This follows immediately from the definition of constructivity of a theory and from the deduction theorem for the functional calculus of Heyting.

Moreover

4.12. If \( \alpha \) is a constructive closed formula, then for every formula \( \beta \) of the form \( \beta = \exists \gamma \ldots \exists \gamma \) where \( \gamma \) contains no quantifiers and \( \exists \gamma \) is in either \( \gamma \) or \( \exists \gamma \), there exists a finite sequence \( \gamma_1, \ldots, \gamma_m \) of the formulas without quantifiers such that \( \alpha \rightarrow \beta \) is provable if and only if at least one of the formulas \( \alpha \rightarrow \gamma_1, \ldots, \alpha \rightarrow \gamma_m \) is provable.

This follows immediately from the deduction theorem for the functional calculus of Heyting and from 4.10.

References


Elementarily definable analysis

by

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The purpose of this paper is to give a strict mathematical shape to some ideas expressed by H. Weyl in "Das Kontinuum" [4]. Weyl proposes a restriction of the logical methods of analysis to the elementarily definable ones. A notion is elementarily definable if it is definable by means of the quantifiers bounding the integral variables only. A strict definition will be given later. It is very interesting to note how many theorems of the classical analysis can be obtained by means of elementarily methods. It is shown in this paper that the classical analysis of continuous functions can be reproduced in an elementary manner. The problem of how many theorems from the theory of non continuous real functions can be obtained in an elementary way remains open. Some counter examples are given in the sequel.

To begin with the problem arises how to define elementary definability. There are at least two answers:

1. A mathematical notion \( A \) is elementarily definable if it is definable by means of an elementary definition

\[
A(f,...,x,...) = \exists(...f,...,x,...,)
\]

2. A notion \( A \) is elementarily definable if there exists a finite set of elementary conditions such that \( A \) is the unique object which satisfies those conditions.

We shall call the first the narrower, the second the broader concept of elementary definability. In this paper we shall consider the narrower notion.

1. Elementary definability in the arithmetic of integers

We shall introduce the notion of elementary definability in the arithmetic of integers. Let \( I \) be the set of all integers (positive, negative and zero). Let \( N \) be the set of non negative integers (natural numbers). The variables \( x, y, z, p, g \) will stand for the integers, the variables \( a, k, l, m \) will represent natural numbers. The letters \( f, g, h \), will be used to denote the functions defined over the set \( I \) and assuming the integral values.
A functional $\Phi(f_1, \ldots, f_n)(x_1, \ldots, x_n)$ defined over the integral functions $f_1, \ldots, f_n$ and over the integers $x_1, \ldots, x_n$ and assuming integral values is said to be \textit{elementarily definable} if there exists a functional relation $R(f_1, \ldots, f_n, x_1, \ldots, x_n)$ such that $R \in \mathcal{D}$ and
\[
\Phi(f_1, \ldots, f_n)(x_1, \ldots, x_n) = (\mu z)[R(f_1, \ldots, f_n, x_1, \ldots, x_n)].
\]
The class of elementarily definable functions, as well as the class of elementarily definable functionals, and the class of elementarily definable relations will be denoted by the same symbol $\mathcal{D}$.

It is evident that the classes of elementarily definable functions and functionals are closed under the operations of substitution of a constant elementary function or a constant number, and of identification of the variables. The computable functions and functionals \footnote{For the definition of computable functionals see \cite{1}.} are elementarily definable.

\textbf{Theorem 1.1.} If $\Phi, f \in \mathcal{D}$ and
\[
R(f_1, \ldots, f_n, x_1, \ldots, x_n) = \Phi(f_1, \ldots, f_n)(x_1, \ldots, x_n) = x_1,
\]
\[
R_i(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) = x_1,
\]
then $R_i$ and $R_e$ are elementarily definable relations.

\textbf{Proof.} If $f \in \mathcal{D}$ then from the definition it follows that there exists a relation $R \in \mathcal{D}$ such that
\[
(f(x_1, \ldots, x_n)) = (\mu z)[R(x_1, \ldots, x_n)].
\]
The relation $R_i$ can be defined as follows
\[
R_i(x_1, \ldots, x_n) = [R(x_1, \ldots, x_n) \wedge \prod_{x \in \tau} (R(x, x_1, \ldots, x_n) \rightarrow z > x)]
\]
\[
\vee [x = 0 \wedge \prod_{x \in \tau} (R(x, x_1, \ldots, x_n) \rightarrow z < x) \wedge R(z, x_1, \ldots, x_n)].
\]
In the case of $\Phi$ the proof is similar.

\textbf{Theorem 1.2.} If $R_i \in \mathcal{D}$ is a relation of the form $R_i(g, h_1, \ldots, h_n, x_1, \ldots, x_m)$, where $g$ represents any function of one argument, $\Phi(f_1, \ldots, f_n)(x) = y$ is a functional of the class $\mathcal{D}$, and
\[
R_i(f_1, \ldots, f_n, h_1, \ldots, h_n, x_1, \ldots, x_m) = R_i(\Phi(f_1, \ldots, f_n), h_1, \ldots, h_n, x_1, \ldots, x_m),
\]
then $R \in \mathcal{D}$.

\textbf{Proof by induction.} If $R_i$ is an initial relation, from the Theorem 1.1 it follows that $R \in \mathcal{D}$.
Now let us suppose that

\footnote{The notion of functional will be explained later.}
From (1) it follows that the relation (4) belongs to \( \mathcal{D} \) and the superposition (3) is elementarily definable.

Now some remarks on the notation. If the functional \( \Phi \) has the form

\[
\Phi(f_1, \ldots, f_m)(x_1, \ldots, x_n) = y,
\]

then it can be considered as a function

\[
\Phi(f_1, \ldots, f_m)(x_1, \ldots, x_{n-1}, y) = y
\]

of one argument \( x_m \) with \( m-1 \) parameters. For example a function of two arguments \( g(x, y) \) can be considered as a sequence of functions:

\[
g_1(y) = g(x, y)
\]

A relation or a functional can be defined over the functions of two arguments. The symbols representing functions of two arguments will often be written in parentheses of the form: \( (\, ) \).

Theorem 1.3 can also be extended in such a manner that if \( R \in \mathcal{D} \) and \( g_1(x) = g_1(x, x) \) and

\[
S[g_1, f_1, \ldots, f_m](x_1, \ldots, x_m, z) = R[g_1, f_1, \ldots, f_m](x_1, \ldots, x_m)
\]

then \( S \in \mathcal{D} \).

There are three recursive functions \( Pa', Fr', Sc' \) which will often be used in the following:

\[
Pa'(x, y) = (x + y)^p + x, \quad Fr'(x) = x - \sqrt{x}^p, \quad Sc'(z) = Fr' - Fr'(z)
\]

\( Pa', Fr', Sc' \) are the pairing functions. They have the following important properties:

\[
Pr[Pa'(x, y)] = x, \quad Sc'[Pa'(x, y)] = y, \quad Pa'[Fr'(z), Sc'(z)] = z,
\]

for any \( x, y, z \in \mathbb{N} \).

By means of \( Pa', Fr', Sc' \) we can define a triplet of pairing functions for all integers:

\[
Pa(x, y) = \begin{cases} 2 \cdot Pa'(x, y) & \text{if } x, y > 0, \\ 2 \cdot Pa'(x, y) + 1 & \text{if } x > 0 \text{ and } y < 0, \\ -2 \cdot Pa'(x, y) & \text{if } x < 0 \text{ and } y > 0, \\ -2 \cdot Pa'(x, y) + 1 & \text{if } x < 0 \text{ and } y < 0. \end{cases}
\]

\[
Fr(z) = \begin{cases} Fr'(z) & \text{if } z > 0, \\ -Fr'(z) & \text{if } z < 0. \end{cases}
\]

\[
Sc(z) = \begin{cases} Sc'(z) & \text{if } z \text{ is even}, \\ -Sc'(z) & \text{if } z \text{ is odd}. \end{cases}
\]
The pairing functions \( Pa, F, Se \) have the pairing properties
\[
F(Pa(x, y)) = x, \quad Se(Pa(x, y)) = y, \quad Pa(F(z), Se(z)) = z,
\]
for any \( x, y, z \in I \).

