We admit in this scheme only those formulae Ξ in which each quantifier bounds a variable of the lowest type. Other axioms are the familiar ones.

A relation ϱ is said to be definable in a system if there exists in this system a theorem of the form (α) such that the expression $\Xi(\ldots X, Y, Z, \ldots)$ is a possible definiens of the relation ϱ *i. e.* $\varrho(\ldots x, y, z, \ldots)$ if and only if $\ldots x, y, z, \ldots$ satisfy the formula Ξ.

From this definition it follows that a relation is definable in S if and only if it is elementarily definable. Hence from Theorem 3.3 it follows that the existence of a non Borelian set is unprovable in the system S. But the general question, whether the class $\mathfrak{D}$ constitutes the model of the system S remains open, because we cannot decide whether the axiom of extensionality is satisfied in the domain $\mathfrak{D}$. There remains also another task: to verify whether the theory of continuous functions can be deduced in S. Perhaps this theory can be obtained in S without the use of the axiom of extensionality.

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Reçu par la Rédaction le 8. 2. 1954

On uniformization of functions (I)

by

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Let I be the unit interval $0 \leqslant x \leqslant 1$, and let $\mathfrak{F}$ be the class of all continuous mappings f of I into itself such that $f(0)=0$ and $f(1)=1$. If $f, \varphi \in \mathfrak{F}$, then $f\varphi \in \mathfrak{F}$ too. The symbol $f\varphi$ denotes always the superposition of f and φ.

We shall prove the following

THEOREM I [1]. *If $f_1, f_2, \ldots, f_n \in \mathfrak{F}$ are functions such that*

(α) *for each $i=1,2,\ldots,n$, there is a sequence $0=x_0<x_1<x_2<\ldots<x_r$ such that f_i is either non-decreasing or non-increasing in every interval $\langle x_{j-1}, x_j\rangle$, $j=1,2,\ldots,r$,*

then there exist functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathfrak{F}$ such that

$$f_1\varphi_1 = f_2\varphi_2 = \ldots = f_n\varphi_n. \tag{1}$$

Theorem I has the following simple interpretation. There are n paths which are going to the top of a mountain. The paths need not always go upwards, some segments of the paths may be directed downwards. On each of the paths a tourist is climbing. Theorem I asserts that the tourists can climb to the top of the mountain in such a way that, at every moment, all of them are on the same level (of course, it may happen that, in some time intervals, some of the tourists must return from the previously covered segments of the paths). To make it clear, let us suppose that the paths are the curves

$$p_1(x),\ p_2(x),\ \ldots,\ p_n(x),$$

where $p_j(x)$ $(j=1,2,\ldots,n)$ is a mapping of I into the three-dimensional space. Let $f_j(x)$ be the height (the third coordinate) of the point $p_j(x)$.

[1]) K. Zarankiewicz, *Un théorème sur l'uniformisation de fonctions continues et son application à la démonstration du théorème de F. J. Dyson sur les transformations de la surface sphérique*, Bull. Acad. Pol. Sc. Cl. III 2 (1954), p. 117-120.

During the print of this paper the authors found out that a theorem similar to Theorems I and II was proved by T. Homma, *A theorem on continuous functions*, Kōdai Math. Sem. Reports 1 (1952), p. 13-16.

Homma's hypothesis about $f_1, f_2, \ldots, f_n$ is other than that in this paper. The example on p. 340 is also given in Homma's paper.

Let us suppose that the equations of the movements of the tourists are respectively

$$x=\varphi_1(t), \qquad x=\varphi_2(t), \qquad \ldots, \qquad x=\varphi_n(t),$$

i. e. that the j-th tourist is at the point $p_j(\varphi_j(t))$ in the moment t. If the functions $\varphi_1,\varphi_2,\ldots,\varphi_n$ satisfy the condition (1), then all the tourists are on the same level at every moment.

The proof of Theorem I is topological. The method of proof makes it possible to obtain a theorem stronger than Theorem I (see Theorem II below). Condition (α) can be weakened. However, some additional hypotheses about $f_1,f_2,\ldots,f_n$ are necessary. Theorem I is false (even in the case $n=2$) if we assume only the continuity of $f_1,f_2,\ldots,f_n$.

