This follows immediately from the definition of constructivity of a theory and from the deduction theorem for the functional calculus of Heyting.

Moreover

4.12. *If α is a constructive closed formula, then for every formula β of the form $\beta = \overset{(1)}{\underset{x_{p_1}}{\Xi}} \ldots \overset{(n)}{\underset{x_{p_n}}{\Xi}} \gamma$ where γ contains no quantifiers and $\overset{(i)}{\Xi}$ is either the sign Σ or Π, there exists a finite sequence $\gamma_1, \ldots, \gamma_m$ of the formulas without quantifiers such that $\alpha \rightarrow \beta$ is provable if and only if at least one of the formulas $\alpha \rightarrow \gamma_1, \ldots, \alpha \rightarrow \gamma_m$ is provable.*

This follows immediately from the deduction theorem for the functional calculus of Heyting and from 4.10.

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Elementarily definable analysis

by

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The purpose of this paper is to give a strict mathematical shape to some ideas expressed by H. Weyl in "Das Kontinuum" [4]. Weyl proposes a restriction of the logical methods of analysis to the elementarily definable ones. A *notion* is *elementarily definable* if it is definable by means of the quantifiers bounding the integral variables only. A strict definition will be given later. It is very interesting to note how many theorems of the classical analysis can be obtained by means of elementary methods. It is shown in this paper that the classical analysis of continuous functions can be reproduced in an elementary manner. The problem of how many theorems from the theory of non continuous real functions can be obtained in an elementary way remains open. Some counter examples are given in the sequel.

To begin with the problem arrises how to define elementary definability. There are at least two answers:

1. A mathematical *notion* A is *elementarily definable* if it is definable by means of an elementary definition

$$A(f, \ldots x, \ldots) \equiv \Xi(\ldots f, \ldots x, \ldots).$$

2. A *notion* A is *elementarily definable* if there exists a finite set of elementary conditions such that A is the unique object which satisfies those conditions.

We shall call the first the *narrower*, the second the *broader concept* of elementary definability. In this paper we shall consider the narrower notion.

1. Elementary definability in the arithmetic of integers

We shall introduce the notion of elementary definability in the arithmetic of integers. Let $\boldsymbol{I}$ be the set of all integers (positive, negative and zero). Let $\boldsymbol{N}$ be the set of non negative integers (natural numbers). The variables x, y, z, p, q will stand for the integers, the variables n, k, l, m will represent natural numbers. The letters $f, g, h,$ will be used to denote the functions defined over the set $\boldsymbol{I}$ and assuming the integral values.

The letters a, b, c will represent the real numbers, and the small Greek letters will represent the real functions. The capital Greek letters will stand for functionals [1]), and the capital Latin letters for sets and relations. We shall consider the relations:

$$R(f_1, \dots, f_k, x_1, \dots, x_n)$$

among k integral functions and n integers. A *relation* is said to be *elementarily definable* if it is definable by a formula containing the quantifiers bounding the integral variables only. A strict definition is the following:

The class $\mathfrak{D}$ of *elementarily definable relations* is the least class of relations containing the initial relations: $x = y+1$, $x = 0$, $x > 0$, $x = y^z$ and $z \geqslant 0$, $y = f_i(x_1, \dots, x_n)$ (the last is the relation between one function f and $n+1$ integers: $x_1, \dots, x_n, y$) and closed under the logical operations of propositional calculus and under the operations of general quantifier bounding the integral variable *i. e.*

$$\text{If} \quad R_1, R_2 \in \mathfrak{D} \quad \text{and} \quad S_1(x_1, \dots, x_k, f_1, \dots, f_n) \equiv \sim\big(R_1(x_1, \dots, x_k, f_1, \dots, f_n)\big),$$

$$S_2(x_1, \dots, x_k, f_1, \dots, f_n) \equiv \big(R_1(x_1, \dots, x_l, f_1, \dots, f_m) \vee R_2(x_j, \dots, x_k, f_p, \dots, f_n)\big),$$

$$S_3(x_1, \dots, x_k, f_1, \dots, f_n) \equiv \prod_{x_0 \in I} R_1(x_0, x_1, \dots, x_k, f_1, \dots, f_n), \quad \text{then} \quad S_1, S_2, S_3 \in \mathfrak{D}.$$

If $R \in \mathfrak{D}$ is a relation only among the numbers $R(x_1, \dots, x_n)$, then we shall say that R is an *elementary number-theoretical relation*. If R is a relation between n-functions and k-integers, we shall say that R is a *functional relation*.

Let

$$(\mu x)[R(\dots x \dots)] = \begin{cases} \text{the least integer } x \text{ such that } R(\dots x \dots), \\ 0 \text{ if the least integer } x \text{ such that } R(\dots x \dots) \text{ does not exist.} \end{cases}$$

The second case contains two possibilities: either there are infinitely many negative integers x such that $R(\dots x \dots)$, or for each integer x it is true that $\sim\big(R(\dots x \dots)\big)$.

A *function* f defined over the set of all integers and assuming the integral values is said to be *elementarily definable* if there exists a number-theoretical elementarily definable relation $R \in \mathfrak{D}$ such that

$$f(x_1, \dots, x_n) = (\mu x_0)[R(x_0, x_1, \dots, x_n)].$$

[1]) The notion of functional will be explained later.

A *functional* $\Phi\langle f_1, \dots, f_k\rangle(x_1, \dots, x_n)$ defined over the integral functions $f_1, \dots, f_k$ and over the integers $x_1, \dots, x_n$ and assuming integral values is said to be *elementarily definable* if there exists a functional relation $R(f_1, \dots, f_k, x_0, x_1, \dots, x_n)$ such that $R \in \mathfrak{D}$ and

$$\Phi\langle f_1, \dots, f_k\rangle(x_1, \dots, x_n) = (\mu x_0)[R(f_1, \dots, f_k, x_0, x_1, \dots, x_n)].$$

The class of elementarily definable functions, as well as the class of elementarily definable functionals, and the class of elementarily definable relations will be denoted by the same symbol $\mathfrak{D}$.

It is evident that the classes of elementarily definable functions and functionals are closed under the operations: of substitution of a constant elementary function or a constant number, and of identification of the variables. The computable functions and functionals [2]) are elementarily definable.

THEOREM 1.1. *If $\Phi, f \in \mathfrak{D}$ and*

$$R_1(f_1, \dots, f_k, x_1, \dots, x_n) \equiv \Phi\langle f_1, \dots, f_k\rangle(x_2, \dots, x_n) = x_1,$$
$$R_2(x_1, \dots, x_n) \equiv f(x_2, \dots, x_n) = x_1,$$

then R_1 and R_2 are elementarily definable relations.

Proof. If $f \in \mathfrak{D}$ then from the definition it follows that there exists a relation $R \in \mathfrak{D}$ such that

$$f(x_2, \dots, x_n) = (\mu x_1)[R(x_1, x_2, \dots, x_n)].$$

The relation R_2 can be defined as follows

$$R_2(x_1, \dots, x_n) \equiv [R(x_1, \dots, x_n) \wedge \prod_{z \in I} \{R(z, x_2, \dots, x_n) \to z \geqslant x_1\}]$$
$$\vee [x_1 = 0 \wedge \prod_{u \in I} \{R(u, x_2, \dots, x_n) \to \sum_{z \in I} z < u \wedge R(z, x_2, \dots, x_n)\}].$$

In the case of Φ the proof is similar.

THEOREM 1.2. *If $R_1 \in \mathfrak{D}$ is a relation of the form $R_1(g, h_1, \dots, h_k, x_1, \dots, x_m)$, where g represents any function of one argument, $\Phi\langle f_1, \dots, f_n\rangle(x) = y$ is a functional of the class $\mathfrak{D}$, and*

$$R(f_1, \dots, f_n, h_1, \dots, h_k, x_1, \dots, x_m) \equiv R_1(\Phi\langle f_1, \dots, f_n\rangle, h_1, \dots, h_k, x_1, \dots, x_m),$$

then $R \in \mathfrak{D}$.

Proof by induction. If R_1 is an initial relation, from the Theorem 1.1 it follows that $R \in \mathfrak{D}$. Now let us suppose that

[2]) For the definition of computable functionals see [1].

$$(1)\quad \begin{aligned}&(\alpha)\ R_1(g,h_1,\dots,h_k,x_1,\dots,x_m)\equiv\sim\big(R_0(g,h_1,\dots,h_k,x_1,\dots,x_m)\big)\ \text{or}\\ &(\beta)\ R_1(g,h_1,\dots,h_k,x_1,\dots,x_m)\equiv R_0(g,h_1,\dots,h_k,x_1,\dots,x_m)\\ &\qquad\qquad\qquad\qquad \vee R_0'(g,h_1,\dots,h_k,x_1,\dots,x_m)\ \text{or}\\ &(\gamma)\ R_1(g,h_1,\dots,h_k,x_1,\dots,x_m)\equiv\prod_{x_{m+1}}R_0(g,h_1,\dots,h_k,x_1,\dots,x_{m+1})\end{aligned}$$

and for R_0, R_0' it is true that the following relations $R_2, R_2'\in\mathfrak{D}$:

$$(2)\quad R_2(f_1,\dots,f_n,h_1,\dots,h_k,x_1,\dots,x_m)\equiv R_0(\Phi\langle f_1,\dots,f_n\rangle,h_1,\dots,h_k,x_1,\dots,x_m),$$

$$(3)\quad R_2'(f_1,\dots,f_n,h_1,\dots,h_k,x_1,\dots,x_m)\equiv R_0'(\Phi\langle f_1,\dots,f_n\rangle,h_1,\dots,h_k,x_1,\dots,x_m).$$

From (2) and (3) it follows that the relation R can be elementarily defined by means of the relations R_2, R_2': (α) using the operation of negation, or (β) using the operation of logical sum, or (γ) using the operation of quantification.

The operation of identification of the variables presents no dificultes.

THEOREM 1.3. *The class $\mathfrak{D}$ of functions, and the class $\mathfrak{D}$ of functionals are closed under the operations of superposition (the substitution of function or functional for a number-variable and for a function-variable).*

Proof. Let $\Phi_1,\Phi_2\in\mathfrak{D}$ this means that there are such $R_1,R_2\in\mathfrak{D}$ that

$$(1)\quad \begin{aligned}\Phi_1\langle g,f_1,\dots,f_k\rangle(x_1,\dots,x_l)&=(\mu y)[R_1(g,f_1,\dots,f_k,x_1,\dots,x_l,y)],\\ \Phi_2\langle h_1,\dots,h_n\rangle(x)&=(\mu y)[R_2(h_1,\dots,h_n,x,y)].\end{aligned}$$

From the Theorem 1.2 it follows that the relation

$$R_3(h_1,\dots,h_n,f_1,\dots,f_k,x_1,\dots,x_l,y)\equiv R_1(\Phi_2\langle h_1,\dots,h_n\rangle,f_1,\dots,f_k,x_1,\dots,x_l,y)$$

is elementarily definable. Hence the superposition

$$(2)\quad \begin{aligned}&\Phi_1\langle\Phi_2\langle h_1,\dots,h_n\rangle,f_1,\dots,f_k\rangle(x_1,\dots,x_l)\\ &\quad=(\mu y)[R_1(\Phi_2\langle h_1,\dots,h_n\rangle,f_1,\dots,f_k,x_1,\dots,x_l,y)]\end{aligned}$$

is a functional elementarily definable.