Other functions which will often be used in the sequel are
\[
[\alpha]_0 = \mu \cdot \chi(x \cdot 1 + x + 1 > 0),
\]
\[
\text{sgn}(\alpha) = \begin{cases} -1 & \text{when } a < 0, \\ 0 & \text{when } a = 0, \\ 1 & \text{when } a > 0. \\ \end{cases}
\]
\[
|\alpha| = a \cdot \text{sgn}(\alpha).
\]

The division \( a \div b \) is regarded as defined for each \( a, b \in I \). If \( b = 0 \) then \( a \div b = 0 \).

**Theorem 1.4.** If \( x, x', y, y' \in x' \in D \) and \( \Phi \) is defined by means of the induction-scheme
\[
\Phi(x, y) = \chi(x + y + 1 > 0) \\
\Phi(x_1, y_1, x_2, y_2) = \chi(x_1 + x_2 > 0) \cdot \chi(y_1 + y_2 > 0) \\
\Phi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = \chi(x_1 + x_2 + x_3 > 0) \cdot \chi(y_1 + y_2 + y_3 > 0)
\]

then \( \Phi \in D \).

**Proof.** The relation
\[
R(x_1, y_1, x_2, y_2) = \chi(x_1 + x_2 > 0) \\
\Phi(x_1, y_1, x_2, y_2) = \chi(x_1 + x_2 > 0) \cdot \chi(y_1 + y_2 > 0)
\]

can be elementarily defined. To simplify the proof suppose that \( \Phi \)
defined only for \( a > 0 \) and assumes the values \( y > 0 \).

\[
E(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = \sum_{x < y} \chi(x_1 + x_2 + x_3 > 0) \cdot \chi(y_1 + y_2 + y_3 > 0)
\]

where \( \chi(x, y) = \mu \chi(x \cdot 1 + x + 1) \), \( p_a \) the \( a \)-th prime number.

Hence
\[
\Phi(x_1, y_1, x_2, y_2) = \chi(x_1 + x_2 > 0) \\
\Phi(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = \chi(x_1 + x_2 + x_3 > 0) \cdot \chi(y_1 + y_2 + y_3 > 0)
\]

A similar scheme of induction for the functions does not exceed class \( D \) of functions.

On the other hand, it is easy to prove that the general scheme of induction leads out of class \( D \).

---

2. The elementarily definable concepts of analysis

It is well known that a real number \( a \) can be represented in the arithmetic of integers:

A. by an integral function \( f \) such that
\[
\left| a - \frac{f(n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for each } n \in N;
\]

B. by an integral function \( f \) such that
\[
a = \sum_{n=0}^{\infty} \frac{f(n)}{2^n} \quad \text{and} \quad 0 < f(n) < 2 \quad \text{for } n > 0;
\]

C. by an integral function \( f \) such that
\[
a < \frac{p}{q+1} = f(Pa(p, q)) = 0 \quad \text{for any } p, q \in I;
\]

D. by an integral function \( f \) such that
\[
|a - \frac{f(n)}{Se(f(n)) + 1}| < \frac{1}{n+1} \quad \text{for each } n \in N;
\]

E. by an integral function \( f \) such that
\[
|a - \frac{f(n)}{Se(f(n)) + 1}| < \frac{1}{m+1} \quad \text{for } n > Se(f(m)) \text{ and } m \in N.
\]

Let us denote the relations expressed in A-E by \( A(a,f), B(a,f), C(a,f), D(a,f), E(a,f) \).

**Theorem 2.1.** If \( P \) and \( Q \) are some of the relations \( A, B, C, D, E \), then there exists a functional \( \Theta_{Q} \in D \) such that if \( P(a,f) \) then \( Q(a,\Theta_{Q}(f)) \).

**Proof.** Let
\[
\Phi(f(a)) = (\mu a) \sum_{k=1}^{a} \frac{1}{k+1} \cdot \frac{1}{k+1} - \frac{a+1}{a+1},
\]

\[
\Theta_{Q}(p) = \Phi(f(p)),
\]

\[
\Theta_{Q}(f(a+1)) = \Phi(f(a+1)) - 2 \cdot \Phi(f(a)),
\]

\[
\Theta_{Q}(f(a)) = \begin{cases} 0 & \text{when } \sum_{k=1}^{a} \frac{1}{k+1} \cdot \frac{1}{k+1} > \frac{a}{a+1}, \\
1 & \text{in the other case},
\end{cases}
\]
A real number $a$ is elementarily definable if there exists a function $f \in \mathcal{D}$ such that

$$\frac{a - f(n)}{n+1} < \frac{1}{n+1} \quad \text{for each } n \in \mathbb{N}.$$  

A real sequence $(a_k)$ is elementarily definable if there exists a function $f \in \mathcal{D}$ such that

$$a_k - f(n,k) < \frac{1}{n+1} \quad \text{for each } n, k \in \mathbb{N}.$$  

A real function $\varphi$ of one argument is elementarily definable over the set $\Lambda$ of real numbers if there exists a functional $\Phi \in \mathcal{D}$ such that for any $a \in A$ and $f$

$$\text{if } \left| a - f(n) \right| < \frac{1}{n+1} \text{ for each } n \in \mathbb{N},$$

then

$$\left| \varphi(a) - \Phi(f)(n) \right| < \frac{1}{n+1} \quad \text{for each } n \in \mathbb{N}.$$  

Similarly, we can determine when a function $\varphi$ of two or more arguments is elementarily definable: it is when there exists a functional $\Phi \in \mathcal{D}$ such that for all $a_1, \ldots, a_h$

$$\text{if } A(a_1, f), A(a_2, f), \ldots, A(a_h, f),$$

then

$$A(\varphi(a_1), \ldots, \varphi(a_h)) \equiv \Lambda(\varphi(a_1), \ldots, \varphi(a_h)).$$

The class of elementarily definable real numbers as well as the class of elementarily definable real sequences and the class of elementarily definable real functions will be denoted by the same symbol $\mathcal{D}$. The elementarily definable numbers, sequences, and functions will be the object of our investigations.

From theorem 2.3 it follows

**Theorem 2.4.** $a \in \mathcal{D} \iff f(a) \in \mathcal{D}$.  

**Theorem 2.5.** If the function $\varphi$ is determined over the set $X$, then

$$\varphi \in \mathcal{D} \iff \sum_{\varphi \in \mathcal{D}, \varphi \in X} \prod_{a \in X} \Lambda(\varphi(a), \Phi(f(a)).$$

From theorem 2.1 it follows at once

**Theorem 2.6.** A real number $a \in \mathcal{D}$ if and only if there exists an integral function $f \in \mathcal{D}$ such that $\Lambda(a, f)$, or $B(a, f)$, or $C(a, f)$, or $D(a, f)$, or $E(a, f)$. 

Now we introduce the concepts of elementarily definable analysis.
A real sequence \((a_n) \in \mathcal{D}\) if and only if there exists an integral function \(f \in \mathcal{D}\) of two arguments such that with \(f(k,n) = (k,n)\) it is true that for all \(k \in \mathbb{N}\) \(A(a_n, f_k)\) or for all \(k \in \mathbb{N}\) \(B(a_n, f_k)\), for all \(k \in C(a_n, f_k)\), or for all \(k \in D(a_n, f_k)\), or for all \(k \in E(a_n, f_k)\).

A real function \(\varphi \in \mathcal{D}\) if and only if there exists an integral functional \(\Phi \in \mathcal{D}\) such that \(A(f(k,n), \Phi(x))\) for all \(x \in X\) and each integral function \(f\) provided that \(X\) and \(\Phi\) are some of the relations \(A, B, C, D, E\) and \(X\) is the set of arguments of \(\varphi\).