For instance, let $x_n=1/2-1/4n$ for $n=1,2,\ldots$, let

$$f_1(x_n)=\frac{1}{2}-\frac{(-1)^n}{4n}, \qquad f_1(0)=0, \qquad f_1\left(\frac{1}{2}\right)=\frac{1}{2}, \qquad f_1(1)=1,$$

and let f_1 be linear in all remaining intervals. Let

$$f_2(0)=0, \qquad f_2\left(\frac{1}{4}\right)=\frac{1}{2}=f_2\left(\frac{1}{2}\right), \qquad f_2(1)=1,$$

and let f_2 be linear in all remaining intervals. Then there exist no functions $\varphi_1,\varphi_2\in\mathfrak{F}$ such that $f_1\varphi_1=f_2\varphi_2$. In fact, let us suppose that $f_1\varphi_1=f_2\varphi_2$, $\varphi_1,\varphi_2\in\mathfrak{F}$. Let $\{t_n\}$ be an increasing sequence of points such that $x_n=\varphi_1(t_n)$, and $t_0=\lim t_n$. We have

$$f_2(\varphi_2(t_n))=f_1(\varphi_1(t_n))=f_1(x_n)=\frac{1}{2}-\frac{(-1)^n}{4n}.$$

Hence $f_2(\varphi_2(t_n))>1/2$ for $n=1,3,5,\ldots$ and $f_2(\varphi_2(t_n))<1/2$ for $n=2,4,6,\ldots$ Consequently $\varphi_2(t_n)>1/2$ for $n=1,3,5,\ldots$ and $\varphi_2(t_n)<1/4$ for $n=2,4,6,\ldots$ The function $\varphi_2\in\mathfrak{F}$ is not continuous in the point t_0, which contradicts the definition of $\mathfrak{F}$.

The proof of Theorem I is by induction with respect to n. Let us first consider the case $n=2$.

Let $f_1,f_2\in\mathfrak{F}$. We are to prove that, under some additional hypotheses about f_1,f_2 (*e. g.* that f_1,f_2 satisfy the condition (α)), there exist functions $\varphi_1,\varphi_2\in\mathfrak{F}$ such that

(2) $$f_1\varphi_1=f_2\varphi_2.$$

The pair of functions φ_1,φ_2 can be geometrically interpreted as a plane curve Γ

(3) $$x=\varphi_1(t), \qquad y=\varphi_2(t), \qquad t\in I.$$

The condition that

(4) $$\varphi_1,\varphi_2\in\mathfrak{F}$$

means geometrically that the curve Γ lies in the unit square $Q=I\times I$, and that the origin of Γ and the end point of Γ are respectively the points

$$p_0=(0,0) \quad \text{and} \quad p_1=(1,1).$$

The condition (2) means that the curve Γ lies in the set

(5) $$A=\underset{(x,y)}{E}\big(f_1(x)=f_2(y)\big)\subset Q.$$

Thus we are to prove that (under additional hypotheses about f_1 and f_2) the compact set A is arcwise connected between the points $p_0,p_1\in A$. The example given above shows that additional hypotheses about f_1 and f_2 are necessary, *i. e.* that the set A is not always arcwise connected between p_0 and p_1.

On the other hand:

(i) *For arbitrary functions $f_1,f_2\in\mathfrak{F}$ the set A defined by* (5) *is connected between p_0 and p_1, i. e. p_0 and p_1 lie in the same component A_0 of A.*

Let $I_0=I\times(0)$, $I_1=I\times(1)$, $I^0=(0)\times I$ and $I^1=(1)\times I$ be the sides of Q, and let $\Phi(x,y)=f_1(x)-f_2(y)$. Obviously

$$A=\underset{(x,y)}{E}\,\Phi(x,y)=0.$$

Suppose that A is not connected between p_0 and p_1. Then there exists an arc Γ_0 with end points q_1 and q_2 such that

$$q_1\in I_0+I^1, \qquad q_2\in I_1+I^0, \qquad \text{and} \qquad \Gamma_0A=0.$$

If $(x,y)\in I_0+I^1$, then $\Phi(x,y)\geqslant 0$. In fact, either $y=0$ or $x=1$. In the first case $f_1(x)\geqslant 0$ and $f_2(y)=0$; in the second $f_1(x)=1$ and $f_2(y)\leqslant 1$.