Let us consider other superposition:

$$(3)\quad \begin{aligned}&\Phi\langle g,f_1,\dots,f_k\rangle\big(\Phi_2\langle h_1,\dots,h_n\rangle(x),x_2,\dots,x_l\big)\\ &\quad=(\mu y)\Big[R_1\Big((g,f_1,\dots,f_k,(\mu z)[R_2(h_1,\dots,h_n,x,z)],x_2,\dots,x_l,y\Big)\Big].\end{aligned}$$

From the definition of minimum it follows that the following equivalence is true:

$$(4)\quad \begin{aligned}&R_1\big(g,f_1,\dots,f_k,(\mu z)[R_2(h_1,\dots,h_n,x,z)],x_2,\dots,x_l,y\big)\\ &\equiv\{\sum_z R_1(g,f_1,\dots,f_k,z,x_2,\dots,x_l,y)\wedge R_2(h_1,\dots,h_n,x,z)\\ &\wedge\prod_v R_2(h_1,\dots,h_n,x,v)\to z\leqslant v\}\vee\{R_1(g,f_1,\dots,f_k,0,x_2,\dots,x_l,y)\\ &\qquad\wedge\prod_v R_2(h_1,\dots,h_n,x,v)\to\sum_u R_2(h_1,\dots,h_n,x,u)\wedge u<v\}.\end{aligned}$$

From (1) it follows that the relation (4) belongs to $\mathfrak{D}$ and the superposition (3) is elementarily definable.

Now some remarks on the notation. If the functional Φ has the form

$$\Phi\langle f_1,\dots,f_n\rangle(x_1,\dots,x_m)=y,$$

then it can be considered as a function

$$\Phi\langle f_1,\dots,f_n\rangle(x_1,\dots,x_{m-1})(x_m)=y$$

of one argument x_m with $m-1$ parameters. For example a function of two arguments $g(x,y)$ can be considered as a sequence of functions: $g_x(y)=g(x,y)$.

A relation or a functional can be defined over the functions of two arguments. The symbols representing functions of two arguments will often be written in parentheses of the form: $\|\ \|$.

Theorem 1.3 can also be extended in such a manner that if $R\in\mathfrak{D}$ and $g_z(x)=g(z,x)$ and

$$S\|g\|\langle f_2,\dots,f_n\rangle(x_1,\dots,x_m,z)\equiv R\langle g_z,f_2,\dots,f_n\rangle(x_1,\dots,x_m),$$

then $S\in\mathfrak{D}$.

There are three recursive functions Pa', Fi', Se' which will often be used in the following:

$$Pa'(x,y)=(x+y)^2+x,\qquad Fi'(z)=z-[\sqrt{z}]^2,\qquad Se'(z)=[\sqrt{z}]-Fi'(z).$$

Pa', Fi', Se' are the pairing functions. They have the following important properties:

$$Fi'\big(Pa'(x,y)\big)=x,\qquad Se'\big(Pa'(x,y)\big)=y,\qquad Pa'\big(Fi'(z),Se'(z)\big)=z,$$

for any $x,y,z\in\boldsymbol{N}$.

By means of Pa', Fi', Se' we can define a triplet of pairing functions for all integers:

$$Pa(x,y)=\begin{cases}2\cdot Pa'(|x|,|y|) & \text{if } x,y\geqslant 0,\\ 2\cdot Pa'(|x|,|y|)+1 & \text{if } x>0 \text{ and } y<0,\\ -2\cdot Pa'(|x|,|y|) & \text{if } x<0 \text{ and } y>0,\\ -2\cdot Pa'(|x|,|y|)+1 & \text{if } x<0 \text{ and } y<0.\end{cases}$$

$$Fi(z)=\begin{cases}Fi'\left[\dfrac{|z|}{2}\right] & \text{if } z\geqslant 0,\\[2ex] -Fi'\left[\dfrac{|z|}{2}\right] & \text{if } z<0.\end{cases}$$

$$Se(z)=\begin{cases}Se'\left[\dfrac{|z|}{2}\right] & \text{if } z \text{ is even},\\[2ex] -Se'\left[\dfrac{|z|}{2}\right] & \text{if } z \text{ is odd}.\end{cases}$$

The pairing functions Pa, Fi, Se have the pairing properties

$$Fi\big(Pa(x,y)\big)=x, \qquad Se\big(Pa(x,y)\big)=y, \qquad Pa\big(Fi(z),Se(z)\big)=z,$$

for any $x,y,z \in \boldsymbol{I}$.

Other functions which will often be used in the sequel are

$$[a]=(\mu x)[x\in \boldsymbol{I}\cdot x+1>a],$$

$$\operatorname{sgn}(a)=\begin{cases}-1 & \text{when } a<0,\\ 0 & \text{when } a=0,\\ 1 & \text{when } a>0.\end{cases}$$

$$|a|=a\cdot \operatorname{sgn}(a).$$

The division a/b is regarded as defined for each $a,b\in \boldsymbol{I}$. If $b=0$ then $a/b=0$.

THEOREM 1.4. *If $\chi^1, \chi^2, \Psi \in \mathfrak{D}$ and Φ is defined by means of the induction-scheme*

$$\Phi\langle g_1,\dots,g_l\rangle(x,0)=\Psi\langle g_1,\dots,g_l\rangle(x),$$

$$\Phi\langle g_1,\dots,g_l\rangle(x,n+1)=\chi^1\langle g_1,\dots,g_l\rangle\big(x,n,\Phi\langle g_1,\dots,g_l\rangle(x,n)\big),$$

$$\Phi\langle g_1,\dots,g_l\rangle\big(x,-(n+1)\big)=\chi^2\langle g_1,\dots,g_l\rangle\big(x,n,\Phi\langle g_1,\dots,g_l\rangle(x,-n)\big),$$

then $\Phi\in\mathfrak{D}$.

Proof. The relation

$$R(g_1,\dots,g_l,x,u,y)\equiv \Phi\langle g_1,\dots,g_l\rangle(x,u)=y$$

can be elementarily defined. To simplify the proof suppose that is Φ defined only for $n\geqslant 0$ and assumes the values $y\geqslant 0$.

$$R(g_1,\dots,g_l,x,n,y)\equiv \sum_{z\in N}\operatorname{Exp}(0,z)=\Psi\langle g_1,\dots,g_l\rangle(x)\wedge \operatorname{Exp}(n,z)=y$$

$$\wedge \prod_{i<n}\operatorname{Exp}(i+1,z)=\chi^1\langle g_1,\dots,g_l\rangle\big(x,i,\operatorname{Exp}(i,z)\big),$$

where $\operatorname{Exp}(n,x)=(\mu k)\big[k\geqslant 0\wedge\sim\big((p_n)^{k+1}|x\big)\big]$, $p_n=$ the nth prime number.

Hence

$$\Phi\langle g_1,\dots,g_l\rangle(x,u)=(\mu y)[R(g_1,\dots,g_l,x,u,y)].$$

A similar scheme of induction for the functions does not exceed class $\mathfrak{D}$ of functions.

On the other hand, it is easy to prove that the general scheme of induction leads out of class $\mathfrak{D}$ [3]).

[3]) The function universal for the class $\mathfrak{D}$ is definable by means of the general scheme of induction. Cf. section 3, Theorem 3.6.

2. The elementarily definable concepts of analysis

It is well known that a real number a can be represented in the arithmetic of integers:

A. by an integral function f such that

$$\left|a-\frac{f(n)}{n+1}\right|<\frac{1}{n+1} \quad \text{for each} \quad n\in \boldsymbol{N};$$

B. by an integral function f such that

$$a=\sum_{n=0}^{\infty}\frac{f(n)}{2^n} \quad \text{and} \quad 0\leqslant f(n)<2 \quad \text{for} \quad n>0;$$

C. by an integral function f such that

$$a<\frac{p}{q+1}\equiv f\big(Pa(p,q)\big)=0 \quad \text{for any} \quad p,q\in \boldsymbol{I};$$

D. by an integral function f such that

$$\left|a-\frac{Fi\big(f(n)\big)}{Se\big(f(n)\big)+1}\right|<\frac{1}{n+1} \quad \text{for each} \quad n\in \boldsymbol{N};$$

E. by an integral function f such that

$$\left|a-\frac{Fi\big(f(n)\big)}{Fi\,Se\big(f(n)\big)+1}\right|<\frac{1}{m+1} \quad \text{for} \quad n>Se\,Se\,f(m) \quad \text{and} \quad n,m\in \boldsymbol{N}.$$

Let us denote the relations expressed in A-E, by $\mathrm{A}(a,f)$, $\mathrm{B}(a,f)$, $\mathrm{C}(a,f)$, $\mathrm{D}(a,f)$, $\mathrm{E}(a,f)$.

THEOREM 2.1. *If* P *and* Q *are some of the relations* A, B, C, D, E, *then there exists a functional $\Theta^{\mathrm{P}}_{\mathrm{Q}}\in\mathfrak{D}$ such that if $\mathrm{P}(a,f)$ then $\mathrm{Q}(a,\Theta^{\mathrm{P}}_{\mathrm{Q}}\langle f\rangle)$.*

Proof. Let

$$\Phi\langle f\rangle(n)=(\mu x)\Big[\sum_{l\in N}\prod_{k>l}0\leqslant\frac{f(k)+1}{k+1}-\frac{x}{2^n}<\frac{1}{2^n}\Big],$$

$$\Theta^{\mathrm{A}}_{\mathrm{B}}\langle f\rangle(0)=\Phi\langle f\rangle(0),$$

$$\Theta^{\mathrm{A}}_{\mathrm{B}}\langle f\rangle(n+1)=\Phi\langle f\rangle(n+1)-2\cdot\Phi\langle f\rangle(n),$$

$$\Theta^{\mathrm{B}}_{\mathrm{C}}\langle f\rangle(x)=\begin{cases}0 & \text{when} \quad \displaystyle\sum_{l\in N}\prod_{k>l}\frac{Fi(x)}{Se(x)+1}>\sum_{n=0}^{k}\frac{f(n)}{2^n}+\frac{1}{l+1},\\ 1 & \text{in the other case,}\end{cases}$$

$$\Theta_{\mathrm{D}}^{\mathrm{C}}\langle f\rangle(n) = (\mu x)\Big[f(x)\neq 0 .\, f\Big((\mu y)\Big[\frac{Fi(y)}{Se(y)+1} = \frac{Fi(x)}{Se(x)+1} + \frac{1}{n+2}\Big]\Big) = 0\Big],$$

$$\Theta_{\mathrm{E}}^{\mathrm{D}}\langle f\rangle(n) = Pa\big(Fi\,f(n), Pa(Se\,f(n), n)\big),$$

$$\Theta_{\mathrm{A}}^{\mathrm{E}}\langle f\rangle(n) = (\mu x)\Big[\sum_{l\in N}\prod_{m>l}\prod_{k>Se\,Se\,f(m)}\Big|\frac{Fi\,f(k)}{Fi\,Se\,f(k)+1} - \frac{x}{n+1}\Big| < \frac{2}{3(n+1)}\Big].$$