From theorem 2.2 it follows that the functions mentioned in theorem 2.2 are elementarily definable, and

**Theorem 2.7.** The set of elementarily definable real numbers is a field of numbers.

The set of possible elementary definitions is enumerable. Hence the set of elementarily definable numbers as well as the set of elementarily definable sequences and the set of elementarily definable real functions are enumerable. Computable numbers and sequences are elementarily definable.

Polynomials with elementarily definable coefficients are elementarily definable functions accordingly to Theorem 2.2. Continuous elementarily definable function assumes elementarily definable value e.g. 0 at an elementarily definable point (as shall be shown in the following). Hence the roots of an elementarily definable polynomial are elementarily definable. This means that the field of elementarily definable numbers is algebraically closed in the field of real numbers. Similarly each analytical function with an elementarily definable sequence of coefficients is an elementarily definable real function. Trigonometric functions and all so called elementary functions of analysis are elementarily definable because they can be presented as elementarily definable sequences.

**Theorem 2.8.** For any two real functions \(\varphi, \psi\) if \(\varphi, \psi \in \mathcal{D}\) then the superposition \(\varphi \psi \in \mathcal{D}\); if \((a_n) \in \mathcal{D}\) then \((\varphi(a_n)) \in \mathcal{D}\).

**Proof.** If \(\varphi\) is defined over the set \(X\) then over the set \(Y\), and for \(a \in X\) \(\varphi(a) \in Y\), and for each \(a, b\)

\[
\begin{align*}
\text{if } & A(a, f), \text{ then } A(\varphi(a), \Phi(f)), \\
\text{if } & B(b, g), \text{ then } A(\psi(b), \Psi(g)),
\end{align*}
\]

then for each \(a \in X\) if \(A(a, f)\), then \(A(\varphi(a), \Phi(f))\). Hence the function \(\varphi \psi\) is elementarily defined by means of the functional \(\Phi(f)(x) = \Psi(\Phi(f)(x))\) which belongs to \(\mathcal{D}\) according to 1.3.

If \((a_n) \in \mathcal{D}\) and \(\Lambda(\omega, f_0)\), then by (1) \(A(\varphi(a_n), \Phi(f_0))\). The sequence \((\varphi(a_n))\) is elementarily defined by means of the elementarily defined integral function \(\Phi(f_0)(x)\).

**Example.** \((a_n + b_0), (a_n - b_0), (\{a_n\}, \{\{a_n\}\})\) are elementarily definable sequences, if \((a_n), (b_n) \in \mathcal{D}\) (Theorem 2.2).

**Theorem 2.9.** If \(a \in I\) for each \(k \in \mathbb{N}\), then \((a_n) \in \mathcal{D}\) if and only if there exists an integral function \(g \in \mathcal{D}\) such that \(a_k = g(k)\) for \(k \in \mathbb{N}\). If \(\varphi\) is defined over the set \(I\), and \(g(x) \in I\) for \(x \in I\), then \(\varphi \in \mathcal{D}\) if and only if there exists an integral function \(g \in \mathcal{D}\) such that \(g(x) = g(x)\) for \(x \in I\).

**Proof.** If \(a_n \in I\)

\[
(1) \quad \left| a_n - \frac{f(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for each } n \in \mathbb{N}
\]

then \(a_k = f(k,0) = g(k)\). Conversely, if \(a_k = g(k)\) then the function \(f(k,n)\) satisfies the condition (1).

If \(\varphi \in \mathcal{D}\), then there exists a functional \(\Phi \in \mathcal{D}\) such that

\[
(2) \quad A(a, f), \quad (B(b, g), \Psi(g)).
\]

Let \(f(0) = f(x, n) = x(n+1)\). It is true that \(\Lambda(\varphi, f_0)\) for \(x \in I\). If \(\varphi\) is defined over the set \(I\), and \(\varphi(x) \in I\) for \(x \in I\), then from (2) it follows that

\[
(3) \quad \left| g(x) - \frac{\Phi(f_0(x))}{n+1} \right| < \frac{1}{n+1}.
\]

Hence \(g(x) = \Phi(f_0(0)) = g(x)\) for \(x \in \mathcal{D}\) according to Theorem 1.2. Conversely, if \(g(x) = g(x)\), then the functional \(\Phi(f_0(n)) = g(f(0)) \cdot (n+1)\) satisfies the condition (2) for \(a \in I\).

**Theorem 2.10.** There are the relations \(R_1, R_2, R_3, R_4 \in \mathcal{D}\) such that if \(A(a, f), A(b, g)\) and \(\Lambda(a_n, h_n)\) for each \(n \in \mathbb{N}\) then

\[
R_1(f, g) = a \cdot b, \quad R_2(h, g, n) = a_n \cdot b_n, \quad R_3(h, n, x, y) = a_n < x, \quad R_4(h, g) = [a_n] \cdot b.
\]

**Proof.**

\[
(1) \quad a < b = \sum_{k \in N} \sum_{n \in N} \frac{h_k(n)}{n+1} > \frac{1}{n+1},
\]

\[
(2) \quad a_n < b = \sum_{k \in N} \sum_{n \in N} \frac{h_k(n)}{n+1} > n+1,
\]

\[
(3) \quad a_n < \frac{x}{y+1} = \sum_{k \in N} \sum_{n \in N} \frac{h_k(n)}{n+1} \cdot \frac{1}{n+1} < \frac{x}{y+1}.
\]
In the following we shall consider the functionals defined over the functions of two arguments. To avoid the confusion of the types of function variables we shall write the variables standing for functions of two arguments in parentheses \([f,g]\). Hence for example if \(f(n,y) = f_y(n)\), then we can write \(\langle f, g \rangle\) but \([f]f\) (cf. the remarks on notations on p. 315).

From Theorem 2.2 it follows that there exists a functional \(\Phi \in \mathcal{D}\) such that \(\Phi(\langle f, g \rangle)(n) = \Phi(\langle \Phi(f), g \rangle)(n)\) and

(4) \(\Lambda(\langle a_k, b \rangle, \Phi(h, g))\).

Hence from Theorem 1.3 it follows that there exists a functional \(\Psi \in \mathcal{D}\) such that

(5) \(\Phi(\langle h, g \rangle) = \Psi(h) \Phi(g)(n,x)\).

From (3), (4), and (5) we find that the relation

\[\left| a_n - b \right| < \frac{x}{y+1} = \mathbb{E}_n(\Psi) \mathbb{E}_n(g)(n,x,y)\]

is elementarily definable. Hence \(R_n \in \mathcal{D}\) because

\(\{a_k\} \rightarrow b = \prod_{n \in N} \prod_{k \in N} |a_k - b| < \frac{1}{k+1}\).

**Theorem 2.11.** There exists a functional \(\Phi \in \mathcal{D}\) such that \(\Lambda(a_k, f_k)\) for each \(k \in N\) and \(\{a_k\} \rightarrow a\), then \(\Lambda(a, \Phi(f))\).

**Proof.**

\(\Phi(f) = (\mu) \sum_{x \in X} \prod_{x \in X} |a_k - b| < \frac{2}{x+1}\).

From Theorems 2.10 and 2.2 it follows that the relation

\(R(f, k, n, x) = \left| a_k - b \right| < \frac{2}{x+1}\)

is elementarily definable. Hence \(\Phi \in \mathcal{D}\).

**Theorem 2.12.** If \(\{a_k\} \in \mathcal{D}\) and \(\{a_k\} \rightarrow a\), then \(a \in \mathcal{D}\).

**Proof.** From Theorem 2.11 by means of the definitions.

**Theorem 2.13.** There exists a functional \(\Omega \in \mathcal{D}\) such that \(\Lambda(a_k, f_k)\) for each \(n \in N\) and the sequence \(\{a_k\}\) is limited, then the sub-sequence \(a_{\Omega(1)}\) is convergent.

