pour obtenir l'existence d'une droite D_0 telle que $P(D_0)=0$; il en résulte que le plan mené par D_0 et L coupe V suivant une section dont le centre de gravité se trouve sur L, c. q. f. d.

Il est évident que le théorème reste vrai quand on entend par centre de gravité d'une section le centre de gravité du contour suivant lequel elle tranche S.

Envisageons maintenant un corps convexe dont la surface S est assujettie à des conditions de régularité garantissant en tout point P de S l'existence de rayons principaux de courbure, $R_1(P)$ et $R_2(P)$ $(R_1 \geqslant R_2)$, et leur continuité comme fonctions de P sur S. En écrivant $x=R_1(P)$, $y=R_2(P)$ on définit une représentation de S sur le plan cartésien (x,y). D'après (II), il y a sur S deux points antipodiques, P et Q, ayant une image commune (x_0,y_0). Il s'ensuit que les deux courbures principales en P et en Q sont respectivement égales. On peut rendre ce fait plus intuitif au dépens de la précision du langage en disant:

(13) *Il y a sur S deux antipodes telles que leurs voisinages infiniment proches sont congruents.*

Remarquons maintenant que l'on peut définir l'antipodisme sur S par la condition que la corde PQ passe par un point fixe O donné d'avance à l'intérieur de V. Considérons une suite $\{O_n\}$ de tels points convergente vers un point P_0 situé sur S. D'après (13), on trouve sur S une suite de points $\{P_n\}$ et une autre $\{Q_n\}$ tels que leurs voisinages respectifs sont congruents et que la corde P_nQ_n contient O_n. Comme O_n tend vers P_0 — pour $n\to\infty$ — on peut extraire de $\{P_n\}$ ou de $\{Q_n\}$ une suite partielle $\{R_{n_j}\}$ tendant vers P_0. Appelons T_{n_j} l'antipode de R_{n_j} par rapport à O_{n_j} — on peut extraire de $\{T_{n_j}\}$ une suite partielle de points, $\{T'_i\}$, tendant vers un point T' de S, leurs antipodes respectifs R'_i (par rapport aux O'_i) tendent toujours vers P_0. Il faut distinguer deux cas: 1^0 $T'\neq P_0$, 2^0 $T'=P_0$. Dans le premier cas il existe sur S un point différent de P_0 dont le voisinage est congruent à celui de P_0, dans le second cas il y a dans tout voisinage de P_0 deux points dont les voisinages respectifs sont congruents. Comme P_0 est arbitraire, on peut dire:

(14) *Les points de S peuvent être divisés en deux catégories: la première consiste de paires de points aux voisinages respectivement congruents, la deuxième de limites de telles paires.*

Exemple. Supposons que la Terre est un ellipsoïde de rotation et que le Pôle Nord soit plus applati que le Pôle Sud. Alors les Pôles appartiennent à la deuxième catégorie, tous les autres points à la première.

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Algebraic models of axiomatic theories *)

by

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Let $\mathcal{S}$ be a sentential calculus which contains the signs of disjunction, of conjunction and of implication, and perhaps some other sentential operators. We suppose that all theorems of the positive logic are theorems in $\mathcal{S}$. The system $\mathcal{S}$ determines a first order [1]) functional calculus $\mathcal{S}^*$.

The subject of the papers [10], [12] was the general algebraic method of the examination of a non-specified functional calculus $\mathcal{S}^*$, with applications to the special functional calculi: of the two-valued logic $\mathcal{S}^*_\varkappa$, of Heyting $\mathcal{S}^*_\chi$, of Lewis $\mathcal{S}^*_\lambda$, of the positive logic $\mathcal{S}^*_\pi$ and of the minimal logic $\mathcal{S}^*_\mu$.

In this paper I shall apply the method mentioned above to the study of theories formalized on the basis of a logical calculus, which may be either a functional calculus $\mathcal{S}^*$, or a functional calculus $\mathcal{S}^*$ with equality. The theories with functions are included.

The first part of this paper contains the general definition of the model of a formalized theory based on a functional calculus $\mathcal{S}^*$ (or $\mathcal{S}^*$ with equality) where $\mathcal{S}^*$ is not exactly specified. This definition is closely related to the general notion of satisfiability introduced in [10]. If $\mathcal{S}^*$ is the classical functional calculus, this definition is a generalization of the definition of the model in the usual sense, called here the *semantic model* [2]). Known theorems on models of elementary axiomatic theories based on the classical logic hold also in the general case [3]).

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[1]) For the exact description of the systems $\mathcal{S}$ and $\mathcal{S}^*$ see [10], §§ 1, 2.

[2]) See [8], p. 356.

[3]) The results of the first part establish an easy generalization of the results of [10] (see also L. Henkin [1]). They can also be regarded as generalizations of some investigations of J. Łoś [4]. The majority of them are necessary for the second part, containing the essential results of this paper.

The second part of the paper treats of axiomatic theories based on the Heyting functional calculus $\mathcal{S}^*_\chi$. A theory $\mathcal{T}$ is said to be *constructive* provided that:

1° if $\sum_{x_p} a$ is a theorem in $\mathcal{T}$, then there exists a term ξ such that the formula $a\binom{\xi}{x_p}$ which results from a by the substitution of ξ for x_p [4]) is a theorem in $\mathcal{T}$;

2° if $a+\beta$ is a theorem in $\mathcal{T}$, then either a or β is a theorem in $\mathcal{T}$.

If a theory $\mathcal{T}$ is constructive, one can associate with each formula $a=\Xi\beta$ (where Ξ is a sequence of quantifiers and β contains no quantifiers) a sequence $a_1, a_2, \ldots$ of formulas without quantifiers such that a is a theorem if and only if at least one of the formulas $a_1, a_2, \ldots$ is a theorem. This remark may be regarded as an analogue of Herbrand's [5]) theorem.

I shall formulate a necessary and sufficient condition for a theory $\mathcal{T}$ to be constructive. This condition has a purely algebraic form. I shall prove that the theory $\mathcal{T}$ is constructive whenever all axioms of $\mathcal{T}$ (except the axioms of equality) belong to the least set Z^0 of formulas of $\mathcal{T}$ such that: 1° Z^0 contains all elementary formulas of $\mathcal{T}$, 2° if $\beta, \gamma \in Z^0$, then $\beta\cdot\gamma \in Z^0$, 3° if $\gamma \in Z^0$, then $\beta\to\gamma$, $\neg\beta$, $\prod_{x_p}\gamma \in Z^0$. In particular, every theory whose axioms are equalities is constructive. For instance, the theories of groups, of rings, of Boolean algebras are constructive. More generally, if we eliminate the sign $+$ and Σ from all axioms of a general theory $\mathcal{T}$, that do not belong to Z^0, using de Morgan's laws, we obtain a weaker theory $\mathcal{T}'$ which is constructive. In this way one can obtain a constructive fragment of arithmetic.

The method used in the second part is similar to that used in [12], and is due essentially to Tarski and McKinsey [6]).

As an application I obtain the theorem 4.11, which is stronger than the fundamental theorem (χ) of [12] about $\mathcal{S}^*_\chi$.

§ 1. Elementary axiomatic theories

Let $\mathcal{S}$ [7]) be a fixed consistent system of sentential calculus containing:

(a) the *disjunction sign* $+$, the *conjunction sign* $\cdot$, the *implication sign* $\to$;

(b) some other *binary sentential operators* $o_1, \ldots, o_r$;

[4]) We assume that the necessary changes in the bound occurrences of variables of a were performed before the operation of substitution.

[5]) See [3].

[6]) Cf. [5], [6].

[7]) See [10], § 1.

(c) some *unary sentential operators* $o^1, \ldots, o^s$.

The set of operators mentioned in (b) or (c) may be empty. The rules of inference in $\mathcal{S}$ are *modus ponens*, and the *rule of replacement of equivalent parts*. We suppose that all theorems of the positive sentential calculus are theorems of $\mathcal{S}$.

The system $\mathcal{S}$ determines in an obvious way a system $\mathcal{S}^*$ [8]) of the first order *functional calculus* with the following rules of inference: *modus ponens*, the *rule of replacement of equivalent parts*, the *rule of substitution for individual variables*, and the four known *rules for quantifiers*. The *theorems* in $\mathcal{S}^*$ are all substitutions of theorems of $\mathcal{S}$ and all their consequences.

The system $\mathcal{S}$ determines also a kind of abstract algebras (called *$\mathcal{S}$-algebras* [9])) with algebraical operations corresponding to the logical sentential operators $+, \cdot, \to, o_1, \ldots, o^s$. The $\mathcal{S}$-algebras which are the matrices of the system $\mathcal{S}$, are relatively pseudocomplemented lattices (with the sum (join) $a+b$, and the product (meet) $a\cdot b$) having the unit element e, which is the distinguished element corresponding to the logical value of truth. If an $\mathcal{S}$-algebra is a complete lattice, it is called an *$\mathcal{S}^*$-algebra*. We shall suppose that the system $\mathcal{S}$ has the following property (E) [10]): given a denumerable set of infinite sums and products in an $\mathcal{S}$-algebra A, $a_n = \sum_i a_{ni}$, $b_n = \prod_i b_{ni}$, there is an isomorphism (with respect to all the operations $+, \cdot, \to, o_1, \ldots, o^s$) of A into an $\mathcal{S}^*$-algebra which preserves all these sums and products.