The left-hand functionals can be defined by substitutions *e. g.*

$$\Theta_{\mathrm{C}}^{\mathrm{A}}\langle f\rangle(x) = \Theta_{\mathrm{C}}^{\mathrm{B}}\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f\rangle\rangle(x).$$

THEOREM 2.2. *If φ is one of the following real functions: $a+b$, $a-b$, $a\cdot b$, a/b, $|a|$, $[a]$, then there exists a functional $\Phi\in\mathfrak{D}$ such that for any a, b and f, g if $\mathrm{A}(a,f)$ and $\mathrm{A}(b,g)$ then $\mathrm{A}(\varphi(a),\Phi\langle f\rangle)$ (or $\mathrm{A}(\varphi(a,b),\Phi\langle f,g\rangle)$).*

Proof.

$$\Phi^{(1,2)}\langle f,g\rangle(n) = (\mu x)\Big[\sum_{l\in N}\prod_{k>l}\Big|\frac{f(k)\mp g(k)}{k+1} - \frac{x}{n+1}\Big| < \frac{2}{3(n+1)}\Big],$$

$$\Phi^{(3)}\langle f,g\rangle(n) = (\mu x)\Big[\sum_{l\in N}\prod_{k>l}\Big|\frac{f(k)\cdot g(k)}{(k+1)^2} - \frac{x}{n+1}\Big| < \frac{2}{3(n+1)}\Big],$$

$$\Phi^{(4)}\langle f,g\rangle(n) = (\mu x)\Big[\sum_{l\in N}\prod_{k>l}\Big|\frac{f(k)}{g(k)} - \frac{x}{n+1}\Big| < \frac{2}{3(n+1)}\Big],$$

$$\Phi^{(5)}\langle f\rangle(n) = |f(n)|,$$

$$\Phi^{(6)}\langle f\rangle(n) = (n+1)\cdot(\mu x)\Big[\sum_{l\in N}\prod_{k>l} 0 < \frac{f(k)+1}{k+1} - x < 1\Big].$$

Let $f^a(n) = (\mu x)[|x/(n+1) - a| < 1/(n+1)]$. It is evident that for every real number a $\mathrm{A}(a,f^a)$. This means that

$$\Big|\frac{f^a(n)}{n+1} - a\Big| < \frac{1}{n+1} \quad \text{for each} \quad n\in\boldsymbol{N}.$$

THEOREM 2.3. *There exists a functional $\Phi_7\in\mathfrak{D}$ such that if $\mathrm{A}(a,f)$ then $f^a(n) = \Phi_7\langle f\rangle(n)$.*

Proof.

$$\Phi_7\langle f\rangle(n) = (\mu x)\Big[\sum_{l\in N}\prod_{k>l}\Big|\frac{x}{n+1} - \frac{f(l)}{l+1}\Big| < \frac{1}{n+1}\Big].$$

Now we introduce the concepts of elementarily definable analysis.

A *real number* a is *elementarily definable* if there exists a function $f\in\mathfrak{D}$ such that

$$\Big|a - \frac{f(n)}{n+1}\Big| < \frac{1}{n+1} \quad \text{for each} \quad n\in\boldsymbol{N}.$$

A *real sequence* $\{a_k\}$ is *elementarily definable* if there exists a function $f\in\mathfrak{D}$ such that

$$\Big|a_k - \frac{f(n,k)}{n+1}\Big| < \frac{1}{n+1} \quad \text{for each} \quad n,k\in\boldsymbol{N}.$$

A *real function φ of one argument* is *elementarily definable* over the set A of real numbers if there exists a functional $\Phi\in\mathfrak{D}$ such that for any $a\in A$ and f

$$\text{if} \quad \Big|a - \frac{f(n)}{n+1}\Big| < \frac{1}{n+1} \quad \text{for each} \quad n\in\boldsymbol{N},$$

$$\text{then} \quad \Big|\varphi(a) - \frac{\Phi\langle f\rangle(n)}{n+1}\Big| < \frac{1}{n+1} \quad \text{for each} \quad n\in\boldsymbol{N}.$$

Similarly we can determine when a *function φ of two or many arguments* is *elementarily definable*: it is when there exists a functional $\Phi\in\mathfrak{D}$ such that for all $a_1,\dots,a_k$, $f_1,\dots,f_k$

$$\text{if} \quad \mathrm{A}(a_1,f_1),\ \mathrm{A}(a_2,f_2),\dots,\ \mathrm{A}(a_k,f_k), \quad \text{then} \quad \mathrm{A}\big(\varphi(a_1,\dots,a_k),\Phi\langle f_1,\dots,f_k\rangle\big).$$

The class of elementarily definable real numbers as well as the class of elementarily definable real sequences and the class of elementarily definable real functions will be denoted by the same symbol $\mathfrak{D}$. The elementarily definable numbers, sequences and functions will be the object of our investigations.

From theorem 2.3 it follows

THEOREM 2.4. $a\in\mathfrak{D}\equiv f^a\in\mathfrak{D}$.

THEOREM 2.5. *If the function φ is determined over the set X, then*

$$\varphi\in\mathfrak{D}\equiv\sum_{\Phi\in\mathfrak{D}}\prod_{a\in X}\mathrm{A}\big(\varphi(a),\Phi\langle f^a\rangle\big),$$

$$\varphi\in\mathfrak{D}\equiv\sum_{\Phi\in\mathfrak{D}}\prod_{a\in X} f^{\varphi(a)} = \Phi\langle f^a\rangle.$$

From theorem 2.1 it follows at once

THEOREM 2.6. *A real number $a\in\mathfrak{D}$ if and only if there exists an integral function $f\in\mathfrak{D}$ such that $\mathrm{A}(a,f)$, or $\mathrm{B}(a,f)$, or $\mathrm{C}(a,f)$, or $\mathrm{D}(a,f)$, or $\mathrm{E}(a,f)$.*

A real sequence $\{a_k\} \in \mathfrak{D}$ if and only if there exists an integral function $f \in \mathfrak{D}$ of two arguments such that with $f_k(n) = f(k,n)$ it is true that for all $k \in \boldsymbol{N}$ $\mathrm{A}(a_k, f_k)$, or for all k $\mathrm{B}(a_k, f_k)$, or for all k $\mathrm{C}(a_k, f_k)$, or for all k $\mathrm{D}(a_k, f_k)$, or for all k $\mathrm{E}(a_k, f_k)$.

A real function $\varphi \in \mathfrak{D}$ if and only if there exists an integral functional $\Phi \in \mathfrak{D}$ such that if $\mathrm{P}(a,f)$, then $\mathrm{Q}\big(\varphi(a), \Phi\langle f\rangle\big)$, for all $a \in X$ and each integral function f, provided that P and Q are some of the relations $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$ and X is the set of arguments of φ.

From theorem 2.2 it follows that the functions mentioned in theorem 2.2 are elementarily definable, and

THEOREM 2.7. *The set of elementarily definable real numbers is a field of numbers.*

The set of possible elementary definitions is enumerable. Hence the set of elementarily definable numbers as well as the set of elementarily definable sequences and the set of elementarily definable real functions are enumerable. Computable numbers and sequences are elementarily definable.

Polynomials with elementarily definable coeficients are elementarily definable functions accordingly to Theorem 2.2. Continuous elementarily definable function assumes elementarily definable value *e. g.* 0 at an elementarily definable point (as shall be shown in the following). Hence the roots of an elementarily definable polynomial are elementarily definable. This means that the field of elementarily definable numbers is algebraically closed in the field of real numbers. Similarly each analytical function with an elementarily definable sequence of coeficients is an elementarily definable real function. Trigonometrical functions and all so called elementary functions of analysis are elementarily definable because they can be presented as elementarily definable sequences.

THEOREM 2.8. *For any two real functions φ, ψ, if $\varphi, \psi \in \mathfrak{D}$ then the superposition $\varphi\psi \in \mathfrak{D}$; if $\{a_n\} \in \mathfrak{D}$ then $\{\varphi(a_n)\} \in \mathfrak{D}$.*

Proof. If φ is defined over the set X, ψ over the set Y, and for $a \in X$, $\varphi(a) \in Y$, and for each a, b

$$(1) \qquad \begin{aligned} &\text{if } \mathrm{A}(a,f), \text{ then } \mathrm{A}\big(\varphi(a), \Phi\langle f\rangle\big), \\ &\text{if } \mathrm{A}(b,g), \text{ then } \mathrm{A}\big(\psi(b), \Psi\langle g\rangle\big), \end{aligned}$$

then for each $a \in X$ if $\mathrm{A}(a,f)$, then $\mathrm{A}\Big(\psi\big(\varphi(a)\big), \Psi\langle\Phi\langle f\rangle\rangle\Big)$. Hence the function $\psi\varphi$ is elementarily defined by means of the functional $\Theta\langle f\rangle(x) = \Psi\langle\Phi\langle f\rangle\rangle(x)$ which belongs to $\mathfrak{D}$ according to 1.3.

If $\{a_n\} \in \mathfrak{D}$ and $\mathrm{A}(a_n, f_n)$, then by (1) $\mathrm{A}\big(\varphi(a_n), \Phi\langle f_n\rangle\big)$. The sequence $\{\varphi(a_n)\}$ is elementarily defined by means of the elementarily defined integral function $\Phi\langle f_n\rangle(x)$.

For example, $\{a_n + b_n\}$, $\{a_n \cdot b_n\}$, $\{a_n - b_n\}$, $\{a_n/b_n\}$, $\{|a_n|\}$, $\{[a_n]\}$ are elementarily definable sequences, if $\{a_n\}, \{b_n\} \in \mathfrak{D}$ (Theorem 2.2).

THEOREM 2.9. *If $a \in \boldsymbol{I}$ for each $k \in \boldsymbol{N}$, then $\{a_k\} \in \mathfrak{D}$ if and only if there exists an integral function $g \in \mathfrak{D}$ such that $a_k = g(k)$ for $k \in \boldsymbol{N}$. If φ is defined over the set $\boldsymbol{I}$, and $\varphi(x) \in \boldsymbol{I}$ for $x \in \boldsymbol{I}$, then $\varphi \in \mathfrak{D}$ if and only if there exists an integral function $g \in \mathfrak{D}$ such that $\varphi(x) = g(x)$ for $x \in \boldsymbol{I}$.*

Proof. If $a_k \in \boldsymbol{I}$ and

$$(1) \qquad \left| a_k - \frac{f(k,n)}{n+1} \right| < \frac{1}{n+1} \quad \text{for each} \quad n \in \boldsymbol{N}$$

then $a_k = f(k,0) = g(k)$. Conversely, if $a_k = g(k)$ then the function $f(k,n) = g(k)\cdot(n+1)$ satisfies the condition (1).