**Proof.** The functional \(\Omega\) is defined as follows:

\(\Phi(\langle f \rangle)(k) = (\mu) \sum_{x \in X} \prod_{x \in X} |a_k - b| < \frac{2}{x+1}\)

\(\Omega(f)(k) = (\mu) \left[ \sum_{x \in X} \prod_{x \in X} |a_k - b| < \frac{2}{x+1}\right]\).

The limit of the sub-sequence \(a_{\Omega(1)}\) is the upper limit of the set of points of accumulation of the numbers of the sequence \(\{a_k\}\). From theorem 2.10 it follows that the functional \(\Phi \in \mathcal{D}\) because the relations

\[a_n < \frac{x}{k+1} \quad \text{and} \quad a_n > \frac{x}{k+1}\]

are elementarily definable. Hence likewise from Theorem 2.10 we find that the relation

\[a_n < \frac{\Phi(\langle f \rangle)(k)}{k+1}\]

is elementarily definable, also \(\Omega \in \mathcal{D}\).

**Theorem 2.14.** If \(\{a_n\} \in \mathcal{D}\) and the sequence \(\{a_n\}\) is limited, there exists a sub-sequence \(\{a_{\Omega(1)}\} \in \mathcal{D}\) such that \(\lim (a_{\Omega(1)}) = \lim (a_n)\).

**Proof.** From Theorems 2.12 and 2.13 by means of the definitions.

**Theorem 2.15.** There exists a functional \(\Phi \in \mathcal{D}\) such that if \(\{a_k\}\) is limited and \(a = \sup(a_k)\) and \(\Lambda(a_k, f_k)\), then \(\Phi(a_k)\).

**Proof.**

\[\Phi(a_k) = (\mu) \left[ \sum_{x \in X} \prod_{x \in X} |a_k - b| < \frac{2}{x+1}\right]\]

**Theorem 2.16.** There exists a functional \(\Psi \in \mathcal{D}\) such that if \(\Lambda(a_k, f_k)\), then the sequence \(\{a_{\Omega(1)}\}\) is monotonically.

**Proof.** We can distinguish the following cases:

1. The sequence \(\{a_k\}\) is unlimited when
   - \(\prod_{x \in X} a_k < x\)
   - \(\prod_{x \in X} a_k > x\)

2. The sequence \(\{a_k\}\) is limited when
   - \(\sum_{x \in X} a_k < y\)
   - \(\sum_{x \in X} a_k > y\)

In this case we shall consider the subsequence \(b_n = a_{\Omega(1)}\). The sequence \(\{b_n\}\) contains an increasing subsequence if

\[\sum_{x \in X} a_k < b_n < \lim b_n\]

the sequence \(\{b_n\}\) contains a decreasing subsequence if

\[\sum_{x \in X} a_k > b_n > \lim b_n\]

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the sequence \((b_n)\) contains a constant subsequence if
\[(c')\]

Now we define the functional \(\Psi\) in the following manner:
\[
\Psi[f](0) =
\begin{cases}
0 & \text{if (a) or (b)}, \\
\Omega[f][\{m, b, a < \lim b_n\}] & \text{if (c'),} \\
\Omega[f][\{m, b, b > \lim b_n\}] & \text{if (c'''),} \\
\Omega[f][\{m, b, b = \lim b_n\}] & \text{if (c')}.
\end{cases}
\]

From Theorem 1.4 it follows that \(\Psi \in \mathcal{D}\). It is evident that the sequence \(a_{\psi f}(0)\) is decreasing in cases (a) and (c'), increasing in cases (b) and (c'), and constant in case (c'').

3. Elementarily definable sets of real numbers

We introduce the concept of elementarily definable sets as well as the concept of elementarily definable sequences of sets.

The class of the elementarily definable sets of real numbers will be denoted by the same symbol \(\mathcal{D}\) as the class of definable real functions.

(1) \(Z \in \mathcal{D}\) iff there exists a relation \(R \in \mathcal{D}\) such that for all \(a\) and \(f\) if \(A(a,f)\) then \(a \in \{Z(a,f)\}\) if and only if \(R(f,a)\).

(In this section the indices \(z\) in the symbols \(Z\) of sequences of sets are regarded as running throughout the set \(\mathcal{I}\) of all integers.)

THEOREM 3.1. The condition \(Z \in \mathcal{D}\) is equivalent to the following:

(2) There exists \(S \in \mathcal{D}\) such that, for all \(a\), \(a \in \{Z(a,Z)\}\) if and only if \(B(f)[S(f,a)]\).

Proof. \(Z \in \mathcal{D} \Rightarrow (2)\). If \(Z \in \mathcal{D}\), then there exists a relation \(R \in \mathcal{D}\) which satisfies the condition (1). We set \(S=R\). It is evident that \(A(a,f)\).

Hence the condition (2) is fulfilled.

(2) \(\Rightarrow Z \in \mathcal{D}\). Let \(R(f)=S(f)[f]\). Hence by Theorem 2.3 the condition (1) is fulfilled.

THEOREM 3.2. The sets and sequences \(\mathcal{D}\) constitute an enumerable additive Boolean algebra.

Proof. If \(Z_1, Z_2\) and \(R_1, R_2\) satisfy the condition (1), then the sets
\[
-Z_1 \cup Z_2
\]
and \(Z_1 \cup Z_2\) are determined by the relation
\[
\sim (R_1(f), R_2(f) \lor R_2(f)).
\]

If \(Z \in \mathcal{D}\), then the relation \(S(f)=\sum_{x \in \omega} R(f,x)\) determines the set \(\sum_{x \in \omega} Z_x\).

THEOREM 3.3. The sets \(\mathcal{D}\) are Borelian sets of finite degree.

Proof. From 3.1 and from the definition of elementarily definable relations it follows that the class \(\mathcal{D}\) of sets and sequences of sets is the smallest class that contains the following initial sequences:

(a) \(Z_{x_1-1} = E_x^0[f_0(x_1) - x_1]\),
(b) \(Z_{x_1-1} = E_x^0[x_1 - x_1 + 1]\),
(c) \(Z_{x_1-1} = E_x^0[x_1 = x_1] \land \forall x_1 \neq 0\),
(d) \(Z_{x_1-1} = E_x^0[x_1 > 0]\),

(where \(1 \leq i, j, k \leq n\)), and is closed under the following operations on sequences:

A. The operation of addition of two sequences, which leads from two sequences \(\{Z_{x_1-1}\}\) and \(\{Z_{x_2-1}\}\) to the sequence
\[
Z_{x_1-1} = Z_{x_1-1} + Z_{x_2-1}.
\]

B. The operation of complementation, which leads from a sequence \(\{Z_{x_1-1}\}\) to the sequence \(Z_{x_1-1} = -Z_{x_1-1}\).

C. The operation of enumerable summation, which leads from a sequence \(\{Z_{x_1-1}\}\) to the set (if \(k=1\)) or to the sequence
\[
Z_{x_1-1} = \sum_{x_1=0}^k Z_{x_1-1}.
\]

The sequences of type (a) contain the sets which are segments \([x_1(x_1+1), (x_1+1)(x_2+1)]\). The sequences of type (b)-(d) contain sets which are: The whole set of real numbers if the indices satisfy the arithmetic condition, or the empty set if they do not. Also the initial sets are Borelian of finite degree. Hence each set \(Z \in \mathcal{D}\) is Borelian of finite degree.

4) It is easy to show that for each positive integer \(n\) there exists an elementarily definable set of \(n\)th degree, which is not of the degree \(n-1\). See for example [19], p. 278.
THEOREM 3.5. The functions $D$ are measurable (B) of finite degree.