Assume the following notations. Let I_0 always denote the set of all positive integers; Λ — the empty set; I — a fixed set of integers, such that $I_0 \subset I$; I'_k, I''_l (for every $k, l \in I_0$) — some fixed sets of positive integers which can be empty. We always suppose that there exists $k \in I_0$ such that $I'_k \neq \Lambda$. We suppose also, that the condition $I''_l \neq \Lambda$, for some $l \in I_0$, implies that $1 \in I'_2$.

An elementary theory $\mathcal{T}(\mathfrak{A})$ based on the system of logic $\mathcal{S}^*$ and on the set of axioms $\mathfrak{A}$ can briefly be described as follows:

The primitive symbols of $\mathcal{T}(\mathfrak{A})$ are parentheses and

individual variables x_i, where $i \in I_0$;

individual constants x_i, where $i \in I - I_0$;

functors with l arguments f^l_n (*i. e.* symbols for functions from individuals to individuals) where $l \in I_0$ and $n \in I''_l$;

predicates (*i. e.* symbols for relations) *with k arguments* F^k_m, where $k \in I_0$ and $m \in I'_k$;

[8]) See [10], § 2.

[9]) See [10], § 3.

[10]) See [10], p. 69.

sentential operators of the system $\mathcal{S}$, mentioned in (a), (b), (c);

quantifiers $\sum\limits_{x_i}$ and $\prod\limits_{x_i}$ where $i \in I_0$.

The relation F_1^2 will play an outstanding part in our consideration. More exactly, F_1^2 is the *sign of equality* of the system $\mathcal{T}(\mathfrak{A})$ (see the axioms (*) below).

Since for $l \in I_0$, I_l'' may be the empty set, it is possible that the theory $\mathcal{T}(\mathfrak{A})$ contains no functors.

Among the expressions which can be constructed from these signs we distinguish *terms* and *formulas*.

The set J_0 of all terms is the least set such that

(i) $x_i \in J_0$ for $i \in I$;

(ii) if $\xi_1, \ldots, \xi_l \in J_0$, then $f_n^l(\xi_1, \ldots, \xi_l) \in J_0$ for $l \in I_0$ and $n \in I_l''$. In the case of $I_l'' = \Lambda$ for each $l \in I_0$, the set J_0 is the set of all x_i where $i \in I$.

If $\xi \in J_0$ and $x_{i_1}, \ldots, x_{i_l}$ are all individual variables which appear in ξ and $i_1 < i_2 < \ldots < i_l$, then we shall write also $\xi(x_{i_1}, \ldots, x_{i_l})$.

The set T of all formulas in $\mathcal{T}(\mathfrak{A})$ is the least set fulfilling the following conditions:

(i) $F_m^k(\xi_1, \ldots, \xi_k) \in T$ where $\xi_1 \ldots, \xi_k \in J_0$, $k \in I_0$ and $m \in I_k'$;

(ii) if $\alpha, \beta \in T$ and $i \in I_0$, then $(\alpha + \beta) \in T$, $(\alpha \cdot \beta) \in T$, $(\alpha \rightarrow \beta) \in T$, $(\alpha o_k \beta) \in T$ $(k = 1, \ldots, r)$, $(o^k \alpha) \in T$ $(k = 1, \ldots, s)$, $\left(\sum\limits_{x_i} \alpha\right) \in T$, $\left(\prod\limits_{x_i} \alpha\right) \in T$.

We shall write, for brevity, $(\alpha \equiv \beta)$ instead of $\big((\alpha \rightarrow \beta) \cdot (\beta \rightarrow \alpha)\big)$.

In writing formulas we shall practice the omission of the parentheses, the rules being that

1° each of the operators $\cdot, +, \equiv$ binds less strongly than the previous one;

2° each of the operators o^k binds an expression more strongly than any of the binary operators;

3° the quantifiers bind more strongly than any of the operators mentioned in 1° and 2°.

We assume that the notion of *free* and *bound occurrence* of an individual variable is familiar. A formula $\alpha \in T$ is said to be *closed* if it contains no free occurrence of an individual variable x_i $(i \in I_0)$.

We assume that the set $\mathfrak{A}$ of all axioms of $\mathcal{T}(\mathfrak{A})$ consists of some closed formulas belonging to T. If the sign of equality F_1^2 appears among the primitive signs of $\mathcal{T}(\mathfrak{A})$ then the set $\mathfrak{A}$ contains the following set $\mathfrak{E}$ of the axioms of equality:

$$(*) \qquad \begin{gathered} \prod_{x_1} F_1^2(x_1, x_1) \\ \Xi\left(F_1^2(x_1, x_2) \rightarrow \left(\alpha \rightarrow \alpha\binom{x_2}{x_1}\right)\right) \end{gathered}$$

for $\alpha \in T$, where $\alpha\binom{x_2}{x_1}$ results from α by the substitution[4]) of x_2 for x_1, and where Ξ is written instead of the sequence of the quantifiers $\prod\limits_{x_j}$ binding all free occurrences of individual variables in the next formula.

The set of all axioms of $\mathcal{T}(\mathfrak{A})$ except the axioms of equality will be denoted by $\mathfrak{A}_0$.

The set $K(\mathfrak{A})$ of theorems of $\mathcal{T}(\mathfrak{A})$ is the least set such that

1° $K(\mathfrak{A})$ contains all axioms of $\mathcal{T}(\mathfrak{A})$;

2° $K(\mathfrak{A})$ contains all substitutions of all theorems of $\mathcal{S}$;

3° if $\alpha \in K(\mathfrak{A})$ and β is obtained from α by the admissible replacement of all free occurrences of x_i $(i \in I_0)$ by a term ξ, then $\beta \in K(\mathfrak{A})$; we shall always assume that the necessary changes in the bound occurrences of variables of α were performed before the operation of substitution;

4° if $\alpha \in K(\mathfrak{A})$, $(\alpha \rightarrow \beta) \in K(\mathfrak{A})$ then $\beta \in K(\mathfrak{A})$;

5° if $(\alpha \rightarrow \prod\limits_{x_i} \beta) \in K(\mathfrak{A})$ then $(\alpha \rightarrow \beta) \in K(\mathfrak{A})$; if $(\sum\limits_{x_i} \alpha \rightarrow \beta) \in K(\mathfrak{A})$ then $(\alpha \rightarrow \beta) \in K(\mathfrak{A})$;

6° if there is no free occurrence of x_i in α (in β), $i \in I_0$ and if $(\alpha \rightarrow \beta) \in K(\mathfrak{A})$, then $(\alpha \rightarrow \prod\limits_{x_i} \beta) \in K(\mathfrak{A})$ $\big($then $(\sum\limits_{x_i} \alpha \rightarrow \beta) \in K(\mathfrak{A})\big)$;

7° if $\alpha \in K(\mathfrak{A})$ and $(\gamma \equiv \delta) \in K(\mathfrak{A})$ and if γ is a part of α, then the formula β obtained from α by replacing the part γ by δ is also in $K(\mathfrak{A})$.

If $\alpha \in K(\mathfrak{A})$ we shall write $\mathfrak{A} \vdash \alpha$.

The theory $\mathcal{T}(\mathfrak{A})$ is *consistent* if there is a formula $\alpha \in T$ such that $\alpha \operatorname{non} \in K(\mathfrak{A})$.

§ 2. The Lindenbaum algebra $L(\mathfrak{A})$

Let $\mathcal{T}(\mathfrak{A})$ be a consistent theory, based on a fixed system of logic $\mathcal{S}^*$. For every $\alpha \in T$ let $|\alpha|$ denote the class of all $\beta \in T$ such that $\mathfrak{A} \vdash \alpha \equiv \beta$. Let $L(\mathfrak{A})$ be the set of all cosets $|\alpha|$ where $\alpha \in T$. We define in $L(\mathfrak{A})$ the algebraical operations $+, \cdot, \rightarrow, o_1, \ldots, o^s$ as follows: $|\alpha| o |\beta| = |\alpha o \beta|$ if o is one of the binary logical operations of $\mathcal{S}$ and $o|\alpha| = |o\alpha|$ if o is one of the unary operations of $\mathcal{S}$. Since the relation $\mathfrak{A} \vdash \alpha \equiv \beta$ between α and β is a congruence relation in the sense of modern algebra, the definition of operations in $L(\mathfrak{A})$ is correct. The element $|\alpha|$ where $\alpha \in K(\mathfrak{A})$ will be denoted by e.

2.1. *The algebra* $\langle L(\mathfrak{A}); e; +, \cdot, \rightarrow o_1, \ldots, o^s \rangle$ *is an* $\mathcal{S}$*-algebra; more precisely, it is a relatively pseudocomplemented lattice with the unit element* e.

The algebra $L(\mathfrak{A})$ is analogous to the algebra $L(R)$[11] of [10].

2.2. $|a| \subset |\beta|$ *if and only if* $\mathfrak{A} \vdash a \rightarrow \beta$. $|a| = e$ *if and only if* $\mathfrak{A} \vdash a$.