If $\varphi \in \mathfrak{D}$, then there exists a functional $\Phi \in \mathfrak{D}$ such that

$$(2) \qquad \text{if} \quad \mathrm{A}(a,f), \quad \text{then} \quad \mathrm{A}(\varphi, \Phi\langle f\rangle).$$

Let $f_x(n) = f(x,n) = x\cdot(n+1)$. It is true that $\mathrm{A}(x, f_x)$ for $x \in \boldsymbol{I}$. If φ is defined over the set $\boldsymbol{I}$, and $\varphi(x) \in \boldsymbol{I}$ for $x \in \boldsymbol{I}$, then from (2) it follows that

$$\left| \varphi(x) - \frac{\Phi\langle f_x\rangle(n)}{n+1} \right| < \frac{1}{n+1}.$$

Hence $\varphi(x) = \Phi\langle f_x\rangle(0) = g(x)$ where $g \in \mathfrak{D}$ according to Theorem 1.2. Conversely, if $\varphi(x) = g(x)$, then the functional $\Phi\langle f\rangle(n) = g\big(f(0)\big)\cdot(n+1)$ satisfies the condition (2) for $a \in \boldsymbol{I}$.

THEOREM 2.10. *There are the relations $R_1, R_2, R_3, R_4 \in \mathfrak{D}$ such that if $\mathrm{A}(a,f)$, $\mathrm{A}(b,g)$ and $\mathrm{A}(a_n, h_n)$ for each $n \in \boldsymbol{N}$ then*

$$R_1(f,g) \equiv a < b, \qquad R_2(h,g,n) \equiv a_n < b,$$

$$R_3(h,n,x,y) \equiv a_n < \frac{x}{y+1}, \qquad R_4(h,g) \equiv \{a_n\} \to b.$$

Proof.

$$(1) \qquad a < b \equiv \sum_{k \in N} \prod_{n > k} (k+1)\big(g(n) - f(n)\big) > n+1,$$

$$(2) \qquad a_n < b \equiv \sum_{k \in N} \prod_{m > k} (k+1)\big(g(m) - h_n(m)\big) > m+1,$$

$$(3) \qquad a_n < \frac{x}{y+1} \equiv \sum_{k \in N} \prod_{m > k} \frac{h_n(m)}{m+1} + \frac{1}{k+1} < \frac{x}{y+1}.$$

In the following we shall consider the functionals defined over the functions of two arguments. To avoid the confusion of the types of function variables we shall write the variables standing for functions of two arguments in parentheses $\| \ \|$. Hence for example if $f(n,y)=f_n(y)$, then we can write $\langle f_n\rangle$ but $\|f\|$ (cf. the remarks on notations on p. 315).

From Theorem 2.2 it follows that there exists a functional $\Phi^8 \in \mathfrak{D}$ such that $\Phi^8\langle f,g\rangle(n)=\Phi^5\langle\Phi^2\langle f,g\rangle\rangle(n)$ and

(4) $$\mathrm{A}(|a_n-b|,\Phi\langle h_n,g\rangle).$$

Hence from Theorem 1.3 it follows that there exists a functional $\Psi^8 \in \mathfrak{D}$ such that

(5) $$\Phi^8\langle h_n,g\rangle(x)=\Psi^8\|h\|\langle g\rangle(n,x).$$

From (3), (4), and (5) we find that the relation

$$|a_n-b|<\frac{x}{y+1}\equiv R_3(\Psi^8\|h\|\langle g\rangle,n,x,y)$$

is elementarily definable. Hence $R_4 \in \mathfrak{D}$ because

$$\{a_n\}\to b\equiv \prod_{k\in N}\sum_{l\in N}\prod_{n>l}|a_n-b|<\frac{1}{k+1}.$$

THEOREM 2.11. *There exists a functional $\Phi \in \mathfrak{D}$ such that if $\mathrm{A}(a_k,f_k)$ for each $k \in \boldsymbol{N}$ and $\{a_k\}\to a$, then $\mathrm{A}(a,\Phi\|f\|)$.*

Proof.

$$\Phi\|f\|(n)=(\mu x)\left[\sum_{l\in N}\prod_{k>l}\left|a_k-\frac{x}{n+1}\right|<\frac{2}{3(n+1)}\right].$$

From Theorems 2.10 and 2.2 it follows that the relation

$$R(f,k,n,x)\equiv\left|a_k-\frac{x}{n+1}\right|<\frac{2}{3(n+1)}$$

is elementarily definable. Hence $\Phi \in \mathfrak{D}$.

THEOREM 2.12. *If $\{a_k\} \in \mathfrak{D}$ and $\{a_k\}\to a$, then $a \in \mathfrak{D}$.*

Proof. From Theorem 2.11 by means of the definitions.

THEOREM 2.13. *There exists a functional $\Omega \in \mathfrak{D}$ such that if $\mathrm{A}(a_n,f_n)$ for each $n \in \boldsymbol{N}$ and the sequence $\{a_n\}$ is limited, then the sub-sequence $a_{\Omega\|f\|(n)}$ is convergent.*

Proof. The functional Ω is defined as follows:

$$\Phi\|f\|(k)=(\mu x)\left[\sum_{l\in N}\prod_{n>l}a_n\leqslant\frac{x}{k+1}\right],$$

$$\Omega\|f\|(k)=(\mu n)\left[n \in \boldsymbol{N}.\ \frac{\Phi\|f\|(k)-2}{k+1}<a_n\leqslant\frac{\Phi\|f\|(k)}{k+1}\right].$$

The limit of the sub-sequence $a_{\Omega\|f\|(n)}$ is the upper limit of the set of points of accumulation of the numbers of the sequence $\{a_n\}$. From theorem 2.10 it follows that the functional $\Phi \in \mathfrak{D}$ because the relations

$$a_n\leqslant\frac{x}{k+1};\qquad a_n>\frac{x-1}{k+1}$$

are elementarily definable. Hence likewise from Theorem 2.10 we find that the relation

$$a_n<\frac{\Phi\|f\|(k)}{k+1}\equiv R_3(f,n,\Phi\|f\|(k),k)$$

is elementarily definable, also $\Omega \in \mathfrak{D}$.

THEOREM 2.14. *If $\{a_n\} \in \mathfrak{D}$ and the sequence $\{a_n\}$ is limited, there exists a sub-sequence $\{a_{n_k}\} \in \mathfrak{D}$ such that $\lim\{a_{n_k}\} \in \mathfrak{D}$.*

Proof. From Theorems 2.12 and 2.13 by means of the definitions.

THEOREM 2.15. *There exists a functional $\Phi_9 \in \mathfrak{D}$ such that if $\{a_n\}$ is limited and $a=\sup\{a_k\}$ and $\mathrm{A}(a_n,f_n)$, then $f^a=\Phi_9\|f\|$.*

Proof.

$$\Phi^*\|f\|(n)=(\mu x)\left[\prod_{k\in N}a_k<\frac{x}{n+1}\right],$$

$$\Phi_9\|f\|(n)=(\mu x)\left[\sum_{k\in N}\prod_{l>k}\left|\frac{x}{n+1}-\frac{\Phi^*\|f\|(l)}{l+1}\right|<\frac{1}{n+1}\right].$$

THEOREM 2.16. *There exists a functional $\Psi \in \mathfrak{D}$ such that if $\mathrm{A}(a_n,f_n)$, then the sequence $\{a_{\Psi\|f\|(n)}\}$ is monotonic.*

Proof. We can distinguish the following cases:

(1) The sequence $\{a_n\}$ is unlimited when

(a) $\prod_{x\in I}\sum_{n\in N}a_n<x$ or (b) $\prod_{x\in I}\sum_{n\in N}a_n>x$,

(2) the sequence $\{a_n\}$ is limited when

(c) $$\sum_{x,y\in I}\prod_{n\in N}x<a_n<y.$$

In this case we shall consider the subsequence $b_n=a_{\Omega\|f\|(n)}$. The sequence $\{b_n\}$ contains an increasing subsequence if

(c′) $$\sum_{l\in N}\prod_{n>l}\sum_{k>n}b_l<b_k<\lim b_n;$$

the sequence $\{b_n\}$ contains a decreasing subsequence if

(c″) $$\sum_{l\in N}\prod_{n>l}\sum_{k>n}b_l>b_k>\lim b_n;$$

the sequence $\{b_n\}$ contains a constant subsequence if

(c''') $$\prod_{n\in N}\sum_{k>n} b_k = \lim b_n .$$

Now we define the functional Ψ in the following manner:

$$\Psi\|f\|(0) = \begin{cases} 0 & \text{if (a) or (b),}\\ \Omega\|f\|\big((\mu k)[b_k < \lim b_n]\big) & \text{if (c'),}\\ \Omega\|f\|\big((\mu k)[b_k > \lim b_n]\big) & \text{if (c''),}\\ \Omega\|f\|\big((\mu k)[b_k = \lim b_n]\big) & \text{if (c''').}\end{cases}$$

$$\Psi\|f\|(n+1) = \begin{cases} (\mu k)[a_k < -n \wedge a_k < a_{\Psi\|f\|(n)}] & \text{if (a),}\\ (\mu k)[a_k > n \wedge a_k > a_{\Psi\|f\|(n)}] & \text{if (b),}\\ \Omega\|f\|\big((\mu k)[a_{\Psi\|f\|(n)} < b_k < \lim b_n]\big) & \text{if (c'),}\\ \Omega\|f\|\big((\mu k)[a_{\Psi\|f\|(n)} > b_k > \lim b_n]\big) & \text{if (c''),}\\ (\mu k)[k > \Psi\|f\|(n) \wedge a_k = \lim b_n] & \text{if (c''').}\end{cases}$$

From Theorem 1.4 it follows that $\Psi \in \mathfrak{D}$. It is evident that the sequence $a_{\Psi\|f\|(n)}$ is decreasing in cases (a) and (c''), increasing in cases (b) and (c'), and constant in case (c''').

3. Elementarily definable sets of real numbers

We introduce the concept of elementarily definable sets as well as the concept of elementarily definable sequences of sets.

The class of the elementarily definable sets of real numbers will be denoted by the same symbol $\mathfrak{D}$ as the class of definable real functions.

(1) *$Z \in \mathfrak{D}$ ($\{Z_x\} \in \mathfrak{D}$) if and only if there exists a relation $R \in \mathfrak{D}$ such that for all a and f if $\mathrm{A}(a,f)$ then $a \in Z$ ($a \in Z_x$) if and only if $R(f)$ $\big(R(f,x)\big)$.*

(In this section the indices x in the symbols $\{Z_x\}$ of sequences of sets are regarded as running throughout the set I of all integers.)