Proof. Let $\Phi$ be such a functional that $\Phi \in D$ and $\phi^{(n)} = \Phi(f_n)$. Let $F$ be a closed set. There exists an increasing sequence $F_1 \supset F_2 \supset F_3 \ldots$ of closed sets such that $F = \bigcap F_i$ and each set $F_i$ is the sum of recursive sequence of disjoint closed segments with rational endpoints. Each set $F_i$ is also computable. Let $R_i$ be such a computable relation that $b \in F_i = R_i \langle \phi^{(n)} \rangle$. Hence

$$a \in \Phi^{-1}(F_i) = \Phi(a) \in F_i = R_i \langle \phi^{(n)} \rangle = R_i \langle \Phi(f_n) \rangle.$$ 

It is evident that $\Phi^{-1}(F_1) = \bigcap \Phi^{-1}(F_i)$. Putting $a \in A_1 = R_i \langle \Phi(f_n) \rangle$ we obtain that there exists such a number $u \in N$ that each set $A_i \subset A_i = \bigcap \Phi^{-1}(F_i)$. Hence the product $\bigcap A_i = \Phi^{-1}(F)$ belongs to the Borelian multiplicative class of $n+1$ degree. This means that $\Phi$ is measurable (B) of $n+1$ degree. The number $u$ can be evaluated by the number $k$ of quantifiers contained in the definition of the functional $\Phi$. Namely $n < k$. It is evident because the relations $R_i$ are computable.

Let $\eta_1, \eta_2$ be two recursive functions such that $\delta_\eta_1(n)/\delta_\eta_2(n)$ is the sequence of rational numbers. We set

$$W_{\eta_1}(x) = W_{\eta_2}(x) = f^{(n)}(x) = (\eta_{\eta_1}(x), \eta_{\eta_2}(x)).$$

The function $W$ is also recursive. It is evident that $W_{\eta_1}(x) = W_{\eta_2}(x)$.

THEOREM 3.4. $A(n, W_{\eta_1}(x), A(x, y), W_{\eta_2}(x), \phi^{(n)}) = W_{\eta_1}(x, y)$.

THEOREM 3.5. If $X$ is a segment and $a, b$ are the endpoints of $X$, $a < b$, then $X \in D$ if and only if $a, b \in D$.

Proof. From Theorem 2.10 it follows that there exists a relation $R(a, b) \in D$ such that $e \in R(a, b) \iff e(a, b) \in D$.

Conversely, if $X \in D$, then there exists a relation $R(a, b) \in D$ such that $e \in R(a, b) \iff e(a, b) \in D$. Let

$$F(a) = (\mu x \in [\mathcal{R}(W_{\eta_1}(a, x))]) \quad \text{and} \quad G(a) = (\mu x \in [\mathcal{R}(W_{\eta_2}(a, x))].$$

Hence $a = \lim F(a)/(n+1)$, $b = \lim G(a)/(n+1)$ and $a, b \in D$ by means of Theorem 2.12.

THEOREM 3.6. There exists a number $a \in D$ such that the set $(a)$ belongs to $D$.

Proof. The universal function $G(a) = \phi^{(n)}(F(a, x))$ for the elementally definable functions is the unique function $g$ which satisfies the following condition $U(g)$:

$$U(g) = \bigcap a \in [\mathcal{R}(g(F(a, x))) = x + 1$$

$$\land \bigcap a \in [\mathcal{R}(g(F(a, x))) = \mathcal{R}(F(a, x)) = F(a, x) - S(a, x)$$

$$\land \bigcap a \in [\mathcal{R}(g(F(a, x))) = \mathcal{R}(F(a, x)) = F(a, x) - S(a, x)$$

$$\land \bigcap a \in [\mathcal{R}(g(F(a, x))) = \mathcal{R}(F(a, x)) = F(a, x) - S(a, x)$$

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$$\land \bigcap a \in [\mathcal{R}(g(F(a, x))) = \mathcal{R}(F(a, x)) = F(a, x) - S(a, x)$$

From the definition of the class $D$ of relations it follows that the class $D$ of functions is the smallest class of functions containing the initial functions $a, b, +, x \in y, [x^y]$, and closed under the operations of substitution and under the operation of minimum. Hence by Theorem 3.1 of Grzegorczyk [2] it follows that the class $D$ of the functions of one argument belonging to the class $D$ is the smallest class containing the initial functions $a, b, +, x \in y, [x^y]$, and closed under the following operations:

1. of superposition,
2. if $f$ belongs to $D_1$, then $P(F(a, x))$ belongs to $D_1$,
3. if $f \in D_1$, then $P(F(a, x))$ belongs to $D_1$,
4. if $f \in D_1$, then $P(F(a, x))$ belongs to $D_1$. 

(0)
The function \( g \) satisfying \( U(g) \) is well-defined for all numbers \( \varepsilon = F(n, z) \). For \( a < 0 \) \( \Theta_a(x) = g(F(n, z)) \) is equal to one of the initial functions (0). For \( n > 0 \) the function \( \Theta_a \) is obtained from the preceding functions by means of one of the four operations. Hence the function \( \Theta_0(x) = g(F(n, z)) \) is universal for all functions of the class \( D \).

Now we shall define two functionals \( \Theta_0, \Theta_1 \in D \) which establish a one-to-one correspondence between the set of integral functions of one argument and the set of natural functions of one argument assuming only two values, 0 and 1. For this purpose we shall define 5 functionals:

\[
\begin{align*}
\Theta_0(f)(n) &= \begin{cases} 
0 & \text{if } n \text{ is even}, \\
1 & \text{if } n \text{ is odd}.
\end{cases} \\
\Theta_1(f)(n) &= \begin{cases} 
0 & \text{if } f(n) = 0, \\
1 & \text{if } f(n) = 1.
\end{cases}
\end{align*}
\]

Now we define the set \( Z : a \in Z = U(\langle \Theta_0(f) \rangle) \).

The set \( Z \) has the following properties:

1. There exists exactly one real number \( a \), such that \( a \in Z \). From the definition of the set \( U \) it follows that there exists exactly one function \( f \) such that \( U(g) \).

Hence the universal function \( \Phi_0(\Theta_0\langle \Theta_1(g) \rangle) \) such that

\[
U(\langle \Theta_0(\Phi_0(\Theta_1(g))) \rangle) = g,
\]

because Theorems 2.1 and 2.3 involve

\[
\Theta_0(\Phi_0(\Theta_1(g))) = g.
\]

2. Number \( a \), such that \( Z = \{a\} \) is not elementarily definable. Number \( a \) is the unique number for which \( F' = \Phi_0(\Theta_1(g)) \). When \( a \in D \), then \( F' \in D \), and \( a \in D \), but this is impossible because the universal function for a class of functions cannot belong to that class.

3. 0 < \( a < 1 \). From the definition of the functional \( \Theta_0 \) it follows that

\[
\Theta_0(f)(n) = \begin{cases} 
0 & \text{if } f(n) = 0, \\
1 & \text{if } f(n) = 1.
\end{cases}
\]

4. From Theorem 2.3 it follows that the open set \( V = (0, 1) \) is elementarily definable. It contains two components \( (0, a) \) and \( (a, 1) \), but these components are not elementarily definable, according to Theorem 3.5.

Hence this proof implies the following

**Theorem 3.5.** There exists an open set \( V \in D \) such that the components of \( V \) are not elementarily definable sets.

