfrak{F}$ *there exist a co tinuum* A_0 *and mappings* $\psi_1, \psi_2, \dots, \psi_n$ *of* A_0 *onto* I *such that*

$$f_1\psi_1 = f_2\psi_2 = \dots = f_n\psi_n.$$

It suffices to put: A_0 = the component of the set A (defined b containing p_0 and p_1, and

$$\psi_i(x_1, x_2, \dots, x_n) = x_i \quad \text{for} \quad (x_1, x_2, \dots, x_n) \in Z_0.$$

Analogously Theorem III implies the following

Corollary II. *For arbitrary functions* $f_1, f_2, \dots, f_n \in \mathfrak{F}_k$ *there ex tinuum* $A_0 \subset E_{nk}$ *and mappings* $\psi_1, \psi_2, \dots, \psi_n$ *of* A_0 *onto* Q_k *such*

1) $S_{k-1} \subset A_0$;
2) $\psi_i(x) = x$ *for* $x \in S_{k-1}$, $i = 1, 2, \dots, n$;
3) S_{k-1} *is homologous to zero in* A_0;
4) $f_1\psi_1 = f_2\psi_2 = \dots = f_n\psi_n$.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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Addendum to "An extension of Sperner's Lemma, with applications to closed-set coverings and fixed points"

(Fundamenta Mathematicae 40 (1953), p. 3-12)

by

F. Bagemihl (Princeton, N. J.)

In Theorem 2, Theorem 3, and Corollary 3, it is sufficient to assume merely that $m \neq 2$, for then, in the proof of Theorem 2, the asserted properties of the numbers $a_h^{(j)}$ imply that, in the natural orientation of the m-plex, $S_2, S_3, \dots, S_m$ all become π-simplexes, so that $\pi \neq \nu + 1$, and hence Theorem 1 applies. (Corollary 3 is true, of course, for *every* m, as is easily seen, for $m > 1$, by making use of the fact that the m-plex is connected but its frontier is not.)