THEOREM 3.1. *The condition $Z \in \mathfrak{D}$ ($\{Z_n\} \in \mathfrak{D}$) is equivalent to the following:*

(2) *There exists $S \in \mathfrak{D}$ such that, for all a, $a \in Z$ ($a \in Z_n$) if and only $S(f^a)$ $\big(S(f^a,n)\big)$.*

Proof. $Z \in \mathfrak{D} \rightarrow (2)$. If $Z \in \mathfrak{D}$, then there exists a relation $R \in \mathfrak{D}$ which satisfies the condition (1). We set $S = R$. It is evident that $\mathrm{A}(a,f^a)$. Hence the condition (2) is fulfilled.

$(2) \rightarrow Z \in \mathfrak{D}$. Let $R(f) \equiv S(\Phi_7\|f\|)$. Hence by Theorem 2.3 the condition (1) is fulfilled.

THEOREM 3.2. *The sets and sequences $\mathfrak{D}$ constitute an enumerable additive Boolean algebra.*

Proof. If Z_1, Z_2 and R_1, R_2 satisfy the condition (1), then the sets $-Z_1$ and $Z_1 \cup Z_2$ are determined by the relation $\sim\big(R_1(f)\big)$, $\big(R_1(f) \vee R_2(f)\big)$.

If $\{Z_n\} \in \mathfrak{D}$, then the relation $S(f) \equiv \sum_{x\in I} R(f,x)$ determines the set $\sum_{n=-\infty}^{\infty} Z_n$.

THEOREM 3.3. *The sets $\mathfrak{D}$ are Borelian sets of finite degree.*

Proof. From 3.1 and from the definition of elementarily definable relations it follows that the class $\mathfrak{D}$ of sets and sequences of sets is the smallest class that contains the following initial sequences:

(α) $$Z_{x_1,\dots,x_k} = \underset{a}{E}\,[f^a(x_i) = x_j],$$

(β) $$Z_{x_1,\dots,x_k} = \underset{a}{E}\,[x_i = x_j + 1],$$

(γ) $$Z_{x_1,\dots,x_k} = \underset{a}{E}\,[x_i = x_j^{x_l} \text{ and } x_l \geqslant 0],$$

(δ) $$Z_{x_1,\dots,x_k} = \underset{a}{E}\,[x_i \geqslant 0],$$

(where $1 \leqslant i,j,l \leqslant k$), and is closed under the following operations on sequences:

A. The operation of addition of two sequences, which leads from two sequences $\{Z'_{x_1,\dots,x_k}\}$ and $\{Z''_{x_1,\dots,x_k}\}$ to the sequence

$$Z_{x_1,\dots,x_k} = Z'_{x_1,\dots,x_k} \cup Z''_{x_1,\dots,x_k}.$$

B. The operation of complementation, which leads from a sequence $\{Z'_{x_1,\dots,x_k}\}$ to the sequence $Z_{x_1,\dots,x_k} = -Z'_{x_1,\dots,x_k}$.

C. The operation of enumerable summation, which leads from a sequence $\{Z'_{x_1,\dots,x_k}\}$ to the set (if $k=1$) or to the sequence

$$Z_{x_2,\dots,x_k} = \sum_{x_1=-\infty}^{\infty} Z'_{x_1,\dots,x_k}.$$

The sequences of type (α), contain the sets which are segments $\big[x_j/(x_i+1), (x_j+1)/(x_i+1)\big)$. The sequences of type (β)-(δ) contain sets which are: The whole set of real numbers if the indices satisfy the arithmetical condition, or the empty set if they do not. Also the initial sets are Borelian of finite degree. Hence each set $Z \in \mathfrak{D}$ is Borelian of finite degree [4].

[4]) It is easy to show that for each positive integer n there exists a Borelian elementarily definable set of nth degree, which is not of the degree $n-1$. See for example [3], p. 278.

THEOREM 3.3′. *The functions* $\mathfrak{D}$ *are mesurable* (B) *of finite degree.*

Proof. Let Φ be such a functional that $\Phi \in \mathcal{D}$ and $f^{\varphi(a)} = \Phi\langle f^a \rangle$. Let F be a closed set. There exists an decreasing sequence $F_1 \subset F_2 \subset F_3, \ldots$ of closed sets such that $F = \prod_i^{\infty} F_i$ and each set F_i is a summe of recursive sequence of disjoint closed segments with rational endpoints. Each set F is also computable. Let R_i be such a computable relation that $b \in F_i \equiv R_i\langle f^b \rangle$. Hence

$$a \in \varphi^{-1}(F_i) \equiv \varphi(a) \in F_i \equiv R_i\langle f^{\varphi(a)} \rangle \equiv R_i\langle \Phi\langle f^a \rangle\rangle .$$

It is evident that $\varphi^{-1}(F) = \varphi^{-1}\Big(\prod_i^{\infty} F_i\Big) = \prod_i^{\infty} \varphi^{-1}(F_i)$. Putting $a \in A_i \equiv R_i\langle \Phi\langle f^a \rangle\rangle$ we obtain that there exists such a number $n \in \boldsymbol{N}$ that each set A_i belongs to the Borelian class of n degree. But $\prod_i^{\infty} A_i = \prod_i^{\infty} \varphi^{-1}(F_i)$. Hence the product $\prod_i^{\infty} \varphi^{-1}(F_i) = \varphi^{-1}(F)$ belongs to the Borelian multiplicative class of $n+1$ degree. This means that φ is mesurable (B) of $n+1$ degree. The number n can be evaluated by the number k of quantifiers contained in the definition of the functional Φ. Namely $n \leqslant k$. It is evident because the relations R_i are computable.

Let j_1, j_2 be two recursive functions such that $r_n = j_1(n)/j_2(n)$ where $\{r_n\}$ is the sequence of rational numbers. We set

$$W_n(k) = W(n,k) = f^{r_n}(k) = (\mu x)\left[\left|\frac{x}{k+1} - \frac{j_1(n)}{j_2(n)}\right| < \frac{1}{k+1}\right],$$

$$N_w(x,y) = (\mu n)\left[y \neq 0 \wedge n \geqslant 0 \wedge \frac{x}{y} = \frac{j_1(n)}{j_2(n)}\right].$$

The function W is also recursive. It is evident that

THEOREM 3.4. $\mathrm{A}(r_n, W_n)$, $\mathrm{A}(x/y, W_{N_w(x,y)})$, $f^{x/y} = W_{N_w(x,y)}$.

THEOREM 3.5. *If X is a segment and a, b are the endpoints of X, $a < b$, then $X \in \mathfrak{D}$ if and only if $a, b \in \mathfrak{D}$.*

Proof. From Theorem 2.10 it follows that there exists a relation $R_5 \in \mathcal{D}$ such that $c \in X \equiv R_5(f^a, f^b, f^c)$.

Conversely, if $X \in \mathfrak{D}$, then there exists a relation $R \in \mathfrak{D}$ such that $c \in X \equiv R(f^c)$. Let

$$F(n) = (\mu x)[R(W_{N_w(x,n+1)})],$$

$$G(n) = (\mu x)[R(W_{N_w(x-1,n+1)}) \wedge \sim R(W_{N_w(x,n-1)})].$$

Hence $a = \lim F(n)/(n+1)$, $b = \lim G(n)/(n+1)$ and $a, b \in \mathfrak{D}$ by means of Theorem 2.12.

THEOREM 3.6. *There exists a number $a \notin \mathfrak{D}$ such that the set $\{a\}$ belongs to $\mathfrak{D}$.*

Proof. The universal function $G_n(x) = g\big(P(n,x)\big)$ for the elementarily definable functions is the unique function g, which satisfies the following condition $U\langle g \rangle$:

$$\begin{aligned}
U\langle g\rangle \equiv\ & \prod_{x\in I} g\big(P(0,x)\big) = x+1 \\
& \wedge \prod_{x\in I} g\big(P(1,x)\big) = |x| \wedge \prod_{x\in I} g\big(P(2,x)\big) = Fi(x) - Se(x) \\
& \wedge \prod_{x\in I} g\big(P(3,x)\big) = [Fi(x)^{Se(x)}] \wedge \prod_{x\in I} g\big(P(4,x)\big) = Fi(x) \\
& \wedge \prod_{x\in I} g\big(P(5,x)\big) = Se(x) \wedge \prod_{x\in I} g\big(P(6,x)\big) = P(x,x) \\
& \wedge \prod_{x\in I} g\big(P(7,x)\big) = P(0,x) \wedge \prod_{x\in I} g\big(P(8,x)\big) = P(x,0) \\
& \wedge \prod_{n>8}\Big\{ \prod_{x\in I} g\big(P(n,x)\big) = g\Big(P\Big(Fi'(n), g\big(P(Fi'Se'(n),x)\big)\Big)\Big) \text{ when } SeSen = 0 \\
& \wedge \prod_{x\in I} g\big(P(n,x)\big) = P\Big(Fi(x), g\Big(P\big(Fi'(n), Se(x)\big)\Big)\Big) \quad \text{when} \quad SeSen = 1 \\
& \wedge \prod_{x\in I} g\big(P(n,x)\big) = P\Big(g\Big(P\big(Fi(n), Fi(x)\big)\Big), Se(x)\Big) \quad \text{when} \quad SeSen = 2 \\
& \wedge \prod_{x\in I} g\big(P(n,x)\big) = (\mu y)\Big[g\Big(P\big(Fi(n), P(x,y)\big)\Big) = 0\Big] \quad \text{when} \quad SeSen > 2\Big\}.
\end{aligned}$$

From the definition of the class $\mathcal{D}$ of relations it follows that the class $\mathcal{D}$ of functions is the smallest class of functions containing the initial functions $x+1$, $|x|$, $x-y$, $[x^y]$, and closed under the operations of substitution and under the operation of minimum. Hence by Theorem 3.1 of Grzegorczyk [2] it follows that the class $\mathcal{D}_1$ of the functions of one argument belonging to the class $\mathcal{D}$ is the smallest class containing the initial functions

$$\text{(0)} \qquad \begin{aligned} & x+1, \quad |x|, \quad Fi(x) - Se(x), \quad [Fi(x)^{Se(x)}], \\ & Fi(x), \quad Se(x), \quad P(0,x), \quad P(x,x), \quad P(x,0), \end{aligned}$$

and closed under the following operations:

1. of superposition,
2. if f belongs to $\mathcal{D}_1$, then $P\Big(Fi(x), f\big(Se(x)\big)\Big)$ belongs to $\mathcal{D}_1$,
3. if $f \in \mathcal{D}_1$, then $P\Big(f\big(Fi(x)\big), Se(x)\Big) \in \mathcal{D}_1$,
4. if $f \in \mathcal{D}_1$, then $(\mu x)\Big[f\big(P(x,y)\big) = 0\Big] \in \mathcal{D}_1$.