**Theorem 3.8.** There exists a non-enumerable set \( Z' \in D \) such that

\[
a \in Z' = U(\langle \Theta_0(\Phi_0(\Theta_1(g))) \rangle).
\]

Proof. It is possible to define in an elementary manner a class \( U' \) of all universal functions for the class \( D \) of functions. The class \( D \) is enumerable. Each universal function for the class \( D \) represents a permutation (perhaps with repetitions) of the elements of the class \( D \). The number of the permutations of an enumerable set is \( 2^\omega \), hence \( U = 2^\omega \). Also the class \( Z' \) of numbers such that

\[
a \in Z' = U(\langle \Theta_0(\Phi_0(\Theta_1(g))) \rangle)
\]

contains only the numbers which do not belong to \( D \), and \( U = 2^\omega \), and \( Z'(0, 1) \).
According to the definition of the class \( U \) of the preceding Theorem the class \( U' \) can be defined as follows:

\[
U'(g) = \{ \prod_{n \in \mathbb{N}} g(P(0,x)) = x+1 \\
\land \prod_{n \in \mathbb{N}} g(P(1,x)) = |x| \land \prod_{n \in \mathbb{N}} g(P(2,x)) = F(x) - S(x) \\
\land \prod_{n \in \mathbb{N}} g(P(3,x)) = [F(x)g(n)] \land \prod_{n \in \mathbb{N}} g(P(4,x)) = F(x) \\
\land \prod_{n \in \mathbb{N}} g(P(5,x)) = S(x) \land \prod_{n \in \mathbb{N}} g(P(6,x)) = P(x, x) \\
\land \prod_{n \in \mathbb{N}} g(P(7,x)) = P(0,x) \land \prod_{n \in \mathbb{N}} g(P(8,x)) = P(x, 0) \\
\land \prod_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}} g(P(v,x)) = g(P(k, g(P(1,x)))) \\
\land \prod_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}} g(P(w,x)) = P(F(x), g(P(k, S(x)))) \\
\land \prod_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}} g(P(u,x)) = P(g(P(k, F(x))), S(x)) \\
\land \prod_{n \in \mathbb{N}} \sum_{x \in \mathbb{Z}} g(P(a,x)) = (\mu f) \left( g(P(k, P(x,y))) = 0 \right) \\
\land \prod_{n \in \mathbb{N}} \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = x+1 \right) \land \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = |x| \right)
\land \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = F(x) - S(x) \right) \lor \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = [F(x)g(n)] \right) \\
\land \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = F(x) \right) \lor \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = S(x) \right) \\
\land \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = P(x,x) \right) \lor \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = P(x, 0) \right) \\
\land \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = P(x,0) \right) \lor \left( \prod_{x \in \mathbb{Z}} g(P(n,x)) = g(P(k, g(P(1,x)))) \right) \\
\land \prod_{x \in \mathbb{Z}} g(P(n,x)) = P(F(x), g(P(k, S(x)))) \\
\land \prod_{x \in \mathbb{Z}} g(P(n,x)) = P(g(P(k, F(x))), S(x)) \\
\land \prod_{x \in \mathbb{Z}} g(P(n,x)) = (\mu f) \left( g(P(k, P(x,y))) = 0 \right)
\right).
\]

There remain several difficult problems concerning elementarily definable sets:

1. Is the closure an elementarily definable set, elementarily definable?

2. Is the class of all elementarily definable real numbers, elementarily definable?

3. Is the measure of an elementarily definable set, an elementarily definable number?

4. The analysis of continuous functions

A continuous function \( f \) defined over a segment \([a, b]\) is elementarily defined by the sequence of its values \( \varphi(r_n) \) at the rational points \( r_n \). This remark was the basis for the not quite strict proofs of H. Weyl stating that the theory of continuous functions can be obtained in an elementary manner. In this section we shall give a strict shape to these ideas.

**Theorem 4.1.** There exists a functional \( \Phi \in \mathcal{D} \) such that if \( f \) is a continuous function defined over the segment \([a, b]\) and \( \Delta \delta \varphi(r_n) \) for \( r_n \in [a, b] \) then \( \Phi \in C(f_3, f_3', f_3''') \) for any \( c \in (a, b) \).

**Proof.** From Theorem 2.10 it follows that the functionals \( \Phi_1, \Phi_3 \) are elementarily definable.

\[
\Phi_1(\langle f_3, f_3', f_3'' \rangle) = (\mu f) \left( \sum_{n=0}^{\infty} \prod_{x \in \mathbb{Z}} g(P(n,x)) \right) = x+1
\]

\[
\Phi_3(\langle f_3, f_3', f_3'' \rangle) = (\mu f) \left( \sum_{n=0}^{\infty} \prod_{x \in \mathbb{Z}} g(P(n,x)) \right) = x+1
\]

From these definitions it follows that setting \( \Phi = \Phi_1, \Phi_3 \) we have

\[
\int_{a}^{b} g(r) \, dr = \left( \sum_{n=0}^{\infty} \prod_{x \in \mathbb{Z}} g(P(n,x)) \right) = x+1
\]

From the continuity it follows that

\[
\varphi(c) = \lim_{r \to c} \varphi(r) = \lim_{r \to c} \int_{a}^{b} g(r) \, dr
\]

from the assumption of the theorem it follows that

\[
\int_{a}^{b} g(r) \, dr = \left( \sum_{n=0}^{\infty} \prod_{x \in \mathbb{Z}} g(P(n,x)) \right) = x+1
\]

whence, from (1), (2), (3), \( \varphi(c) = \lim_{r \to c} \int_{a}^{b} g(r) \, dr \). Also the functional \( \Phi_{1,3} \) satisfies the required condition.

Hence, when we consider the classical operations of analysis it is not necessary to introduce the hyperfunctionals which transform the functionals representing the real functions into other functionals representing the real functions. It suffices to consider the sequences \( \varphi(r_n) \)
and the functionals which transforms the functions \( f \) such that \( A[\varphi(r_n), f_n] \) into other functions. This simplifies the notations of the following theorems.

**Theorem 4.2.** If \( \varphi \) is a continuous function defined over the segment \([a, b] \times \mathcal{D}\), then

\[
\varphi \in \mathcal{D} = [\varphi(r_{\mathcal{D}})] \times \mathcal{D}
\]

where \( r_{\mathcal{D}} \) is a sequence of rationals contained in the segment \([a, b]\).

**Proof.** The first implication results from the definition of definable function, the inverse one from Theorem 4.1.

**Theorem 4.3.** There exists a functional \( \Phi_D \in \mathcal{D} \) such that if \( \varphi \) is a continuous function \( \varphi \) is defined over a segment \([a, b]\) and the values of the function \( \varphi(c) \) for \( c \in [a, b] \) constitute the segment \([a', b']\) and \( A[\varphi(r_n), f_n] \) for \( r_n \in [a, b] \), then \( A[\varphi^{-1}(d), \Phi_D(r)] \in [\langle f', f'' \rangle] \) for \( d \in [a', b'] \) where \( \varphi^{-1}(d) \) is the least \( c \) for which \( \varphi(c) = d \).

**Proof.**

\[
\Phi_D[\langle f', f'' \rangle] = \{ (\mu) \left[ \sum_{i \in N} \sum_{k \in \mathbb{N}} \sum_{\text{all } r_n} a < r_n < b \right. \}
\]

\[
\text{where } n = \frac{x}{a + 1} = \frac{y + 1}{y + 1} \right) \text{ for } \mu \in \mathcal{D}.
\]

Indeed, let us suppose that \( c = \varphi^{-1}(d) \) and \( x = f'(n) \), whence

\[
\left| \frac{x}{a + 1} - c \right| < \frac{1}{a + 1} + \frac{1}{b + 1} \text{ for some } \mu \in \mathcal{D}.
\]

From the continuity it follows that for each \( k \in \mathbb{N} \) there exists such \( r_n \), that

\[
\frac{x}{a + 1} < r_n < c \quad \text{and} \quad |\varphi(r_n) - d| < \frac{1}{b + 1}.
\]

On the other hand, we have

\[
\frac{x}{a + 1} < r_n < c \quad \text{and} \quad \frac{f(m, k)}{b + 1} < \frac{1}{b + 1} \text{ for all } k \in \mathbb{N}.
\]

Hence, from (1), (2), (3),

\[
\left| \frac{x}{a + 1} - r_n \right| < \frac{1}{a + 1} + \frac{1}{b + 1} \quad \text{and} \quad \left| \frac{f(m, k)}{b + 1} - d \right| < \frac{2}{b + 1}.
\]

It suffices to show that \( f'(n) \) is the least \( x \) that satisfies these conditions. Indeed, suppose that \( x < f'(n) \), whence the function \( \varphi \) assumes the value \( d \) in the closed segment

\[
[\frac{x - 1}{a + 1} + \frac{1}{a + 1} + \frac{x + 1}{b + 1} - \frac{1}{b + 1}] \text{ for all } \mu \in \mathcal{D}.
\]

Hence, there exists such \( k \) that, for all \( r_n \) of this segment,

\[
|\varphi(r_n) - d| < \frac{3}{b + 1},
\]

whence, by (3)

\[
\frac{f(m, k)}{b + 1} - d > \frac{2}{b + 1}.
\]

Also \( x \) does not satisfy the required condition.