Analogously, if $(x,y)\in I^0+I_1$, then $\Phi(x,y)\leqslant 0$.

Consequently $\Phi(q_1)\geqslant 0$ and $\Phi(q_2)\leqslant 0$. This implies that $\Phi(q_0)=0$ for a point $q_0\in\Gamma_0$, which contradicts the hypothesis that $\Gamma_0A=0$. Lemma (i) is proved.

By (i) the point p_0 and p_1 are in a component A_0 of A. Now it is obvious that the additional hypotheses about f_1 and f_2 should imply the local connectness of A_0. In fact, A_0 is then arcwise connected between p_0 and p_1, and there exist functions $\varphi_1,\varphi_2\in\mathfrak{F}$ satisfying (2).

Let W be a dense subset of I and let $\mathfrak{F}_W$ be the class of all functions $f\in\mathfrak{F}$ such that

(a) the set $f^{-1}(W)$ is dense in I;

(b) if $y\in W$, then the set $f^{-1}(y)$ contains only a finite number of components, *i. e.* $f^{-1}(y)$ is the sum of a finite number of closed intervals and of isolated points.

Obviously the condition (a) can be replaced by the following equivalent one:

(a′) if the set $f^{-1}(y)$ contains an interval, then $y \in W$ (*i. e.* if $f(x) = y_0 = \text{const}$ in an interval $\langle x_1, x_2 \rangle \subset I$, then $y_0 \in W$).

(ii) *If $f_1, f_2 \in \mathfrak{F}_W$, then each component of the set A defined by (5) is locally connected.*

The proof is based on the following simple lemma, where L_ξ and L^η denote the straight lines

$$L_\xi = \mathop{E}_{(x,y)}(x = \xi), \qquad L^\eta = \mathop{E}_{(x,y)}(y = \eta).$$

(iii) *Let V_1 and V_2 be two dense subsets of I and let X be a compact set contained in Q. If, for every $\xi \in V_1$ and $\eta \in V_2$, the sets XL_ξ and XL^η have a finite number of components, then each component of X is locally connected.*

Let X' be a component of X, and $(x_0, y_0) \in X'$, $\varepsilon > 0$. Let

$$x_0 - \varepsilon < \xi_1 < x_0 < \xi_2 < x_0 + \varepsilon, \qquad \xi \in V_1 \quad \text{or} \quad \xi_i \notin I \qquad (i = 1, 2),$$
$$y_0 - \varepsilon < \eta_1 < y_0 < \eta_2 < y_0 + \varepsilon, \qquad \eta_i \in V_2 \quad \text{or} \quad \eta_i \notin I \qquad (i = 1, 2),$$

let $Q_0 = \mathop{E}_{(x,y)}(\xi_1 \leqslant x \leqslant \xi_2 \text{ and } \eta_1 \leqslant y \leqslant \eta_2)$ and let S_0 be the boundary of Q_0. The set $X'S_0$ has a finite number of components, and consequently the set $Q_0 X'$ has also a finite number of components since each component of $Q_0 X'$ has a non empty intersection with the set S_0. The component X'' of $Q_0 X'$ which contains the point (x_0, y_0) is a connected neighbourhood of (x_0, y_0) relatively to X', and $X'' \subset Q_0$. Since ε, X' and (x_0, y_0) are arbitrary, Lemma (iii) is proved.

To prove (ii) it suffices to put in (iii)

$$X = A, \qquad V_1 = f_1^{-1}(W) \quad \text{and} \quad V_2 = f_2^{-1}(W).$$

Theorem I can be generalized as follows.

THEOREM II. *If $f_1, f_2, \ldots, f_n \in \mathfrak{F}_W$ (where W is a dense subset of I), then there exist functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathfrak{F}$ such that*

$$f_1\varphi_1 = f_2\varphi_2 = \ldots = f_n\varphi_n \in \mathfrak{F}_W. \tag{6}$$

The proof is by induction on n. The case $n = 1$ is trivial. Consider now the case $n = 2$.