The function g satisfying $U\langle g\rangle$ is well-defined for all numbers $z=P(n,x)$. For $n<9$ $G_n(x)=g\big(P(n,x)\big)$ is equal to one of the initial functions (0). For $n>8$ the function G_n is obtained from the preceding functions by means of one of the four operations. Hence the function $G_n(x)=g\big(P(n,x)\big)$ is universal for all functions of the class $\mathcal{D}_1$.

Now we shall define two functionals $\omega, \varpi \in \mathcal{D}$ which establish a one-to-one correspondence between the set of integral functions of one argument and the set of natural functions of one argument assuming only two values, 0 and 1. For this purpose we shall define 6 functionals:

$$\omega_2\langle f\rangle(n)=\begin{cases} f\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even,}\\ f\left(-\dfrac{n+1}{2}\right) & \text{if } n \text{ is odd.}\end{cases}$$

$$\varpi_2\langle f\rangle(x)=\begin{cases} f(2x) & \text{if } x\geqslant 0,\\ f(2|x|\dot{-}1) & \text{if } x<0.\end{cases}$$

$$\omega_1\langle f\rangle(n)=\begin{cases} 2^{f(n)} & \text{if } f(n)\geqslant 0,\\ 3^{|f(n)|} & \text{if } f(n)<0.\end{cases}$$

$$\varpi_1\langle f\rangle(n)=\mathrm{Exp}\big(0,f(n)\big)-\mathrm{Exp}\big(1,f(n)\big).$$

$$\omega_0\langle f\rangle(0)=0,$$

$$\omega_0\langle f\rangle(n+1)=\mathrm{sgn}\Big(f\Big(\Big(\sum_{i\leqslant n}1\dot{-}\omega_0\langle f\rangle(i)\Big)\dot{-}1\Big)\dot{-}\sum_{i\leqslant n}\Big(\omega_0\langle f\rangle(i)\cdot\prod_{i\leqslant j\leqslant n}\omega_0\langle f\rangle(j)\Big)\Big),$$

$$\varpi_0\langle f\rangle(n)=\sum_{i\leqslant k_0}\Big(f(i)\cdot\prod_{i\leqslant j\leqslant k_0}f(j)\Big) \quad \text{for} \quad k_0=(\mu k)\Big[\sum_{i\leqslant k+1}1\dot{-}f(i)=n+2\Big].\ ^{5)}$$

The functional ω_2 transforms a function defined over all integers into a function defined over non negative integers. The functional ω_1 transforms a function defined over all non-negative integers and assuming integral values into a function defined over the set $\boldsymbol{N}$ and assuming the values of the set $\boldsymbol{N}$. And the functional ω_0 transforms a function defined over the set $\boldsymbol{N}$ and assuming the values of the set $\boldsymbol{N}$ into a function defined over the set $\boldsymbol{N}$ and assuming only two values 0 and 1. The functionals $\varpi_2, \varpi_1, \varpi_0$ are the inverse with respect to the functionals $\omega_2, \omega_1, \omega_0$ in such a manner that the following equations are satisfied:

$$\varpi_2\langle\omega_2\langle f\rangle\rangle=f, \qquad \varpi_1\langle\omega_1\langle f\rangle\rangle=f, \qquad \varpi_0\langle\omega_0\langle f\rangle\rangle=f.$$

The functional ω_0 is defined in such a manner that e. g. if the initial values of the function f are 3,5,0,2,4, then the initial values of the function $\omega_0\langle f\rangle$ are 0,1,1,1,0,1,1,1,1,1,0,0,1,1,0,1,1,1,1,0.

[5]) The symbols Π, Σ denote in these formulae limited sums and products.

Setting

$$\omega_0\langle\omega_1\langle\omega_2\langle f\rangle\rangle\rangle(x)=\omega\langle f\rangle(x),$$

$$\varpi_2\langle\varpi_1\langle\varpi_0\langle f\rangle\rangle\rangle(x)=\varpi\langle f\rangle(x),$$

we find that $\varpi\langle\omega\langle f\rangle\rangle(x)=f(x)$.

Now we define the set Z: $a\in Z\equiv U\langle\varpi\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f^a\rangle\rangle\rangle$.

The set Z has the following properties:

1. There exists exactly one real number a, such that $a\in Z$. From the definition of the set U it follows that there exists exactly one function g such that $U\langle g\rangle$. Hence there exists exactly one function $\Phi_7\langle\Theta_{\mathrm{A}}^{\mathrm{B}}\langle\omega\langle g\rangle\rangle\rangle$ such that

$$U\langle\varpi\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle\Phi_7\langle\Theta_{\mathrm{A}}^{\mathrm{B}}\langle\omega\langle g\rangle\rangle\rangle\rangle\rangle\rangle,$$

because Theorems 2.1 and 2.3 involve

$$\varpi\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle\Phi_7\langle\Theta_{\mathrm{A}}^{\mathrm{B}}\langle\omega\langle g\rangle\rangle\rangle\rangle\rangle=g.$$

2. Number a, such that $Z=\{a\}$ is not elementarily definable. Number a is the unique number for which $f^a=\Phi_7\langle\Theta_{\mathrm{A}}^{\mathrm{B}}\langle\omega\langle g\rangle\rangle\rangle$. When $a\in\mathfrak{D}$, then $f^a\in\mathcal{D}$, and $g\in\mathcal{D}$, but this is impossible because the universal function for a class of functions cannot belong to that class.

3. $0<a<1$. From the definition of the functional ω_0 it follows that $[a]=0$ and $a\geqslant 0$. $0,1\in\mathfrak{D}$, whence $0\neq a\neq 1$.

4. From Theorem 3.2 it follows that the open set $V=(0,1)-\{a\}$ is elementarily definable. It contains two components $(0,a)$ and $(a,1)$, but these components are not elementarily definable, according to Theorem 3.5.

Hence this proof implies the following

THEOREM 3.7. *There exists an open set $V\in\mathfrak{D}$ such that the components of V are not elementarily definable sets.*

THEOREM 3.8. *There exists a non-enumerable set $Z'\in\mathfrak{D}$ such that if $a\in Z'$, then $a\notin\mathfrak{D}$.*

Proof. It is possible to define in an elementary manner the class U' of all universal functions for the class $\mathcal{D}$ of functions. The class $\mathcal{D}$ is enumerable. Each universal function for the class $\mathcal{D}$ represents a permutation (perhaps with repetitions) of the elements of the class $\mathcal{D}$. The number of the permutations of a enumerable set is $2^{\aleph_0}$, hence $\overline{\overline{U'}}=2^{\aleph_0}$. Also the class Z' of numbers such that

$$a\in Z'\equiv U'\langle\varpi\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f^a\rangle\rangle\rangle$$

contains only the numbers which do not belong to $\mathfrak{D}$, and $\overline{\overline{Z'}}=2^{\aleph_0}$, and $Z'\subset(0,1)$.

According to the definition of the class U of the preceding Theorem the class U' can be defined as follows:

$$\begin{aligned}
U'\langle g\rangle \equiv \prod_{x\in I} g\big(P(0,x)\big) &= x+1 \\
\wedge \prod_{x\in I} g\big(P(1,x)\big) = |x| \wedge \prod_{x\in I} g\big(P(2,x)\big) &= Fi(x)-Se(x) \\
\wedge \prod_{x} g\big(P(3,x)\big) = [Fi(x)^{Se(x)}] \wedge \prod_{x} g\big(P(4,x)\big) &= Fi(x) \\
\wedge \prod_{x} g\big(P(5,x)\big) = Se(x) \wedge \prod_{x} g\big(P(6,x)\big) &= P(x,x) \\
\wedge \prod_{x} g\big(P(7,x)\big) = P(0,x) \wedge \prod_{x} g\big(P(8,x)\big) &= P(x,0) \\
\wedge \prod_{k,l} \sum_{n} \prod_{x} g\big(P(n,x)\big) &= g\Big(P\big(k,g(P(l,x))\big)\Big) \\
\wedge \prod_{k} \sum_{n} \prod_{x} g\big(P(n,x)\big) &= P\Big(Fi(x),g\big(P(k,Se(x))\big)\Big) \\
\wedge \prod_{k} \sum_{n} \prod_{x} g\big(P(n,x)\big) &= P\Big(g\big(P(k,Fi(x))\big),Se(x)\Big) \\
\wedge \prod_{k} \sum_{n} \prod_{x} g\big(P(n,x)\big) &= (\mu y)\Big[g\big(P(k,P(x,y))\big)=0\Big] \\
\wedge \prod_{n} \Big\{\Big(\prod_{x} g\big(P(n,x)\big) = x+1\Big) &\vee \Big(\prod_{x} g\big(P(n,x)\big) = |x|\Big) \\
\vee \Big(\prod_{x} g\big(P(n,x)\big) = Fi(x)-Se(x)\Big) &\vee \Big(\prod_{x} g\big(P(n,x)\big) = [Fi(x)^{Se(x)}]\Big) \\
\vee \Big(\prod_{x} g\big(P(n,x)\big) = Fi(x)\Big) &\vee \Big(\prod_{x} g\big(P(n,x)\big) = Se(x)\Big) \\
\vee \Big(\prod_{x} g\big(P(n,x)\big) = P(x,x)\Big) &\vee \Big(\prod_{x} g\big(P(n,x)\big) = P(0,x)\Big) \\
\vee \Big(\prod_{x} g\big(P(n,x)\big) = P(x,0)\Big) &\vee \sum_{k,l<n} \Big\{\prod_{x} g\big(P(n,x)\big) = g\Big(P\big(k,g(P(l,x))\big)\Big) \\
\vee \prod_{x} g\big(P(n,x)\big) &= P\Big(Fi(x),g\big(P(k,Se(x))\big)\Big) \\
\vee \prod_{x} g\big(P(n,x)\big) &= P\Big(g\big(P(k,Fi(x))\big),Se(x)\Big) \\
\vee \prod_{x} g\big(P(n,x)\big) &= (\mu y)\Big[g\big(P(k,P(x,y))\big)=0\Big]\Big\}\Big\}.
\end{aligned}$$

There remain several difficult problems concerning elementarily definable sets:

1. Is the closure an elementarily definable set, elementarily definable?

2. Is the class of all elementarily definable real numbers, elementarily definable?

3. Is the measure of an elementarily definable set, an elementarily definable number?

4. The analysis of continuous functions

A continuous function φ defined over a segment $[a,b]$ is elementarily defined by the sequence of its values $\varphi(r_n)$ at the rational points r_n. This remark was the basis for the not quite strict proofs of H. Weyl stating that the theory of continuous functions can be obtained in an elementary manner. In this section we shall give a strict shape to these ideas.