**Theorem 4.4.** There exists a functional \( \Phi_D \in \mathcal{D} \) such that if \( \varphi \) is a continuous function defined over the segment \([a, b]\) and \( A[\varphi(r_n), f_n] \) for \( r_n \in [a, b] \), and

\[
\varphi \text{ is least } c \text{ such that } e \in [a, b], \varphi(c) = \max \varphi,
\]

then

\[
f' = \Phi_D[\langle f', f'' \rangle].
\]

**Proof.** From Theorem 2.10 it follows that there exists a functional \( \Phi_D \in \mathcal{D} \) such that \( r_{\mathcal{D}} f', f'' \) is a sequence of all rationals contained in the segment \([a, b]\). Namely

\[
\Phi_D[\langle f', f'' \rangle] = \{ (\mu) \left[ \sum_{i \in N} \sum_{k \in \mathbb{N}} \sum_{\text{all } r_n} a < r_n < b \right. \}
\]

\[
\text{Let } \Phi_D[\langle f', f'' \rangle] = \{ (\mu) \left[ \sum_{i \in N} \sum_{k \in \mathbb{N}} \sum_{\text{all } r_n} a < r_n < b \right. \}
\]

To the sequence \( \{ \varphi(r_n), f_n \} \) we apply Theorem 2.15 and obtain

\[
f' = \Phi_D[\langle f', f'' \rangle].
\]

where \( d \) is the upper limit of this sequence. Now let us notice that \( e = \varphi^{-1} d \). Hence, from Theorem 4.3 and 4.3, we obtain \( f' = \Phi_D[\langle f', f'' \rangle] \).

Also

\[
\Phi_D[\langle f', f'' \rangle] = \Phi_D[\langle f', f'' \rangle].
\]
Theorem 4.5. There exists a functional $\Phi_\omega \in \mathcal{D}$ such that if
\[
\max_{\omega} \varphi > d > \min_{\omega} \varphi
\]
then there exists a number $c \in [a, b]$ such that $\varphi(c) - d$ and
\[
f' = \Phi_\omega f \left[ \left( f', f'' \right) \right] (k + 1)
\]
where $f$ satisfies the condition $\Lambda(\varphi_0, f_0)$ for all $r_0 \in [a, b]$.

Proof. Let us consider the function $\varphi(c) = |\varphi(c) - d|$. The function $\varphi$ assumes the maximum value $0$ at a point $c$ for which $\varphi(c) = d$. From Theorem 2.2 it follows that the function $\varphi$ is elementarily definable by means of the function $\varphi$. Hence, using Theorem 4.4, we obtain our theorem.

Theorem 4.6. There exists a functional $\Phi_\omega \in \mathcal{D}$ such that if $\varphi$ is a continuous function defined over the segment $[a, b]$, then for any $c, d \in [a, b]$

(1) if $|c - d| < \frac{1}{\Phi_\omega f \left[ \left( f', f'' \right) \right] (k + 1)}$ then $|\varphi(c) - \varphi(d)| < \frac{1}{k + 1}$

where $f$ satisfies the condition

(2) $\Lambda(\varphi_0, f_0)$ for each $r_0 \in [a, b]$.

Proof. The function $\varphi$ being continuous, is uniformly continuous in the segment $[a, b]$. Hence

(3) $\prod_{\lambda \in \Lambda} \sum_{\lambda \in \Lambda} \prod_{\lambda \in \Lambda} \left| \left| f_{\lambda} \right| \right|_{\lambda} \left( a < r_\lambda, r_\lambda < b \right) \left| \left| r_\lambda - r_\lambda \right| \right| < \frac{1}{\lambda^{k + 1}}$

(4) $\rightarrow |\varphi(r_\lambda) - \varphi(r_\lambda)| < \frac{1}{\lambda^{k + 1}}$

From (2) and (3) it follows that

(5) $\prod_{\lambda \in \Lambda} \sum_{\lambda \in \Lambda} \prod_{\lambda \in \Lambda} \left( a < r_\lambda, r_\lambda < b \right) \left| \left| r_\lambda - r_\lambda \right| \right| < \frac{1}{\lambda^{k + 1}}$

Now let

$\Phi_\omega f \left[ \left( f', f'' \right) \right] (k) = (\mu) \prod_{\lambda \in \Lambda} \left| \left| f_{\lambda} \right| \right|_{\lambda} \left( a < r_\lambda, r_\lambda < b \right) \left| \left| r_\lambda - r_\lambda \right| \right| < \frac{1}{\lambda^{k + 1}}$

\[
\rightarrow \sum_{\lambda \in \Lambda} \prod_{\lambda \in \Lambda} \left| \left| f_{\lambda} \right| \right|_{\lambda} \left( a < r_\lambda, r_\lambda < b \right) \left| \left| r_\lambda - r_\lambda \right| \right| < \frac{1}{\lambda^{k + 1}}
\]

Hence it follows that, for any $r_m, r_n, l \in [a, b]$,

(6) $|r_m - r_n| < \frac{1}{\Phi_\omega f \left[ \left( f', f'' \right) \right] (k + 1)}$

\[
\rightarrow \sum_{l \in \Lambda} \prod_{\lambda \in \Lambda} \left| \left| f_{\lambda} \right| \right|_{\lambda} \left( a < r_\lambda, r_\lambda < b \right) \left| \left| r_\lambda - r_\lambda \right| \right| < \frac{1}{\lambda^{k + 1}}
\]

Let us consider two real numbers $c, d \in [a, b]$ such that

(7) $|c - d| < \frac{1}{\Phi_\omega f \left[ \left( f', f'' \right) \right] (k + 1)}$

From the continuity of $\varphi$ it follows that there are two rationals such that

(8) $c < r_m, \quad r_n < d$

(9) $|\varphi(c) - \varphi(r_n)| < \frac{1}{\lambda^{k + 1}}$

(10) $|\varphi(d) - \varphi(r_m)| < \frac{1}{\lambda^{k + 1}}$

From (7) and (8) we obtain

(11) $|r_m - r_n| < \frac{1}{\Phi_\omega f \left[ \left( f', f'' \right) \right] (k + 1)}$

From (6) and (11) it follows that

(12) $\lim_{l \rightarrow \infty} \frac{1}{l + 1} \frac{1}{l + 1} < \frac{1}{\lambda^{k + 1}}$

From (2) we obtain

(13) $\varphi(r_m) = \lim_{l \rightarrow \infty} f_{\lambda}(l + 1)$

(14) $\varphi(r_n) = \lim_{l \rightarrow \infty} f_{\lambda}(l + 1)$

From (9), (10), (12), (13), (14) it follows that $|\varphi(c) - \varphi(d)| < \frac{1}{\lambda^{k + 1}}$. This completes the proof of condition (1).

Theorem 4.7. There exists a functional $\Phi_\omega \in \mathcal{D}$ such that if a function $\varphi$ is differentiable over the segment $[a, b]$ and $\Lambda(\varphi_0, f_0)$ for $r_0 \in [a, b]$, then

$\Lambda(\varphi'(c), \Phi_\omega) \left[ \left( f', f'' \right) \right]$ for $c \in (a, b)$.