2.3. *For every* $a \in T$

$$(*) \qquad \sum_{\xi \in J_0} \left| a\binom{\xi}{x_i} \right| = \left| \sum_{x_i} a \right|,$$

$$(**) \qquad \prod_{\xi \in J_0} \left| a\binom{\xi}{x_i} \right| = \left| \prod_{x_i} a \right|.$$

The proof, similar to that of 4.3 in [10], is omitted. Obviously, $\sum\limits_{\xi \in J_0}$ and $\prod\limits_{\xi \in J_0}$ on the left side of the equalities $(*)$ and $(**)$ are the signs of sum and product in $L(\mathfrak{A})$, respectively, and the signs $\sum\limits_{x_i}$, $\prod\limits_{x_i}$ on the right side of these equalities are quantifiers.

By an analogical reasoning to that used in the proof of 2.3 we obtain also

$$\text{(i)} \qquad \sum_{p \in I} \left| a\binom{x_p}{x_i} \right| = \left| \sum_{x_i} a \right|,$$

$$\text{(ii)} \qquad \prod_{p \in I} \left| a\binom{x_p}{x_i} \right| = \left| \prod_{x_i} a \right|.$$

If no functors occur among the primitive symbols of $\mathfrak{T}(\mathfrak{A})$, the equalities $(*)$ and $(**)$ are reducible to the equalities (i) and (ii) respectively.

According to [10][12] an $\mathcal{S}$-homomorphism ($\mathcal{S}$-isomorphism) h of $L(\mathfrak{A})$ into an $\mathcal{S}$-algebra A is said to be an *$\mathcal{S}^*$-homomorphism* (*$\mathcal{S}^*$-isomorphism*) if h preserves all the sums $(*)$ and all the products $(**)$. An $\mathcal{S}^*$-algebra A is said to be an *$\mathcal{S}^*$-extension* of $L(\mathfrak{A})$ if there is an $\mathcal{S}^*$-isomorphism of $L(\mathfrak{A})$ into A. Obviously such an $\mathcal{S}^*$-extension exists since $\mathcal{S}$ has the property (E).

§ 3. Algebraic models of elementary axiomatic theories

We shall consider a theory $\mathfrak{T}(\mathfrak{A})$ based on a fixed logical calculus $\mathcal{S}^*$. Let J be a non-empty set. Let A be an $\mathcal{S}^*$-algebra. The class of all mappings of the Cartesian product J^k into A will be denoted by $\boldsymbol{F}^k(J, A)$. φ^k, ψ^k will always denote functions belonging to $\boldsymbol{F}^k(J, A)$. Let $\boldsymbol{f}^l(J)$ be the class of all mappings of the Cartesian product J^l into J. The letters τ^l, σ^l will always denote elements of $\boldsymbol{f}^l(J)$.

[11]) Cf. [10], p. 69.

[12]) See [10], p. 70-71.

Every formula $a \in T$ may be interpreted as a (J, A)-*functional*[13]) denoted by $(J, A)\Phi_a$, by regarding

(a) all individual variables x_i $(i \in I_0)$ (individual constants x_i, where $i \in I - I_0$) as variables running over J (as fixed elements of J);

(b) all functors f_n^l $(l \in I_0,\ n \in I_l'')$ as fixed elements of $\boldsymbol{f}^l(J)$;

(c) all k-argument predicates F_m^k as fixed elements of $\boldsymbol{F}^k(J, A)$;

(d) each of the logical operations of $\mathcal{S}$ mentioned in § 1 ((a), (b), (c)) as a corresponding algebraical operation in A;

(e) the logical quantifiers $\sum\limits_{x_i}$ and $\prod\limits_{x_i}$ as the signs of infinite sums $(A)\sum\limits_{x_i \in J}$ and products $(A)\prod\limits_{x_i \in J}$ in the algebra (A), respectively $(i \in I_0)$.

Let

$$(\mathrm{s}_1) \qquad x_i = j_i \in J \qquad (i \in I),$$

$$(\mathrm{s}_2) \qquad f_n^l = \tau_n^l \in \boldsymbol{f}^l(J) \qquad (l \in I_0,\ n \in I_l''),$$

$$(\mathrm{s}_3) \qquad F_m^k = \varphi_m^k \in \boldsymbol{F}^k(J, A) \qquad (k \in I_0,\ m \in I_k')$$

be arbitrary but fixed system of valuations of individual signs, functors and predicates of $\mathfrak{T}(\mathfrak{A})$. This system of valuations will also be denoted by $[\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\}]$.

Let $\{j_i\}^-$ and $\{j_i\}^+$ always denote substitution (s_1) for individual signs of $\mathfrak{T}(\mathfrak{A})$ reduced to $i \in I - I_0$ and to $i \in I_0$ respectively.

The symbol $(J, A)\Phi_a(\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\})$ will denote the value of the functional $(J, A)\Phi_a$ for the values of its arguments fixed above by (s_1), (s_2), (s_3). If a is a closed formula of T, then the value of $(J, A)\Phi_a$ does not depend on the values of x_i, where $i \in I_0$. Therefore we write $(J, A)\Phi_a(\{j_i\}^-, \{\tau_n^l\}, \{\varphi_m^k\})$ instead of $(J, A)\Phi_a(\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\})$.

It is easy to verify that

3.1. [14]) *If* $\beta = a\binom{\xi}{x_p}$, *then*

$$(J, A)\Phi_\beta(\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\}) = (J, A)\Phi_a(\{\bar{j}_i\}, \{\tau_n^l\}, \{\varphi_m^k\})$$

where $\bar{j}_i = j_i$ *if* $i \neq p$, *and* $\bar{j}_p =$ *the value of* ξ *by the substitutions* (s_1), (s_2).

3.2. [15]) *If an* $\mathcal{S}$-*homomorphism* g *of* A *into another* $\mathcal{S}^*$-*algebra* A' *preserves all infinite sums and products, then*

$$g\big((J, A)\Phi_a(\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\})\big) = (J, A')\Phi_a(\{j_i\}, \{\tau_n^l\}, \{g\varphi_m^k\}).$$

[13]) Cf. [10], p. 71. See also [7].

[14]) See [10], 5.4, p. 72.

[15]) See [10], 5.5, p. 72.

The system $\mathcal{M}=[\{j_i\}^-,\{\tau_n^l\},\{\varphi_m^k\}]$ is said to be a *generalized model* of the theory $\mathcal{T}(\mathfrak{A})$ in the algebra A and in the domain J if, for every $a\in\mathfrak{A}$,

$$(J,A)\Phi_a(\{j_i\}^-,\{\tau_n^l\},\{\varphi_m^k\})=e.$$

If among the primitive symbols of $\mathcal{T}(\mathfrak{A})$ the sign of equality F_1^2 does not occur, then a generalized model of $\mathcal{T}(\mathfrak{A})$ will be called a *model* of $\mathcal{T}(\mathfrak{A})$.

Suppose that F_1^2 is the primitive symbol of $\mathcal{T}(\mathfrak{A})$. Then the function φ_1^2 establishes an interpretation of the equality sign F_1^2 in the generalized model $\mathcal{M}$, by the relation $\approx$, defined as follows:

$$j_k\approx j_l \quad \text{if and only if} \quad \varphi_1^2(j_k,j_l)=e.$$

3.3. *The relation $\approx$ is a congruence relation.*

Obviously, it is possible that $\varphi_1^2(j_k,j_l)=e$ for $j_k\neq j_l$.

A (J,A) generalized model $\mathcal{M}$ of a theory $\mathcal{T}(\mathfrak{A})$ with equality is said to be the (J,A) *model*[16]) of $\mathcal{T}(\mathfrak{A})$ if, for every $j_k,j_l\in J$, $\varphi_1^2(j_k,j_l)=e$ if and only if $j_k=j_l$.

3.4. *Let $\mathcal{M}=[\{j_i\}^-,\{\tau_n^l\},\{\varphi_m^k\}]$ be a generalized model of a theory $\mathcal{T}(\mathfrak{A})$ with equality. For every $j\in J$, let $\|j\|$ be the class of all $l\in J$ such that $\varphi_1^2(j,l)=e$. Let $\hat{J}$ be the set of all cosets $\|j\|$, where $j\in J$. Obviously $\overline{\overline{\hat{J}}}\leqslant\overline{\overline{J}}$. Further let*

$$\hat{\tau}_n^l(\|j_1\|,\dots,\|j_l\|)=\|\tau_n^l(j_1,\dots,j_l)\| \quad \textit{for any} \quad j_1,\dots,j_l\in J,$$
$$\hat{\varphi}_m^k(\|j_1\|,\dots,\|j_k\|)=\varphi_m^k(j_1,\dots,j_k) \quad \textit{for any} \quad j_1,\dots,j_k\in J.$$

Then the generalized model $\mathcal{M}=[\{\|j_i\|\}^-,\{\hat{\tau}_n^l\},\{\hat{\varphi}_m^k\}]$ is the $(\hat{J},A)$ model of the theory $\mathcal{T}(\mathfrak{A})$.

The easy proof is omitted.