THEOREM 4.1. *There exists a functional $\Phi_{10}\in\mathfrak{D}$ such that if φ is a continuous function defined over the segment $[a,b]$ and $\mathrm{A}\big(\varphi(r_n),f_n\big)$ for $r_n\in[a,b]$ then $\mathrm{A}\big(\varphi(c),\Phi_{10}\|f\|\langle f^a,f^b,f^c\rangle\big)$ for any $c\in[a,b]$.*

Proof. From Theorem 2.10 it follows that the functionals Φ_{10},Φ_{11} are elementarily definable

$$\Phi_{10}\langle f^a,f^b,f^c\rangle(k)=(\mu n)\Big[n\geqslant 0\wedge a\leqslant r_n\leqslant b\wedge |r_n-c|<\frac{1}{k+1}\Big],$$

$$\Phi_{11}\|f\|\langle f^a,f^b,f^c\rangle(k)=(\mu x)\Big[\sum_{m\in N}\prod_{n>m}\Big|\frac{x}{k+1}-\frac{f\big(\Phi_{10}\langle f^a,f^b,f^c\rangle(n),n\big)}{n+1}\Big|<\frac{2}{3(k+1)}\Big].$$

From these definitions it follows that setting $\Phi(k)=\Phi_{10}\langle f^a,f^b,f^c\rangle(k)$ we have

$$\prod_{k\in N}|r_{\Phi(k)}-c|<\frac{1}{k+1}. \tag{1}$$

From the continuity it follows that

$$\varphi(c)=\lim\varphi(r_{\Phi(k)}); \tag{2}$$

from the assumption of the theorem it follows that

$$\prod_{k\in N}\Big|\varphi(r_{\Phi(k)})-\frac{f\big(\Phi(k),k\big)}{k+1}\Big|<\frac{1}{k+1}, \tag{3}$$

whence, from (1), (2), (3), $\varphi(c)=\lim f\big(\Phi(k),k\big)/(k+1)$. Also the functional Φ_{10} satisfies the required condition.

Hence, when we consider the classical operations of analysis it is not necessary to introduce the hiperfunctionals which transform the functionals representing the real functions into other functionals representing the real functions. It suffices to consider the sequences $\varphi(r_n)$

and the functionals which transforms the functions f such that $\mathrm{A}(\varphi(r_n), f_n)$ into other functions. This simplifies the notations of the following theorems.

THEOREM 4.2. *If φ is a continuous function defined over the segment $[a,b] \in \mathfrak{D}$, then*

$$\varphi \in \mathfrak{D} \equiv \{\varphi(r_n^{(a,b)})\} \in \mathfrak{D}$$

where $r_n^{(a,b)}$ is a sequence of rationals contained in the segment $[a,b]$.

Proof. The first implication results from the definition of definable function, the inverse one from Theorem 4.1.

THEOREM 4.3. *There exists a functional $\Phi_{12} \in \mathfrak{D}$ such that if a continuous function φ is defined over a segment $[a,b]$ and the values of the function $\varphi(c)$ for $c \in [a,b]$ constitute the segment $[a',b']$ and $\mathrm{A}(\varphi(r_n), f_n)$ for $r_n \in [a,b]$, then $\mathrm{A}(\varphi^{-1}(d), \Phi_{12}\|f\|\langle f^a, f^b, f^d\rangle)$ for $d \in [a',b']$ where $\varphi^{-1}(d) =$ the least c for which $\varphi(c) = d$.*

Proof.

$$\Phi_{12}\|f\|\langle f^a, f^b, f^d\rangle(n)$$

$$= (\mu x)\Big[\sum_{l \in N} \prod_{k \in N} \sum_{m \in N} a \leqslant r_m \leqslant b \wedge \left| d - \frac{f(m,k)}{k+1}\right| < \frac{2}{k+1}$$

$$\wedge \left| r_m - \frac{x}{n+1}\right| < \frac{1}{n+1} + \frac{1}{l+1}\Big].$$

Indeed, let us suppose that $c = \varphi^{-1}(d)$ and $x = f^c(n)$, whence

(1) $$\left|\frac{x}{n+1} - c\right| < \frac{1}{n+1} + \frac{1}{l+1} \quad \text{for some } l \in \boldsymbol{N}.$$

From the continuity it follows that for each $k \in \boldsymbol{N}$ there exists such r_m, that

(2) $$\frac{x}{n+1} \leqslant r_m \leqslant c \quad \text{and} \quad |\varphi(r_m) - d| < \frac{1}{k+1}.$$

On the other hand, we have

(3) $$\left|\varphi(r_m) - \frac{f(m,k)}{k+1}\right| < \frac{1}{k+1} \quad \text{for all } k \in \boldsymbol{N}.$$

Hence, from (1), (2), (3),

$$\left|\frac{x}{n+1} - r_m\right| < \frac{1}{n+1} + \frac{1}{l+1} \quad \text{and} \quad \left|\frac{f(m,k)}{k+1} - d\right| < \frac{2}{k+1}.$$

It suffices to show that $f^c(n)$ is the least x that satisfies these conditions. Indeed, suppose that $x < f^c(n)$, whence the function φ assumes the value d in the closed segment

$$\left[\frac{x-1}{n+1} + \frac{1}{l+1}, \frac{x+1}{n+1} - \frac{1}{l+1}\right] \quad \text{for no} \quad l \in \boldsymbol{N}.$$

Hence, there exists such k that, for all r_m of this segment,

$$|\varphi(r_m) - d| > \frac{3}{k+1},$$

whence, by (3)

$$\left|\frac{f(m,k)}{k+1} - d\right| > \frac{2}{k+1}.$$

Also x does not satisfy the required condition.

THEOREM 4.4. *There exists a functional $\Phi_{15} \in \mathfrak{D}$ such that if φ is a continuous function defined over the segment $[a,b]$ and $\mathrm{A}(\varphi(r_n), f_n)$ for $r_n \in [a,b]$, and*

$$c = \textit{least } e \textit{ such that } e \in [a,b], \textit{ and } \varphi(e) = \max_{[a,b]} \varphi,$$

then

$$f^c = \Phi_{15}\|f\|\langle f^a, f^b\rangle.$$

Proof. From Theorem 2.10 it follows that there exists a functional $\Phi_{13} \in \mathfrak{D}$ such that $r_{\Phi_{13}\langle f^a, f^b\rangle(n)}$ is a sequence of all rationals contained in the segment $[a,b]$. Namely

$$\Phi_{13}\langle f^a, f^b\rangle(0) = (\mu m)[a \leqslant r_m \leqslant b],$$

$$\Phi_{13}\langle f^a, f^b\rangle(n+1) = (\mu m)[m > \Phi_{13}\langle f^a, f^b\rangle(n) \wedge a \leqslant r_m \leqslant b].$$

Let

$$\Phi_{14}\|f\|\langle g\rangle(x,y) = f(g(x), y).$$

To the sequence $\{\varphi(r_{\Phi_{13}\langle f^a, f^b\rangle(n)})\}$ we apply Theorem 2.15 and obtain

$$f^d = \Phi_9\|\Phi_{14}\|f\|\langle\Phi_{13}\langle f^a, f^b\rangle\rangle\|,$$

where d is the upper limit of this sequence. Now let us notice that $c = \varphi^{-1}d$. Hence, from Theorem 4.3 and 2.3, we obtain $f^c = \Phi_7\langle\Phi_{12}\|f\|\langle f^a, f^b, f^d\rangle\rangle$. Also

$$\Phi_{15}\|f\|\langle f^a, f^b\rangle$$

$$= \Phi_7\langle\Phi_{12}\|f\|\langle f^a, f^b, \Phi_9\|\Phi_{14}\|f\|\langle\Phi_{13}\langle f^a, f^b\rangle\rangle\|\rangle\rangle.$$

THEOREM 4.5. *There exists a functional $\Phi_{15} \in \mathcal{D}$ such that if*

$$\max_{[a,b]} \varphi \geqslant d \geqslant \min_{[a,b]} \varphi$$

then there exists a number $c \in [a,b]$ such that $\varphi(c) = d$ and

$$f^c = \Phi_{16} \| f \| \langle f^a, f^b, f^d \rangle$$

where f satisfies the condition $\mathrm{A}(\varphi(r_n), f_n)$ for all $r_n \in [a,b]$.

Proof. Let us consider the function $\psi(e) = -|\varphi(e) - d|$. The function ψ assumes the maximum value 0 in a point c for which $\varphi(c) = d$. From Theorem 2.2 it follows that the function ψ is elementarily definable by means of the function φ. Hence, using Theorem 4.4, we obtain our theorem.

THEOREM 4.6. *There exists a functional $\Phi_{17} \in \mathcal{D}$ such that if φ is a continuous function defined over the segment $[a,b]$, then for any $c, d \in [a,b]$*

(1) $$\text{if } |c-d| < \frac{1}{\Phi_{17}\|f\|\langle f^a, f^b\rangle(k)+1} \quad \text{then} \quad |\varphi(c) - \varphi(d)| < \frac{1}{k+1}$$

where f satisfies the condition

(2) $$\mathrm{A}\big(\varphi(r_n), f_n\big) \quad \text{for each} \quad r_n \in [a,b].$$

Proof. The function φ being continuous, is uniformly continuous in the segment $[a,b]$. Hence

(3) $$\prod_{k \in N} \sum_{l \in N} \prod_{m,n \in N} \Big\{\Big(a \leqslant r_m, r_n \leqslant b \wedge |r_m - r_n| < \frac{1}{l+1}\Big) \rightarrow |\varphi(r_m) - \varphi(r_n)| < \frac{1}{3(k+1)}\Big\}.$$

From (2) and (3) it follows that

(4) $$\prod_{k \in N} \sum_{l \in N} \prod_{m,n \in N} \Big\{\Big(a \leqslant r_m, r_n \leqslant b \wedge |r_m - r_n| < \frac{1}{l+1}\Big) \rightarrow \sum_{j \in N} \prod_{i > j} \Big|\frac{f(n,i)}{i+1} - \frac{f(m,i)}{i+1}\Big| < \frac{1}{3(k+1)}\Big\}.$$

Now let

(5) $$\Phi_{17}\|f\|\langle f^a, f^b\rangle(k) = (\mu l)\Big[\prod_{m,n \in N} \Big\{a \leqslant r_m, r_n \leqslant b \wedge |r_m - r_n| < \frac{1}{l+1} \rightarrow \sum_{j \in N} \prod_{i > j} \Big|\frac{f(n,i)}{i+1} - \frac{f(m,i)}{i+1}\Big| < \frac{1}{3(k+1)}\Big\}\Big].$$

Hence it follows that, for any $r_m, r_n \in [a,b]$,

(6) $$\text{if} \quad |r_m - r_n| < \frac{1}{\Phi_{17}\|f\|\langle f^a, f^b\rangle(k)+1}, \quad \text{then} \quad \sum_{j \in N} \prod_{i > j} \Big|\frac{f(n,i) - f(m,i)}{i+1}\Big| < \frac{1}{3(k+1)}.$$

Let us consider two real numbers $c, d \in [a,b]$ such that

(7) $$|c - d| < \frac{1}{\Phi_{17}\|f\|\langle f^a, f^b\rangle(k)+1}.$$

From the continuity of φ it follows that there are two rationals such that

(8) $$c \leqslant r_m, \qquad r_n \leqslant d,$$

(9) $$|\varphi(c) - \varphi(r_m)| < \frac{1}{3(k+1)},$$

(10) $$|\varphi(d) - \varphi(r_n)| < \frac{1}{3(k+1)}.$$

From (7) and (8) we obtain

(11) $$|r_m - r_n| < \frac{1}{\Phi_{17}\|f\|\langle f^a, f^b\rangle(k)+1}.$$

From (6) and (11) it follows that

(12) $$\Big|\lim_{i \to \infty} \frac{f(n,i)}{i+1} - \lim_{i \to \infty} \frac{f(m,i)}{i+1}\Big| < \frac{1}{3(k+1)}.$$

From (2) we obtain

(13) $$\varphi(r_m) = \lim_{i \to \infty} \frac{f(m,i)}{i+1},$$

(14) $$\varphi(r_n) = \lim_{i \to \infty} \frac{f(n,i)}{i+1}.$$

From (9), (10), (12), (13), (14) it follows that $|\varphi(c) - \varphi(d)| < 1/(k+1)$. This completes the proof of condition (1).