Proof. We start from the definition

$\varphi'(c) = \lim_{h \rightarrow 0} \frac{\varphi(c + |h|) - \varphi(c - |h|)}{|h| + g}$. 

Hence
\[ \psi'(c) = \lim_{m \to \infty} \left( \frac{f(m+1)}{m+1} - \frac{f(m)}{m+1} \right) \left( \frac{m+1}{2} \right). \]

On the other hand, it is true that
\[ \psi'(c) = \lim_{m \to \infty} \left( \frac{f(m+1)}{m+1} - \frac{f(m)}{m+1} \right). \]

Hence
\[ \psi'(c) = \lim_{m \to \infty} \left( \frac{f(N_\omega f(m+1), m+1)}{m+1} - \frac{f(N_\omega f(m), m+1)}{m+1} \right). \]

Now from Theorem 3.4 it follows that the conditions of Theorem 2.11 are satisfied, and by the double application of Theorem 2.11 we find that there exists the required functional \( \Phi_\omega \in \mathcal{D} \).

For the end-points of the segment \([a, b]\) we can define the derived function by means of the one-sided differentiation. The corresponding functionals are also elementarily definable.

**Theorem 4.8.** There exists a functional \( \Phi_\omega \in \mathcal{D} \) such that if \( \varphi \) is a function continuous in the segment \([a, b]\) and \( \lambda(p, \varphi, f) \) for \( r_n \in [a, b] \), then
\[ A_\omega \left( \int_a^b \varphi(x) dx \right). \]

**Proof.**
\[ \int_a^b \varphi(x) dx = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{n+1} \varphi(x) = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{n+1} f(N_\omega, i, n+1, b). \]

Hence, by Theorem 3.4, and by the double application of Theorem 2.11, we find that there exists the required functional \( \Phi_\omega \in \mathcal{D} \).

In conformity with the theorems of this section we can say that the classical operations on continuous functions are elementarily definable. Hence these operations do not exceed the class \( \mathcal{D} \). A continuous function \( \varphi \in \mathcal{D} \), defined in the segment \([a, b] \in \mathcal{D} \), assumes the maximum value at a point \( c \in \mathcal{D} \), is elementarily definable, uniformly continuous in this segment, and
\[ \varphi^{-1} \in \mathcal{D}, \quad \int_a^b \varphi(x) dx \in \mathcal{D} \quad \text{for} \quad a < c < b. \]

For the functions elementarily definable but not continuous the theorems analogous to the familiar theorems of analysis are not true. For example

**Theorem 4.9.** There exists a real function \( \varphi \in \mathcal{D} \) which assumes only two values 0 and 1, and assumes the value 1 at one point which is not elementarily definable.

**Proof.** Let \( Z \) be the set defined in the proof of Theorem 3.4. We set
\[ \varphi(a) = \begin{cases} 0 & \text{for } a \notin Z, \\ 1 & \text{for } a \in Z, \end{cases} \]

and \( \Phi(f, a) = \begin{cases} 1 & \text{if } U(\omega(f, a)) \geq 0, \\ 0 & \text{if } U(\omega(f, a)) < 0. \end{cases} \)

It is evident that
\[ f^{(0)} = \varphi \cdot f. \]

**Hence \( \varphi \notin \mathcal{D} \).**

**Theorem 4.10.** There exists a one-to-one mapping function \( \varphi \in \mathcal{D} \) such that the set of its values is elementarily definable, and \( \varphi^{-1} \notin \mathcal{D} \).

**Proof.** Let
\[ \varphi(n) = \begin{cases} 2n & \text{for } n \in Z, \\ 0 & \text{for } n \notin Z, \end{cases} \]

and \( \Phi(f, a) = \begin{cases} 1 & \text{if } E(n(f, a)) \leq 0, \\ 2 & \text{if } E(n(f, a)) > 0. \end{cases} \)

Hence \( f^{(0)} = \varphi \cdot f \). Then \( \varphi \notin \mathcal{D} \). Thus the function \( \varphi \) establishes a one-to-one correspondence between the segment \([0, 1]\) and the set \([0, 1] - Z + [2] \).

The function \( \varphi^{-1} \notin \mathcal{D} \) because \( \varphi^{-1}(0) = a \in Z \), and \( a \notin \mathcal{D} \), and for any function \( \varphi \in \mathcal{D} \) it is true that if \( c \in \mathcal{D} \), then \( \varphi(c) \in \mathcal{D} \).

Finally, we shall give some remarks concerning systems of arithmetic. Let us consider the system \( S \) of Peano's arithmetic formalized in the simple theory of types. We assume that integers are individuals of the lowest type. The familiar arithmetical functions \( + \) and \( \cdot \) can be the primitive notions. We state a restriction in the formulation of the axiom of definability
\[ \sum_{X, Y, Z, \ldots} E(X, Y, Z, \ldots) = \exists(\ldots, X, Y, Z, \ldots). \]
We admit in this scheme only those formulae $\Sigma$ in which each quantifier bounds a variable of the lowest type. Other axioms are the familiar ones.

A relation $\varphi$ is said to be definable in a system if there exists in this system a theorem of the form $(x)$ such that the expression $\Sigma(\ldots X, Y, Z, \ldots)$ is a possible definition of the relation $\varphi$ i.e. $\varphi(\ldots x, y, z, \ldots)$ if and only if $\exists x, y, z, \ldots$ satisfy the formula $\Sigma$.

From this definition it follows that a relation is definable in $S$ if and only if it is elementarily definable. Hence from Theorem 3.3 it follows that the existence of a non Borelian set is unprovable in the system $S$. But the general question, whether the class $\mathcal{D}$ constitutes the model of the system $S$ remains open, because we cannot decide whether the axiom of extensionality is satisfied in the domain $\mathcal{D}$. There remains also another task: to verify whether the theory of continuous functions can be deduced in $S$. Perhaps this theory can be obtained in $S$ without the use of the axiom of extensionality.

References


On uniformization of functions (I)

R. Sikorski and K. Zarankiewicz (Warszawa)

Let $I$ be the unit interval $0 < x < 1$, and let $f$ be the class of all continuous mappings $f$ of $I$ into itself such that $f(0) = 0$ and $f(1) = 1$. If $f \in f$, then $f \in f$ too. The symbol $f$ denotes always the superposition of $f$ and $\varphi$.

We shall prove the following

Theorem 1'. If $f_1, f_2, \ldots, f_n \in f$ are functions such that
(a) for each $i = 1, 2, \ldots, n$, there is a sequence $0 = a_0 < a_1 < a_2 < \ldots < a_r$ such that $f_i$ is either non-decreasing or non-increasing in every interval $(a_{j-1}, a_j)$, $j = 1, 2, \ldots, r$;

then there exist functions $f_1, f_2, \ldots, f_n \in f$ such that

(1) $f_1 f_2 \cdots f_n = f_1 \cdots f_n$.

Theorem I has the following simple interpretation. There are $a$ paths which are going to the top of a mountain. The paths need not always go upwards, some segments of the paths may be directed downwards. On each of the paths a tourist is climbing. Theorem I asserts that the tourists can climb to the top of the mountain in such a way that, at every moment, all of them are on the same level (of course, it may happen that, in some time intervals, some of the tourists must return from the previously covered segments of the paths). To make it clear, let us suppose that the paths are the curves $p_1(x), p_2(x), \ldots, p_n(x)$, where $p_j(x)$ is a mapping of $I$ into the three-dimensional space. Let $f_i(x)$ be the height (the third coordinate) of the point $p_i(x)$.


During the print of this paper the authors found out that a theorem similar to Theorems I and II was proved by T. Hoorna. A theorem on continuous functions, Fund. Math. Sem. Reports 1 (1932), p. 13-14.

Homma's hypothesis about $f_1, f_2, \ldots, f_n$ is other than that in this paper. The example on p. 340 is also given in Homma's paper.

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