By (i) and (ii) there is a simple arc Γ: $x = \varphi_1(t)$, $y = \varphi_2(t)$, $t \in I$ joining the points p_0 and p_1 and lying in the set A. Thus we have

$$f_1\varphi_1 = f_2\varphi_2 \quad \text{and} \quad \varphi_1, \varphi_2 \in \mathfrak{F}.$$

If $y \in W$, the set

$$f_1^{-1}(y) \times f_2^{-1}(y) \tag{7}$$

has a finite number of components, which are isolated points, rectangles or segments. More exactly, the set (7) contains a segment or a rectangle if and only if one of the sets $f^{-1}(y)$ or $f^{-1}(y)$ contains a segment. Since the number of such y's is at most enumerable, there is a finite or enumerable sequence $F_1, F_2, \ldots$ of disjoint segments and rectangles which contains all the components (of all the sets (7)) possessing more than one point.

Consequently, we can suppose additionally that

(c) the intersection of Γ with any of components of the sets (7) is either a segment, or a point, or empty.

If not, we can modify, by induction on n, the functions φ_1 and φ_2 in some disjoint subintervals of I in such a way that ΓF_n is either a segment, or a point, or empty for $n = 1, 2, \ldots$

Now we shall prove that the function

$$f = f_1\varphi_1 = f_2\varphi_2$$

belongs to $\mathfrak{F}_W$.

Suppose that $y_0 = f(x) = f_1(\varphi_1(x)) = f_2(\varphi_2(x))$ for $x_1 \leqslant x \leqslant x_2$ $(x_1 < x_2)$. If $f_1^{-1}(y_0)$ contains an interval, then $y_0 \in W$ by (a′). Suppose that $f_1^{-1}(y_0)$ contains no interval. Then $\varphi_1(x) = \text{const}$ for $x_1 \leqslant x \leqslant x_2$. Since Γ is a simple arc, the function φ_2 is either increasing or decreasing in the interval $\langle x_1, x_2 \rangle$. Consequently $f_2(t) = y_0 = \text{const}$ for t belonging to the interval $\varphi_2(\langle x_1, x_2 \rangle)$. Therefore, by (a′), $y_0 \in W$. Hence the function f satisfies the condition (a′).

Now let $y \in W$. We have

$$f^{-1}(y) = \mathop{E}_{x}\Big((\varphi_1(x), \varphi_2(x)) \in f_1^{-1}(y) \times f_2^{-1}(y)\Big).$$

By (c) the set $\Gamma \cdot (f^{-1}(y) \times f_2^{-1}(y))$ is the sum of a finite number of segments and points. Consequently the set $f^{-1}(y)$ is the sum of finite number of segments and points, *i. e.* f has the property (b).

Consequently $f \in \mathfrak{F}_W$, which completes the proof in the case $n = 2$.

Suppose now that Theorem II is true for an integer $n-1 > 1$. We shall prove it for the number n.

Let $f_1, f_2, \ldots, f_n \in \mathfrak{F}_W$. By the induction hypothesis there exist functions $\bar\varphi_1, \bar\varphi_2, \ldots, \bar\varphi_{n-1} \in \mathfrak{F}$ such that

$$f = f_1\bar\varphi_1 = f_2\bar\varphi_2 = \ldots = f_{n-1}\bar\varphi_{n-1} \in \mathfrak{F}_W.$$

Apply now Theorem II (the proved case of $n = 2$) to the functions $f, f_n \in \mathfrak{F}_W$. We infer that there are functions $\varphi, \varphi_n \in \mathfrak{F}$ such that

$$f\varphi = f_n\varphi_n \in \mathfrak{F}_W,$$

i. e.

$$f_1\bar\varphi_1\varphi = f_2\bar\varphi_2\varphi = \ldots = f_{n-1}\bar\varphi_{n-1}\varphi = f_n\varphi_n.$$

Setting $\varphi_i = \overline{\varphi}_i \varphi$ for $i=1,2,\dots,n-1$, we obtain (6), q. e. d.

Theorem I follows immediately from Theorem II where $W=I$. More generally, we infer that the condition (α) in Theorem I can be replaced by any of the following conditions:

(α_1) *$f_i^{-1}(y)$ has a finite number of components for every $y \in I$ and $i=1,2,\dots,n$;*

(α_2) *the functions f_i $(i=1,2,\dots,n)$ are of bounded variation and, for every $y \in I$, the set $f^{-1}(y)$ contains no interval;*

(α_3) *the functions f_i $(i=1,2,\dots,n)$ are of bounded variation; if $f_j(x) = y_0 = \mathrm{const}$ in an interval $x_1 \leqslant x \leqslant x_2$, then the sets $f_i^{-1}(y_0)$ have a finite number of components.*

The case (α_1) follows from Theorem II where $W=I$.