Consider a system $[\{j_i\},\{\tau_n^l\},\{\varphi_m^k\}]$ of valuations for primitive symbols of $\mathcal{T}(\mathfrak{A})$. Suppose that $\mathcal{M}=[\{j_i\}^-,\{\tau_n^l\},\{\varphi_m^k\}]$ is a generalized model of $\mathcal{T}(\mathfrak{A})$ in J and A. Then, instead of $(J,A)\Phi_a(\{j_i\},\{\tau_n^l\},\{\varphi_m^k\})$, we shall write $(J,A,\mathcal{M})\Phi_a(\{j_i\}^+)$.

We shall say that *a theory $\mathcal{T}(\mathfrak{A})$ has a model in a domain J*, if there is an $\mathcal{S}^*$-algebra A such that $\mathcal{T}(\mathfrak{A})$ has a model in J and A. More generally, we shall say that *a theory $\mathcal{T}(\mathfrak{A})$ has a model*, if there is a domain J and an $\mathcal{S}^*$-algebra A such that $\mathcal{T}(\mathfrak{A})$ has a model in J and A.

[16]) If $\mathcal{S}^*$ is the classical functional calculus and A is the two-element Boolean algebra, then the (J,A) pseudomodel determines uniquely in the domain J the semantic model $[\{j_i\},\{\tau_n^l\},\{\varrho_m^k\}]$, where ϱ_m^k are k-argument relations such that $\varrho_m^k(j_{i_1},\dots,j_{i_k})$ holds if and only if $\varphi_m^k(j_{i_1},\dots,j_{i_k})=e$. Hence an algebraic model can be regarded as a generalization of a semantic model.

3.5. *Let $\mathcal{T}(\mathfrak{A})$ be an arbitrary theory and let $\mathcal{M}$ be a generalized model of $\mathcal{T}(\mathfrak{A})$ in a domain J and an $\mathcal{S}^*$-algebra A. Then, given arbitrary $a\in K(\mathfrak{A})$ and arbitrary system of valuations $\{j_i\}^+$ for individual variables in $\mathcal{T}(\mathfrak{A})$, we have $(J,A,\mathcal{M})\Phi_a(\{j_i\}^+)=e$. In other words, if a is a theorem of a theory $\mathcal{T}(\mathfrak{A})$, then $(J,A,\mathcal{M})\Phi_a=e$ identically in every generalized model $\mathcal{M}$ of $\mathcal{T}(\mathfrak{A})$.*

This follows from the definition of a generalized model and from the fact that the class of formulas $\beta\in T$, having the property $(J,A,\mathcal{M})\Phi_\beta=e$ identically in a fixed generalized model $\mathcal{M}$ of $\mathcal{T}(\mathfrak{A})$, is closed under the rules of inference.

3.6. [17]) *Let $\mathcal{T}(\mathfrak{A})$ be a consistent theory. Let h be an $\mathcal{S}^*$-homomorphism of $L(\mathfrak{A})$ into an $\mathcal{S}^*$-algebra L^*. Then, for every $a\in T$,*

$$(J_0,L^*)\Phi_a(\{j_i\},\{\tau_n^l\},\{\varphi_m^k\})=h|a|,$$

where

$$j=x \textit{ for } i\in I,\ \tau_n^l(\xi_1,\dots,\xi_l)=f_n^l(\xi_1,\dots,\xi_l),\quad \varphi_m^k(\xi_1,\dots,\xi_k)=h\big(|F_m^k(\xi_1,\dots,\xi_k)|\big).$$

Consequently, *the system $[\{j_i\}^-,\{\tau_n^l\},\{\varphi_m^k\}]$ is the generalized model of $\mathcal{T}(\mathfrak{A})$ in the domain J_0 and the algebra L^*.* This generalized model will be called *the natural generalized model* and will always be denoted by $\mathcal{N}(\mathfrak{A},h,L^*)$ or briefly by $\mathcal{N}$.

The easy proof by induction with respect to the length of a, based on the definitions of (J,A)-functional Φ_a, of $L(\mathfrak{A})$ and of $\mathcal{S}^*$-homomorphism, and on the lemma 2.3, is omitted.

3.7. *If $\mathcal{T}(\mathfrak{A})$ is a consistent theory with equality (without equality), then $\mathcal{T}(\mathfrak{A})$ has a model in a domain whose cardinal number is not greater than $\aleph_0$ (is $\aleph_0$).*

Indeed, if $\mathcal{T}(\mathfrak{A})$ is a theory without equality, then the natural generalized model $\mathcal{N}(\mathfrak{A},h,L^*)$ is a model in the domain $\overline{\overline{J}}_0=\aleph_0$. In this case $\mathcal{N}$ will be called the *natural model* of $\mathcal{T}(\mathfrak{A})$. If $\mathcal{T}(\mathfrak{A})$ is a theory with equality, then 3.7 follows from 3.6 and 3.4. The model $\hat{\mathcal{N}}$, obtained from $\mathcal{N}$ by the method mentioned in the formulation of 3.4, will also be called the *natural model* of $\mathcal{T}(\mathfrak{A})$.

3.8. *Let $\mathcal{S}^*$ be a logical system with the negation sign $-$, such that the formulas $-(\beta\to\beta)\to a$ are theorems of $\mathcal{S}^*$. If a theory $\mathcal{T}(\mathfrak{A})$, based on the system $\mathcal{S}^*$, has a model, then $\mathcal{T}(\mathfrak{A})$ is consistent.*

Suppose that $\mathcal{T}(\mathfrak{A})$ is not consistent. Let a be an axiom of $\mathcal{T}(\mathfrak{A})$, and let $\mathcal{M}$ be a model of $\mathcal{T}(\mathfrak{A})$ in an algebra A and a domain J. Thus

[17]) Theorem 3.6 is similar to 5.2 of [10], p. 72.

we have $(J,A,\mathcal{M})\Phi_a=e$. Since $\mathcal{T}(\mathfrak{A})$ is not consistent, $\mathfrak{A}\vdash -a$. Hence, by 3.5, $(J,A,\mathcal{M})\Phi_{-a}=e$. On the other hand, $(J,A,\mathcal{M})\Phi_{-a}=-e$. Consequently, $e=-e$, which is impossible, since in $\mathcal{S}$-algebras of the logical systems such that $-(\beta\to\beta)\to a$ are theorems, we have $e\neq -e$ [18]).

3.9. *Let $\mathcal{S}^*$ be the system with the negation sign $-$, satisfying the following condition: $-(\beta\to\beta)\to a$ is a theorem of $\mathcal{S}^*$. Then, if a theory $\mathcal{T}(\mathfrak{A})$ with equality (without equality), based on the system $\mathcal{S}^*$, has a model, it has a model in a domain whose cardinal number is not greater than (is equal to) $\aleph_0$.*

This follows immediately from 3.8 and 3.7.

Notice that theorems 3.6, 3.7, 3.8, 3.9 are analogical to 5.2, 6.2, 7.2 and 7.3 of [10] respectively.

Theorems 3.7 and 3.9 for the special case where $\mathcal{S}^*_\chi$ is the classical functional calculus, are well known Gödel and Skolem-Löwenheim [19]) results. Indeed, Tarski's original definition of satisfiability may be translated into the algebraical language [20]). Consequently, a theory $\mathcal{T}(\mathfrak{A})$ based on $\mathcal{S}^*_\chi$ has a semantic model in a domain J if and only if it has a model in J and the two-element Boolean algebra B_0. Thus the statement mentioned above results from the following theorem:

3.10. *If a theory $\mathcal{T}(\mathfrak{A})$ with equality (without equality) based on the classical functional calculus $\mathcal{S}^*_\chi$ has a model in a domain $J\neq\Lambda$ and in a complete Boolean algebra B, then*

(i) *$\mathcal{T}(\mathfrak{A})$ has a model in a domain whose cardinal number is not greater than $\bar{\bar{J}}$ (in the domain J) and in the two-element Boolean algebra B_0.*

(ii) *$\mathcal{T}(\mathfrak{A})$ has a model in a domain whose cardinal number is not greater than $\aleph_0$ (is equal to $\aleph_0$) and in B_0.*

The proof is analogical to the proof of 9.4 in [10].

Notice moreover that there are theories, based on $\mathcal{S}^*_\chi$, with no semantic models in a domain J, having algebraic models in that domain. For instance, the theory $\mathcal{T}(\mathfrak{A}_0+\mathfrak{E})$ with equality and the single axiom

$$\sum_{x_1}\sum_{x_2}\Big(-F_1^2(x_1,x_2)\cdot\prod_{x_3}\big(F_1^2(x_3,x_1)+F_1^2(x_3,x_2)\big)\Big).$$

Indeed, $\mathcal{T}(\mathfrak{A}_0+\mathfrak{E})$ has no infinite semantic models; on the other hand, the domain of the natural model is denumerable.

[18]) See [10], 7.1, p. 74.
[19]) Cf. for instance, L. Henkin [1].
[20]) Cf. [9], p. 196.

§ 4. Constructive theories

Now we shall consider the theories described in the previous paragraph in the special cases where $\mathcal{S}^*$ is the *functional calculus of Heyting*. Such theories will be denoted by $\mathcal{T}_\chi(\mathfrak{A})$. The set of all formulas in $\mathcal{T}_\chi(\mathfrak{A})$ will be denoted by $T_\chi(\mathfrak{A})$. Further, $L_\chi(\mathfrak{A})$ will denote the Lindenbaum algebra described in § 2, where $\mathcal{S}^*=\mathcal{S}^*_\chi$ (*i. e.* the functional calculus of Heyting). The $\mathcal{S}^*_\chi$-algebras are *complete Heyting algebras* and conversely. The letter H will exclusively denote a Heyting algebra.