THEOREM 4.7. *There exists a functional $\Phi_{18} \in \mathcal{D}$ such that if a function φ is differentiable over the segment $[a,b]$ and $\mathrm{A}\big(\varphi(r_n), f_n\big)$ for $r_n \in [a,b]$, then*

$$\mathrm{A}\big(\varphi'(c), \Phi_{18}\|f\|\langle f^a, f^b, f^c\rangle\big) \quad \text{for} \quad c \in (a,b).$$

Proof. We start from the definition

$$\varphi'(c) = \lim_{h,g \to 0} \frac{\varphi(c+|h|) - \varphi(c-|g|)}{|h|+|g|}.$$

Hence

$$\varphi'(c) = \lim_{m\to\infty}\left(\left(\varphi\left(\frac{f^c(m)+1}{m+1}\right) - \varphi\left(\frac{f^c(m)-1}{m+1}\right)\right)\cdot\left(\frac{m+1}{2}\right)\right).$$

On the other hand, it is true that

$$\varphi\left(\frac{f^c(m)+1}{m+1}\right) = \lim_{l\to\infty}\frac{f\left(N_w\left(f^c(m)+1, m+1\right), l\right)}{l+1},$$

$$\varphi\left(\frac{f^c(m)-1}{m+1}\right) = \lim_{l\to\infty}\frac{f\left(N_w\left(f^c(m)-1, m+1\right), l\right)}{l+1};$$

Hence

$$\varphi'(c) = \lim_{m\to\infty}\lim_{l\to\infty}\frac{\left(f\left(N_w\left(f^c(m)+1, m+1\right), l\right) - f\left(N_w\left(f^c(m)-1, m+1\right), l\right)\right)(m+1)}{2(l+1)}.$$

Now from Theorem 3.4 it follows that the conditions of Theorem 2.11 are satisfied, and by the double application of Theorem 2.11 we find that there exists the required functional $\Phi_{18} \in \mathfrak{D}$.

For the end-points of the segment $[a,b]$ we can define the derived function by means of the one-sided differentiation. The corresponding functionals are also elementarily definable.

THEOREM 4.8. *There exists a functional $\Phi_{19} \in \mathfrak{D}$ such that if φ is a function continuous in the segment $[a,b]$ and $\mathrm{A}\left(\varphi(r_n), f_n\right)$ for $r_n \in [a,b]$, then*

$$\mathrm{A}\left(\int_a^b \varphi(c)\,dc, \Phi_{19}\|f\|\langle f^a, f^b\rangle\right).$$

Proof.

$$\int_a^b \varphi(c)\,dc = \lim_{n\to\infty}\sum_{i=f^a(n)+1}^{f^b(n)}\frac{1}{(n+1)}\cdot\varphi\left(\frac{i}{n+1}\right)$$

$$= \lim_{n\to\infty}\sum_{i=f^a(n)+1}^{f^b(n)}\frac{1}{n+1}\cdot\lim_{k\to\infty}\frac{f\left(N_w(i, n+1), k\right)}{k+1}.$$

Hence, by Theorem 3.4, and by the double application of Theorem 2.11, we find that there exists the required functional $\Phi_{19} \in \mathfrak{D}$.

In conformity with the theorems of this section we can say that the classical operations on continuous functions are elementarily definable. Hence these operations do not exceed the class $\mathfrak{D}$. A continuous function $\varphi \in \mathfrak{D}$, defined in the segment $[a,b] \in \mathfrak{D}$, assumes the maximum value at a point $c \in \mathfrak{D}$, is elementarily definable, uniformly continuous in this segment, and

$$\varphi^{-1} \in \mathfrak{D}, \qquad \varphi' \in \mathfrak{D}, \qquad \int_a^d \varphi(x)\,dx \in \mathfrak{D} \quad \text{for} \quad a \leqslant d \leqslant b.$$

For the functions elementarily definable but not continuous the theorems analogical to the familiar theorems of analysis are not true. For example

THEOREM 4.9. *There exists a real function $\varphi \in \mathfrak{D}$ which assumes only two values 0 and 1, and assumes the value 1 at one point which is not elementarily definable.*

Proof. Let Z be the set defined in the proof of Theorem 3.6. We set

$$\varphi(a) = \begin{cases} 0 & \text{for} \quad a \notin Z, \\ 1 & \text{for} \quad a \in Z, \end{cases}$$

$$\Phi\langle f\rangle(n) = \begin{cases} 1 & \text{if} \quad U\langle\omega\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f\rangle\rangle\rangle, \\ 0 & \text{if} \quad \sim\left(U\langle\omega\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f\rangle\rangle\rangle\right). \end{cases}$$

It is evident that

$$f^{\varphi(a)} \doteq \Phi\langle f^a\rangle.$$

Hence $\varphi \in \mathfrak{D}$.

THEOREM 4.10. *There exists a one-to-one mapping function $\varphi \in \mathfrak{D}$ such that the set of its values is elementarily definable, and $\varphi^{-1} \notin \mathfrak{D}$.*

Proof. Let

$$\varphi(a) = \begin{cases} 2 & \text{for} \quad a \in Z, \\ a & \text{for} \quad a \notin Z. \end{cases}$$

$$\Phi\langle f\rangle(n) = \begin{cases} 2n & \text{if} \quad U\langle\omega\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f\rangle\rangle\rangle, \\ f(n) & \text{if} \quad \sim\left(U\langle\omega\langle\Theta_{\mathrm{B}}^{\mathrm{A}}\langle f\rangle\rangle\rangle\right). \end{cases}$$

Hence $f^{\varphi(a)} = \Phi\langle f^a\rangle$. Then $\varphi \in \mathfrak{D}$. Thus the function φ establishes a one-to-one correspondence between the segment $[0,1]$ and the set $([0,1]-Z)+\{2\}$. The function $\varphi^{-1} \notin \mathfrak{D}$ because $\varphi^{-1}(2) = a \in Z$, and $a \notin \mathfrak{D}$, and for any function $\chi \in \mathfrak{D}$ it is true that if $e \in \mathfrak{D}$, then $\chi(e) \in \mathfrak{D}$.

Finally, we shall give some remarks concerning systems of arithmetic. Let us consider the system S of Peano's arithmetic formalized in the simple theory of types. We assume that integers are individuals of the lowest type. The familiar arithmetical functions $+$ and $\cdot$ can be the primitive notions. We state a restriction in the formulation of the axiom of definability

$$(\alpha) \qquad \sum_R \prod_{X,Y,Z,\ldots} R(X,Y,Z,\ldots) \equiv \Xi(\ldots,X,Y,Z,\ldots).$$

We admit in this scheme only those formulae Ξ in which each quantifier bounds a variable of the lowest type. Other axioms are the familiar ones.

A relation ϱ is said to be definable in a system if there exists in this system a theorem of the form (α) such that the expression $\Xi(\dots X, Y, Z, \dots)$ is a possible definiens of the relation ϱ *i. e.* $\varrho(\dots x, y, z, \dots)$ if and only if $\dots x, y, z, \dots$ satisfy the formula Ξ.

From this definition it follows that a relation is definable in S if and only if it is elementarily definable. Hence from Theorem 3.3 it follows that the existence of a non Borelian set is unprovable in the system S. But the general question, whether the class $\mathfrak{D}$ constitutes the model of the system S remains open, because we cannot decide whether the axiom of extensionality is satisfied in the domain $\mathfrak{D}$. There remains also another task: to verify whether the theory of continuous functions can be deduced in S. Perhaps this theory can be obtained in S without the use of the axiom of extensionality.

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On uniformization of functions (I)

by

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Let I be the unit interval $0 \leqslant x \leqslant 1$, and let $\mathfrak{F}$ be the class of all continuous mappings f of I into itself such that $f(0)=0$ and $f(1)=1$. If $f, \varphi \in \mathfrak{F}$, then $f\varphi \in \mathfrak{F}$ too. The symbol $f\varphi$ denotes always the superposition of f and φ.

We shall prove the following

THEOREM I [1]. *If $f_1, f_2, \dots, f_n \in \mathfrak{F}$ are functions such that*

(α) *for each $i=1,2,\dots,n$, there is a sequence $0=x_0<x_1<x_2<\dots<x_r$ such that f_i is either non-decreasing or non-increasing in every interval $\langle x_{j-1}, x_j \rangle$, $j=1,2,\dots,r$,*

then there exist functions $\varphi_1, \varphi_2, \dots, \varphi_n \in \mathfrak{F}$ such that

$$f_1\varphi_1 = f_2\varphi_2 = \dots = f_n\varphi_n. \tag{1}$$

Theorem I has the following simple interpretation. There are n paths which are going to the top of a mountain. The paths need not always go upwards, some segments of the paths may be directed downwards. On each of the paths a tourist is climbing. Theorem I asserts that the tourists can climb to the top of the mountain in such a way that, at every moment, all of them are on the same level (of course, it may happen that, in some time intervals, some of the tourists must return from the previously covered segments of the paths). To make it clear, let us suppose that the paths are the curves

$$p_1(x),\ p_2(x),\ \dots,\ p_n(x),$$

where $p_j(x)$ $(j=1,2,\dots,n)$ is a mapping of I into the three-dimensional space. Let $f_j(x)$ be the height (the third coordinate) of the point $p_j(x)$.

[1]) K. Zarankiewicz, *Un théorème sur l'uniformisation de fonctions continues et son application à la démonstration du théorème de F. J. Dyson sur les transformations de la surface sphérique*, Bull. Acad. Pol. Sc. Cl. III 2 (1954), p. 117-120.

During the print of this paper the authors found out that a theorem similar to Theorems I and II was proved by T. Homma, *A theorem on continuous functions*, Kōdai Math. Sem. Reports 1 (1952), p. 13-16.

Homma's hypothesis about $f_1, f_2, \dots, f_n$ is other than that in this paper. The example on p. 340 is also given in Homma's paper.