If f_i is of bounded variation, then the set

$$Y_i = \underset{y}{E}\left(\overline{\overline{f_i^{-1}(y)}} \geqslant \aleph_0\right)$$

has measure zero. To obtain the case (α_2) it suffices to set in Theorem II $W = I-(Y_1+Y_2+\dots+Y_n)$. To obtain the case (α_3) it suffices to set in Theorem II $W=I-(Y_1+Y_2+\dots Y_n)+Y$ where Y is the (at most enumerable) set of all numbers y such that, for an integer $i=1,2,\dots,n$, the set $f_i^{-1}(y)$ contains an interval.

Notice that, if $f_1, f_2 \in \mathfrak{F}_W$, the set A need not be connected or locally connected. In fact, let $x_n = 3/4 - 1/4n$. Let

$$f_1(0)=0, \qquad f_1(x_{2k-1}) = \frac{1}{2}+\frac{1}{2k}, \qquad f_1(x_{2k}) = \frac{1}{2} = f_1\left(\frac{3}{4}\right), \qquad f_1(1)=1,$$

and let f_1 be linear in all remaining intervals. Analogously let

$$f_2(0)=0, \qquad f_2(x_{2k-1}) = \frac{1}{2}-\frac{1}{2k}, \qquad f_2(x_{2k}) = \frac{1}{2} = f_2\left(\frac{3}{4}\right), \qquad f_2(1)=1,$$

and let f_2 be linear in all remaining intervals. All the points (x_{2k}, x_{2j}) are isolated points of A, and $f_1, f_2 \in \mathfrak{F}_W$ where $W = I-(1/2)$.

The proof of Theorem II is simpler if we restrict more the class of functions under consideration. In particular, the proof is very simple in the case where the interval I can be divided into subintervals in each of which the functions are linear.

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Reçu par la Rédaction le 15. 3. 1954

On uniformization of functions (II)

by

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I shall give another proof of Theorems I and II from the paper of R. Sikorski and K. Zarankiewicz, *On uniformization of functions (I)* [1].

Both proofs are based on a connectedness property of the set

$$\underset{(x_1,\dots,x_n)}{E}\big(f_1(x_1) = f_2(x_2) = \dots = f_n(x_n)\big)$$

(see Lemmas (i) and (i′)). Theorem III (see p. 349), which seems to be interesting in itself, generalizes this property to the case of mappings $f_1, f_2, \dots, f_n$ of the k-dimensional cube Q_k into itself, the mappings being the identity on the boundary of Q_k.

The second proof makes no use of the principle of induction. It consists in the direct application of the method, used in the first part for the case of two functions, to the general case of n functions. However, this kind of proof requires more advanced topological means.

The second proof is based on the following lemmas which are generalizations of (i) and (ii) respectively.

(i′) *If $f_1, f_2, \dots, f_n \in \mathfrak{F}$, then the set*

$$A = \underset{(x_1,\dots,x_n)}{E}\big(f_1(x_1) = f_2(x_2) = \dots = f_n(x_n)\big) \tag{8}$$

is connected between the points $p_0 = (0,0,\dots,0)$ and $p_1 = (1,1,\dots,1)$ of the n-dimensional Euclidean space.

(ii′) *If $f_1, f_2, \dots, f_n \in \mathfrak{F}_W$ (where W is a dense subset of I), then each component of the set A defined by (8) is locally connected.*

In fact, by (i′) the points p_0 and p_1 lie in a component A_0 of A. By (ii′) A_0 is locally connected. Therefore there exists an arc Γ

$$x_1 = \varphi_1(t), \quad x_2 = \varphi_2(t), \quad \dots, \quad x_n = \varphi_n(t), \qquad t \in I$$

[1] R. Sikorski and K. Zarankiewicz, *On the uniformization of functions (I)*, this volume, p. 339-344. See p. 339 and p. 342. The knowledge of this paper is here assumed.