If $\mathfrak{X}$ is a topological space, then the class of all open subsets of $\mathfrak{X}$ is a Heyting algebra with the following operations:

join $a+b$ and meet $a\cdot b$ are the set-theoretical operations of sum and product;

$a\to b=\mathbf{I}(-a+b)$ where $-a$ is the complement of a and $\mathbf{I}a$ is the interior of a in the space $\mathfrak{X}$;

$\neg a=a\to 0$.

The Heyting algebra of all open subsets of $\mathfrak{X}$ will always be denoted by $\boldsymbol{H}(\mathfrak{X})$.

Since $\mathcal{S}^*_\chi$ has the property (E) and $-(\beta\to\beta)\to a$ is a theorem of $\mathcal{S}^*_\chi$, all theorems of § 2 and § 3 hold in the case of $\mathcal{T}_\chi(\mathfrak{A})$. Moreover, in all the definitions and theorems of § 3 we may restrict the domain of all Heyting algebras to the domain of all Heyting algebras $\boldsymbol{H}(\mathfrak{X})$, where $\mathfrak{X}\neq 0$ is a topological space. This follows from the fact that for $L_\chi(\mathfrak{A})$ there is a topological space $\mathfrak{X}_{\chi\mathfrak{A}}$ with the operation of interior $\mathbf{I}$ such that $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$ is an $\mathcal{S}^*$-extension of $L_\chi(\mathfrak{A})$ [21]).

Let $\mathfrak{x}$ be a fixed element, such that $\mathfrak{x}$ non $\in\mathfrak{X}_{\chi\mathfrak{A}}$ and let $\mathfrak{X}^0_{\chi\mathfrak{A}}=\mathfrak{X}_{\chi\mathfrak{A}}+(\mathfrak{x})$. We shall regard the set $\mathfrak{X}^0_{\chi\mathfrak{A}}$ as a topological space with the following definition of the operation of interior $\mathbf{I}_0$ in $\mathfrak{X}^0_{\chi\mathfrak{A}}$:

(i) $\mathbf{I}_0\mathfrak{X}^0_{\chi\mathfrak{A}}=\mathfrak{X}^0_{\chi\mathfrak{A}}$,

(ii) if $X\subset\mathfrak{X}_{\chi\mathfrak{A}}, Z\subset(\mathfrak{x})$ and $X+Z\neq\mathfrak{X}^0_{\chi\mathfrak{A}}$ then $\mathbf{I}_0(X+Z)=\mathbf{I}(X)$.

Consequently, open subsets of $\mathfrak{X}^0_{\chi\mathfrak{A}}$ are open subsets of $\mathfrak{X}_{\chi\mathfrak{A}}$ and the whole space $\mathfrak{X}^0_{\chi\mathfrak{A}}$. Obviously, the operations of join and meet in $\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$ being the set-theoretical operations of sum and product in $\mathfrak{X}^0_{\chi\mathfrak{A}}$ are the same as in $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$. The operations $a\to b$ and $\neg a$ in $\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$ are not the same as the similar operations in $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$, and to distinguish them from those of $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$ we shall denote them by $a\to_0 b$ and $\neg_0 a$, respectively. Then we have

(iii) $X\to_0 Y=\mathfrak{X}^0_{\chi\mathfrak{A}}$ if and only if $X\subset Y$;

(iv) $\mathfrak{X}^0_{\chi\mathfrak{A}}\to_0 Y=\mathfrak{X}_{\chi\mathfrak{A}}\to Y=Y$ if $Y\subset\mathfrak{X}_{\chi\mathfrak{A}}$;

[21]) See [10], 11.2, p. 86.

(v) $X \to_0 Y = X \to Y$ if $X \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, $Y \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ and X non $\subset Y$;
(vi) $\neg_0 \mathfrak{X}^0_{\chi\mathfrak{A}} = 0$;
(vii) $\neg_0 X = \neg X$ if $X \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ and $X \neq 0$;
(viii) $\neg_0 0 = \mathfrak{X}^0_{\chi\mathfrak{A}}$.

It follows from the definition of topology in $\mathfrak{X}^0_{\chi\mathfrak{A}}$ that there is exactly one open subset of $\mathfrak{X}^0_{\chi\mathfrak{A}}$ which contains the element $\mathfrak{x}$, namely the whole space $\mathfrak{X}^0_{\chi\mathfrak{A}}$. Consequently

(*) *if* $G_k \in \boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$ *and* $\mathfrak{X}^0_{\chi\mathfrak{A}} = \sum_k G_k$, *then there is a* q *such that* $G_q = \mathfrak{X}^0_{\chi\mathfrak{A}}$;

(**) *if* $G_k \in \boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$ *then* $\big(\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\sum_k G_k = \big(\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})\big)\sum_k G_k$ [22].

Let $\overline{\mathfrak{N}}^0 = [\{\iota_i\}, \{\sigma^l_n\}, \{\psi^k_m\}]$ always be the following system of valuations of individual signs, functors and predicates occuring in $\mathcal{T}(\mathfrak{A})$, in the domain J_0 and in the algebra $\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$:

$$\iota_i = x \quad \text{for} \quad i \in I;$$

$$\sigma^l_n(\xi_1, \ldots, \xi_l) = f^l_n(\xi_1, \ldots, \xi_l) \quad \text{where} \quad l \in I_0,\ n \in I''_l;$$

$$\psi^k_m(\xi_1, \ldots, \xi_k) = \begin{cases} \mathfrak{X}^0_{\chi\mathfrak{A}} & \text{if} \quad |F^k_m(\xi_1, \ldots, \xi_k)| = e, \\ h|F^k_m(\xi_1, \ldots, \xi_k)| & \text{if} \quad |F^k_m(\xi_1, \ldots, \xi_k)| \neq e, \end{cases}$$

for $k \in I_0$, $m \in I'_k$, where e is the unit element of $L_\chi(\mathfrak{A})$ and h is the $\mathcal{S}^*_\chi$-isomorphism of $L_\chi(\mathfrak{A})$ into $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$.

Instead of $\big(J_0, \boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_a\big(\{\iota_i\}, \{\sigma^l_n\}, \{\psi^k_m\}\big)$ we shall always write briefly Ψ_a.

4.1. *Given arbitrary* $a \in T_\chi(\mathfrak{A})$, *we have* $\Psi_a = \begin{cases} \mathfrak{X}^0_{\chi\mathfrak{A}} \\ h|a| \end{cases}$. *Moreover, if* $\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}}$, *then* $h|a| = \mathfrak{X}_{\chi\mathfrak{A}}$.

The proof is by induction with respect to the length of a. In the case of $a = F^k_m(\xi_1, \ldots, \xi_k)$ this theorem is obvious. Suppose that $a = \beta + \gamma$. Then $\Psi_a = \Psi_\beta + \Psi_\gamma$. Clearly, $\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}}$ if either $\Psi_\beta = \mathfrak{X}^0_{\chi\mathfrak{A}}$ or $\Psi_\gamma = \mathfrak{X}^0_{\chi\mathfrak{A}}$. Then either $h|\beta| = \mathfrak{X}_{\chi\mathfrak{A}}$ or $h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$. Consequently, $h|a| = \mathfrak{X}_{\chi\mathfrak{A}}$. In the case of $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ and $\Psi_\gamma \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, we have $\Psi_a = h|\beta| + h|\gamma| = h|a|$. If $a = \beta \cdot \gamma$, then $\Psi_a = \Psi_\beta \cdot \Psi_\gamma$. Clearly $\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}}$ if and only if $\Psi_\beta = \Psi_\gamma = \mathfrak{X}^0_{\chi\mathfrak{A}}$. Then $h|\beta| = h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$, thus $h|a| = \mathfrak{X}_{\chi\mathfrak{A}}$. In the case of $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ and $\Psi_\gamma = \mathfrak{X}^0_{\chi\mathfrak{A}}$ we have

$$\Psi_a = h|\beta| \cdot \mathfrak{X}^0_{\chi\mathfrak{A}} = h|\beta| \cdot \mathfrak{X}_{\chi\mathfrak{A}} = h|\beta| \cdot h|\gamma| = h|a|.$$

In the case of $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, $\Psi_\gamma \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ the proof is similar. Suppose now that $a = \beta \to \gamma$. Then $\Psi_a = \Psi_\beta \to_0 \Psi_\gamma$. Consider the case of $\Psi_\beta \subset \Psi_\gamma$; then $\Psi_\beta \to_0 \Psi_\gamma = \mathfrak{X}^0_{\chi\mathfrak{A}}$. If $\Psi_\gamma = \mathfrak{X}^0_{\chi\mathfrak{A}}$, then $h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$; consequently $h|a| = h|\beta| \to h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$. If $\Psi_\gamma \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, then $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, hence $\Psi_\beta = h|\beta|$ and $\Psi_\gamma = h|\gamma|$. Thus $h|a| = h|\beta| \to h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$. Consider now the case of Ψ_β non $\subset \Psi_\gamma$. Then $\Psi_a \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$. Moreover, since $\Psi_\gamma \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, we have $\Psi_\gamma = h|\gamma|$. Suppose that $\Psi_\beta = \mathfrak{X}^0_{\chi\mathfrak{A}}$. Then $h|\beta| = \mathfrak{X}_{\chi\mathfrak{A}}$. Hence

$$\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}} \to_0 \Psi_\gamma = \mathfrak{X}_{\chi\mathfrak{A}} \to \Psi_\gamma = h|\beta| \to h|\gamma| = h|a|.$$

If $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, then $\Psi_\beta = h|\beta|$. Hence $\Psi_a = h|\beta| \to h|\gamma| = h|a|$. Suppose that $a = \neg\beta$. If $\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}}$, then $\neg_0 \Psi_\beta = \mathfrak{X}^0_{\chi\mathfrak{A}}$. Hence $\Psi_\beta = 0 = h|\beta|$. That is $h|a| = \neg h|\beta| = \mathfrak{X}_{\chi\mathfrak{A}}$. Let $\Psi_a \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$. If $\Psi_\beta = \mathfrak{X}^0_{\chi\mathfrak{A}}$, then $h|\beta| = \mathfrak{X}_{\chi\mathfrak{A}}$. Thus

$$\Psi_a = \neg_0 \Psi_\beta = \neg_0 \mathfrak{X}^0_{\chi\mathfrak{A}} = 0 = \neg \mathfrak{X}_{\chi\mathfrak{A}} = \neg h|\beta| = h|a|.$$

If $\Psi_\beta \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$ and $\Psi_\beta \neq 0$, we have $\Psi_\beta = h|\beta|$. Hence $\Psi_a = \neg_0 h|\beta| = \neg h|\beta| = h|a|$. Consider now the case of $a = \sum_{x_p} \beta$. Then

$$\Psi_a = \big(\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\sum_{y_p \in J_0} \Phi_\beta(\{\iota'_i\}, \{\sigma^l_n\}, \{\psi^k_m\})$$

where $\iota'_i = \iota_i$ if $i \neq p$ and $\iota'_p = y_p \in J_0$.

If $\Psi_a = \mathfrak{X}^0_{\chi\mathfrak{A}}$, then on account of (*), there exists such $y_p \in J_0$ that $\Phi_\beta(\{\iota'_i\}, \{\sigma^l_n\}, \{\psi^k_m\}) = \mathfrak{X}^0_{\chi\mathfrak{A}}$. Hence by 3.1 $h\left|\beta\binom{y_p}{x_p}\right| = \mathfrak{X}_{\chi\mathfrak{A}}$. Consequently making use of 2.3 and of the property of $\mathcal{S}^*$-isomorphism h we infer that

$$h|a| = \big(\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})\big)\sum_{y_p \in J_0} h\left|\beta\binom{y_p}{x_p}\right| = \mathfrak{X}_{\chi\mathfrak{A}}.$$

If $\Psi_a \neq \mathfrak{X}^0_{\chi\mathfrak{A}}$, then for every $y_p \in J_0$, $\Phi_\beta(\{\iota'_i\}, \{\sigma^l_n\}, \{\psi^k_m\}) = h|a|$. Hence, by (**), $\Psi_a = \big(\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})\big)\sum_{y_p \in J_0} h\left|\beta\binom{y_p}{x_p}\right| = h|a|$. In the case of $a = \prod_{x_p} \beta$ the proof is similar.

4.2. *Let* $\mathfrak{A} \vdash F^2_1(\xi_1, \xi_2)$. *Then for arbitrary* $a \in T_\chi(\mathfrak{A})$, *involving the term* ξ_1 *and such that every individual variable occuring in* ξ_1 *is free in* a, *we have*

$$\Psi_a = \Psi_{a\binom{\xi_2}{\xi_1}}$$

where $a\binom{\xi_2}{\xi_1}$ *denote the formula obtained from* a *by the replacement of the term* ξ_1 *by* ξ_2.

We recall that the necessary changes in the bound variables of a were performed before the operation of substitution.

[22] The signs $\big(\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})\big)\sum$ and $\big(\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\sum$ denote the infinite sums in the algebra $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$ and $\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$, respectively.

Indeed, consider a formula α as a formula which is obtained from some formula β by substituting the term ξ_1 for an individual variable x_p. Obviously β is a formula with one free occurrence of x_p, and we can suppose that the quantifiers binding the individual variables which appear in ξ_1 and in ξ_2 do not occur in β. It follows from the axioms of equality and $\mathfrak{A}\vdash F_1^2(\xi_1,\xi_2)$ that $\mathfrak{A}\vdash\beta\binom{\xi_1}{x_p}\equiv\beta\binom{\xi_2}{x_p}$. Thus $\mathfrak{A}\vdash\alpha\equiv\alpha\binom{\xi_2}{\xi_1}$. Hence, by a simple inductive argument, $\Psi_\alpha=\Psi_{\alpha\binom{\xi_2}{\xi_1}}$.

4.3. *If $\mathfrak{E}\subset\mathfrak{A}$ and $\alpha\in\mathfrak{E}$ (i. e. if α is one of the axioms of equality of the theory $\mathcal{T}_\chi(\mathfrak{E}+\mathfrak{A}_0)$), then $\Psi_\alpha=\mathfrak{X}^0_{\chi\mathfrak{A}}$.*

Suppose that $\alpha=\prod_{x_1}F_1^2(x_1,x_1)$. For every $\xi\in J_0$ we have $\mathfrak{A}_0+\mathfrak{E}\vdash F_1^2(\xi,\xi)$. Hence $\varphi_1^2(\xi,\xi)=\mathfrak{X}^0_{\chi\mathfrak{A}}$. Consequently, $\Psi_\alpha=\big(\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\prod_{\xi\in J_0}\varphi_1^2(\xi,\xi)=\mathfrak{X}^0_{\chi\mathfrak{A}}$.

Suppose now that $\alpha=\Xi\left(F_1^2(x_1,x_2)\to\left(\alpha_1\to\alpha_1\binom{x_2}{x_1}\right)\right)$. Set $\beta=F_1^2(\xi_1,\xi_2)\to$ $\to\left(\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}\right)$, where ξ_1,ξ_2 are arbitrary terms of $\mathcal{T}_\chi(\mathfrak{A})$. To show that $\Psi_\alpha=\mathfrak{X}^0_{\chi\mathfrak{A}}$ it suffices to prove that $\Psi_\beta=\mathfrak{X}^0_{\chi\mathfrak{A}}$.

Suppose that $\mathfrak{A}\vdash F_1^2(\xi_1,\xi_2)$. Then, on account of 4.2, $\Psi_{\alpha_1}=\Psi_{\alpha_1\binom{\xi_2}{\xi_1}}$. Hence $\Psi_{\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}}=\Psi_{\alpha_1}\dot{\to}\Psi_{\alpha_1\binom{\xi_2}{\xi_1}}=\mathfrak{X}^0_{\chi\mathfrak{A}}$. Consequently, $\Psi_\beta=\mathfrak{X}^0_{\chi\mathfrak{A}}$.

Consider the case where non $\mathfrak{A}\vdash F_1^2(\xi_1,\xi_2)$. Since $\mathfrak{A}\vdash\beta$, we obtain $h|F_1^2(\xi_1,\xi_2)|\subset h\left|\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}\right|$. Further, $\Psi_{F_1^2(\xi_1,\xi_2)}=\varphi_1^2(\xi_1,\xi_2)=h|F_1^2(\xi_1,\xi_2)|\neq\mathfrak{X}_{\chi\mathfrak{A}}$. If $\Psi_{\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}}=\mathfrak{X}^0_{\chi\mathfrak{A}}$, then obviously $\Psi_\beta=\mathfrak{X}^0_{\chi\mathfrak{A}}$. Suppose that $\Psi_{\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}}\neq\mathfrak{X}^0_{\chi\mathfrak{A}}$. Then by 4.1 $\Psi_{\alpha_1\to\alpha_2\binom{\xi_2}{\xi_1}}=h\left|\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}\right|$. Thus $\Psi_\beta=h|F_1^2(\xi_1,\xi_2)|\to_0$ $\to_0 h\left|\alpha_1\to\alpha_1\binom{\xi_2}{\xi_1}\right|=\mathfrak{X}^0_{\chi\mathfrak{A}}$.

A theory $\mathcal{T}_\chi(\mathfrak{A})$ is said to be *constructive* if it fulfills the following conditions:

(I) $\alpha+\beta\in K(\mathfrak{A})$ implies that $\alpha\in K(\mathfrak{A})$ or $\beta\in K(\mathfrak{A})$,

(II) if $\sum_{x_p}\alpha\in K(\mathfrak{A})$, then there exists $\xi\in J_0$ such that $\alpha\binom{\xi}{x_p}\in K(\mathfrak{A})$.

4.4. *Let $\mathcal{T}(\mathfrak{A})$ be a theory such that $\mathcal{N}^0=[\{\iota_i\}^-,\{\sigma_n^l\},\{\psi_m^k\}]$ is the generalized model of $\mathcal{T}_\chi(\mathfrak{A})$. Then $\mathcal{T}_\chi(\mathfrak{A})$ is a constructive theory.*

The proof of 4.4 is similar to that of the fundamental theorem in [12].

Since $\mathfrak{X}_{\chi\mathfrak{A}}$ is the open subset of $\mathfrak{X}^0_{\chi\mathfrak{A}}$, the formula

$$g(A)=A\cdot\mathfrak{X}_{\chi\mathfrak{A}}\quad\text{for}\quad A\in\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$$

defines an $\mathcal{S}_\chi$-homomorphism of the $\mathcal{S}^*_\chi$-algebra $\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})$ onto the $\mathcal{S}^*$-algebra $\boldsymbol{H}(\mathfrak{X}_{\chi\mathfrak{A}})$. This homomorphism preserves all infinite sums and products.

Suppose that $\beta=\sum_{x_p}\alpha\in K(\mathfrak{A})$. Hence, by 3.5,

$$\mathfrak{X}^0_{\chi\mathfrak{A}}=\Psi_\beta=\sum_{\xi\in J_0}\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\alpha(\{\iota_i'\},\{\sigma_n^l\},\{\psi_m^k\})$$

where $\{\iota_i'\}$ denotes the sequence $\{\iota_i\}$ the p-th term of which is replaced by ξ.

By (*) there is a component of the sum equal to $\mathfrak{X}^0_{\chi\mathfrak{A}}$, i. e., there exists such $\xi_0\in J_0$, that

$$\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\alpha(\{\iota_i''\},\{\sigma_n^l\},\{\psi_m^k\})=\mathfrak{X}^0_{\chi\mathfrak{A}}$$

where $\iota_i''=\iota_i$ for $i\neq p$ and $\iota_p''=\xi_0$.

Let $\gamma=\alpha\binom{\xi_0}{x_p}$. By 3.1

$$\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\gamma(\{\iota_i\},\{\sigma_n^l\},\{\psi_m^k\})=\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\alpha(\{\iota_i''\},\{\sigma_n^l\},\{\psi_m^k\})=\mathfrak{X}^0_{\chi\mathfrak{A}}.$$

Hence we obtain by 3.2 and 3.6, on account of $g\psi_m^k(\xi_1,\ldots,\xi_k)$ $=h\big(|F_m^k(\xi_1,\ldots,\xi_k)|\big)$,

$$\mathfrak{X}_{\chi\mathfrak{A}}=g(\mathfrak{X}^0_{\chi\mathfrak{A}})=g\Big(\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\gamma(\{\iota_i\},\{\sigma_n^l\},\{\psi_m^k\})\Big)$$
$$=\big(J_0,\boldsymbol{H}(\mathfrak{X}^0_{\chi\mathfrak{A}})\big)\Phi_\gamma(\{\iota_i\},\{\sigma_n^l\},\{g\psi_m^k\})=h|\gamma|.$$

Consequently, by 2.2, $\gamma=\alpha\binom{\xi}{x_p}\in K(\mathfrak{A})$.

In a similar way we can prove the property (I) of the constructive theories.

It follows immediately from 4.3 and 4.4 that

4.5. *The functional calculus of Heyting with equality is a constructive theory.*

4.6. *Let $\mathcal{T}_\chi(\mathfrak{A})$ be an arbitrary theory and let Z be the set of all formulas $\alpha\in T_\chi(\mathfrak{A})$ satisfying the following condition*

$$\mathfrak{A}\vdash\alpha\ \textit{implies that}\ \Psi_\alpha=\mathfrak{X}^0_{\chi\mathfrak{A}}.$$

Then the least set Z^0 of formulas of $\mathcal{T}_\chi(\mathfrak{A})$ such that:

1° if $\beta=F_m^k(\xi_1,\ldots,\xi_k)$ where $k\in I_0$ and $m\in I_k'$, then $\beta\in Z^0$,

2° if $\beta,\gamma\in Z^0$, then $\beta\cdot\gamma\in Z^0$,

3° if $\gamma\in Z^0$ and *$\beta\in T_{\chi\mathfrak{A}}$, then $\beta\to\gamma\in Z^0$, $\neg\beta\in Z^0$, $\prod_{x_p}\gamma\in Z^0$,*

is contained in Z.

Indeed, if $\beta = F_m^k(\xi_1, \ldots, \xi_k)$ and $\mathfrak{A} \vdash \beta$, then $h|\beta| = h|F_m^k(\xi_1, \ldots, \xi_k)| = \mathfrak{X}_{\chi\mathfrak{A}}$. Thus $\Psi_\beta = \psi_m^k(\xi_1, \ldots, \xi_k) = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

Suppose that $\beta, \gamma \in Z$ and $\alpha = \beta \cdot \gamma$. If $\mathfrak{A} \vdash \alpha$, then $\mathfrak{A} \vdash \beta$ and $\mathfrak{A} \vdash \gamma$. Consequently, $\Psi_\beta = \Psi_\gamma = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Hence $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

Now suppose that $\gamma \in Z$ and $\alpha = \beta \to \gamma$. Let $\mathfrak{A} \vdash \alpha$. Then $h|\beta| \subset h|\gamma|$. If $h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$, then $\Psi_\gamma = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Hence $\Psi_\alpha = \Psi_\beta \to_0 \Psi_\gamma = \mathfrak{X}_{\chi\mathfrak{A}}^0$. In the case of $h|\beta| = \mathfrak{X}_{\chi\mathfrak{A}}$ we have $h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$. Hence $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Let $h|\beta| \neq \mathfrak{X}_{\chi\mathfrak{A}}$ and $h|\gamma| \neq \mathfrak{X}_{\chi\mathfrak{A}}$. Then by 4.1 $\Psi_\beta = h|\beta|$ and $\Psi_\gamma = h|\gamma|$. Since $h|\beta| \subset h|\gamma|$, we have $\Psi_\beta \to_0 \Psi_\gamma = h|\beta| \to_0 h|\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

Suppose that $\gamma \in Z$, $\alpha = \neg\gamma$ and $\mathfrak{A} \vdash \alpha$. Hence $h|\neg\gamma| = \mathfrak{X}_{\chi\mathfrak{A}}$. Thus $h|\gamma| = 0$. Consequently $\Psi_\alpha = \neg_0 0 = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

In the case of $\gamma \in T_{\chi\mathfrak{A}}$, $\alpha = \prod_{x_p} \gamma$ and $\mathfrak{A} \vdash \alpha$, we have $\mathfrak{A} \vdash \gamma\binom{\xi}{x_p}$. Thus $\Psi_{\gamma\binom{\xi}{x_p}} = \mathfrak{X}_{\chi\mathfrak{A}}^0$ for every $\xi \in J_0$. Hence $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

Given a theory $\mathcal{T}_\chi(\mathfrak{A})$, let $Z^0(\mathfrak{A})$ always denote the least set of formulas of $\mathcal{T}_\chi(\mathfrak{A})$ satisfying the following conditions:

1° if $\xi_1, \ldots, \xi_k$ are terms of $\mathcal{T}_\chi(\mathfrak{A})$, then $F_m^k(\xi_1, \ldots, \xi_k) \in Z^0(\mathfrak{A})$ for $k \in I_0$ and $m \in I_k'$,

2° if $\beta, \gamma \in Z^0(\mathfrak{A})$ then $\beta \cdot \gamma \in Z^0(\mathfrak{A})$,

3° if $\gamma \in Z^0(\mathfrak{A})$ and $\beta \in T_{\chi\mathfrak{A}}$ then $\neg\beta \in Z^0(\mathfrak{A})$, $\beta \to \gamma \in Z^0(\mathfrak{A})$ and $\prod_{x_p} \gamma \in Z^0(\mathfrak{A})$.

It follows immediately from 4.6 and from 4.3 that

4.7. *If $\mathcal{T}_\chi(\mathfrak{A})$ is a theory such that $\mathfrak{A} - \mathfrak{E} \subset Z^0(\mathfrak{A})$ (i. e. for every $\alpha \in \mathfrak{A} - \mathfrak{E}$, $\alpha \in Z^0(\mathfrak{A})$), then $\mathcal{T}_\chi(\mathfrak{A})$ is a constructive theory. In particular, if the axioms of a theory $\mathcal{T}_\chi(\mathfrak{A})$ (except the axioms of equality) contain neither the sign of the alternative nor the sign of the existential quantifier then $\mathcal{T}_\chi(\mathfrak{A})$ is a constructive theory.*

The condition which appears in the formulation of 4.7 is essential. Consider for instance the theory containing x_{-1} as the single primitive individual constant and F_1^2 as the single predicate and based on the axioms of equality and the axiom $\sum_{x_1} \neg F_1^2(x_1, x_{-1})$. Obviously this theory is not constructive.

In theorem 4.4 we have formulate in the algebraic language a sufficient condition for a theory $\mathcal{T}_\chi(\mathfrak{A})$ to be constructive. Now we shall prove that this condition is also necessary.

4.8. *If a theory $\mathcal{T}_\chi(\mathfrak{A})$ is constructive, then $\mathfrak{N}^0 = [\{\iota\}^-, \{\sigma_n^l\}, \{\psi_m^k\}]$ is the generalized model of $\mathcal{T}_\chi(\mathfrak{A})$.*

Let Z be the set of all formulas $\alpha \in T_\chi(\mathfrak{A})$ satisfying the following condition: if $\mathfrak{A} \vdash \alpha$, then $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$. To prove 4.8 we shall demonstrate that $T_\chi(\mathfrak{A}) \subset Z$. On account of the proof of the theorem 4.6 it suffices to show that

(i) if $\beta, \gamma \in Z$ then $\beta + \gamma \in Z$,

(ii) if $\beta \in Z$ then $\sum_{x_p} \beta \in Z$.

Suppose that $\alpha = \beta + \gamma$ and $\mathfrak{A} \vdash \alpha$. Since $\mathcal{T}_\chi(\mathfrak{A})$ is constructive, then either $\mathfrak{A} \vdash \beta$ or $\mathfrak{A} \vdash \gamma$. Hence $\Psi_\beta = \mathfrak{X}_{\chi\mathfrak{A}}^0$ or $\Psi_\gamma = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Consequently, $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Now let $\alpha = \sum_{x_p} \beta$. It follows from the constructivity of $\mathcal{T}_\chi(\mathfrak{A})$ that there exists such $\xi \in J_0$ that $\mathfrak{A} \vdash \beta\binom{\xi}{x_p}$. Hence $\Psi_{\beta\binom{\xi}{x_p}} = \mathfrak{X}_{\chi\mathfrak{A}}^0$. Thus $\Psi_\alpha = \mathfrak{X}_{\chi\mathfrak{A}}^0$.

It follows from 4.7 that every theory whose axioms are equalities is constructive. In particular, the theories of groups, of rings, of lattices of Boolean algebras, of closure algebras are constructive. More generally, every elementary theory $\mathcal{T}_\chi(\mathfrak{A})$ can be modified so as to be a constructive one. It suffices to remove the existential quantifiers and the sign of the alternative from the axioms $\alpha \in \mathfrak{A}_0 - Z^0(\mathfrak{A})$ of $\mathcal{T}_\chi(\mathfrak{A})$, by joining the functors [23]) or by the use of de Morgan's laws. In the last case we obviously obtain a theory weaker than $\mathcal{T}_\chi(\mathfrak{A})$.

We intend to apply the results mentioned above to arithmetic in the following form. Consider the system $\mathcal{T}_\chi(\mathfrak{P} + \mathfrak{E})$ of arithmetic, in the description of which we shall use, for convenience, the generally assumed notation. As specific constants of $\mathcal{T}_\chi(\mathfrak{P} + \mathfrak{E})$ let us assume the individual constant **1**, the one-argument functors $'$, $^{\cdot}$ [24]), the two-argument functors **+** and **×**, and the sign of equality **=**. The set of axioms consists of the axioms of equality and of the following formulas:

$$\prod_{x_1} \neg (x_1' \mathbf{=} \mathbf{1}),$$

$$\mathbf{1}^{\cdot} \mathbf{=} \mathbf{1},$$

$$\prod_{x_1} \Big(\neg (x_1 \mathbf{=} \mathbf{1}) \to \big((x_1^{\cdot})' \mathbf{=} x_1\big)\Big),$$

[23]) *I. e.* if $\Phi(x_1, \ldots, x_k, x_{k+1})$ is a formula of $\mathcal{T}_\chi(\mathfrak{A})$, with the free variables $x_1, \ldots, x_{k+1}$ and among the axioms of $\mathcal{T}_\chi(\mathfrak{A})$ appear the formulas

$$\prod_{x_1} \ldots \prod_{x_k} \sum_{x_{k+1}} \Phi(x_1, \ldots, x_k, x_{k+1}),$$

$$\prod_{x_1} \ldots \prod_{x_k} \prod_{x_{k+1}} \prod_{x_{k+2}} \big(\Phi(x_1, \ldots, x_k, x_{k+1}) \cdot \Phi(x_1, \ldots, x_k, x_{k+2}) \to F_1^2(x_{k+1}, x_{k+2})\big),$$

then it is possible to eliminate these axioms by joining a new functor f_m^k and the axiom

$$\prod_{x_1} \ldots \prod_{x_k} \Phi\big(x_1, \ldots, x_k, f_m^k(x_1, \ldots, x_k)\big).$$

[24]) The signs $'$ and $^{\cdot}$ are the signs of successor and antecedent, respectively. To determine the antecedent for the integer **1**, we assume the second axiom.

$$\prod_{x_1}\big((x_1')^{\cdot}\boldsymbol{=}x_1\big),$$

$$\prod_{x_1}\prod_{x_2}\big((x_1'\boldsymbol{=}x_2')\to(x_1\boldsymbol{=}x_2)\big),$$

$$\prod_{x_1}\big((x_1\boldsymbol{+}\mathbf{1})\boldsymbol{=}x_1'\big),$$

$$\prod_{x_1}\prod_{x_2}(x_1\boldsymbol{+}x_2')\boldsymbol{=}(x_1\boldsymbol{+}x_2)',$$

$$\prod_{x_1}\big((x_1\boldsymbol{\times}\mathbf{1})\boldsymbol{=}x_1\big),$$

$$\prod_{x_1}\prod_{x_2}\Big((x_1\boldsymbol{\times}x_2')=\big((x_1\boldsymbol{\times}x_2)\boldsymbol{+}x_1\big)\Big),$$

$$\Xi\Big(a(\mathbf{1})\cdot\prod_{x_j}\big(a(x_j)\to a(x_j')\big)\to\prod_{x_j}a(x_j)\Big),$$

where Ξ is written instead of the quantifiers bounding all free occurrences of individual variables different from x_j which appear in a, and where a is an arbitrary formula of arithmetic belonging to $Z^0(\mathfrak{P}+\mathfrak{E})$.

The fragment of arithmetic described above is a constructive theory.

In this system one can prove many laws of arithmetic in the usual way, for instance: $x_1\boldsymbol{+}x_2=x_2\boldsymbol{+}x_1$, $x_1\boldsymbol{\times}x_2=x_2\boldsymbol{\times}x_1$, $(x_1\boldsymbol{+}x_2)\boldsymbol{+}x_3\boldsymbol{=}x_1\boldsymbol{+}(x_2\boldsymbol{+}x_3)$, $(x_1\boldsymbol{\times}x_2)\boldsymbol{\times}x_3\boldsymbol{=}x_1\boldsymbol{\times}(x_2\boldsymbol{\times}x_3)$, $\neg\big(\sum_{x_3}(x_1\boldsymbol{=}x_2\boldsymbol{+}x_3)\cdot\sum_{x_4}(x_2\boldsymbol{=}x_1\boldsymbol{+}x_4)\big)$, $\neg\big((x_1\boldsymbol{=}x_2)\cdot\sum_{x_3}(x_1\boldsymbol{=}x_2\boldsymbol{+}x_3)\big)$, etc. However, it is impossible to prove that

$$\prod_{x_1}\prod_{x_2}\big((x_1\boldsymbol{=}x_2)+\sum_{x_3}(x_1\boldsymbol{=}x_2\boldsymbol{+}x_3)+\sum_{x_4}(x_2\boldsymbol{=}x_1\boldsymbol{+}x_4)\big).$$

On the other hand, the following formula is provable in $\mathfrak{T}_\chi(\mathfrak{P}+\mathfrak{E})$:

$$\neg\big(\neg(x_1\boldsymbol{=}x_2)\cdot\neg\sum_{x_3}(x_1\boldsymbol{=}x_2\boldsymbol{+}x_3)\cdot\neg\sum_{x_4}(x_2\boldsymbol{=}x_1\boldsymbol{+}x_4)\big).$$

Notice that the axioms of induction with arbitrary a belonging to $\mathfrak{T}_\chi(\mathfrak{P}+\mathfrak{E})$ may be non-constructive. For example, this holds for a of the form $(x_1^{\cdot}\boldsymbol{=}x_1)+\big((x_1^{\cdot})'\boldsymbol{=}x_1\big)$.

4.9. *Let $\mathfrak{T}_\chi(\mathfrak{A})$ be a constructive theory and let β be an arbitrary formula of this theory of the form*

$$(*)\qquad \beta=\overset{(1)}{\underset{x_{p_1}}{\Xi}}\dots\overset{(n)}{\underset{x_{p_n}}{\Xi}}a$$

where a contains no quantifier and $\overset{(i)}{\Xi}$ is either the sign Σ or Π $(i=1,2,\dots,n)$. Then there exists a sequence $a_1,a_2,\dots$ of formulas without quantifiers such that β is a theorem of $\mathfrak{T}_\chi(\mathfrak{A})$ if and only if at least one of the formulas $a_1,a_2,\dots$ is a theorem[25]*). The sequence $a_1,a_2,\dots$ can be determined effectively.*

[25]) The proof is similar to that of (χ') in [12].

The expression $\gamma\binom{\xi(x_{i_1},\dots,x_{i_l})}{x_p}$ has not been uniquely determined. However, it will be uniquely determined as follows: let j_1 be the least positive integer such that $j_1\neq i_k$, $k=1,\dots,l$ and γ contains neither x_{j_1} nor $\prod_{x_{j_1}}$ nor $\sum_{x_{j_1}}$, let j_q $(q=2,\dots,l)$ be the least positive integer satisfying all conditions for j_1 and also $j_q\neq j_r$ $(r=1,\dots,q-1)$. We replace every occurrence of the bound variable x_{i_q} $(q=1,\dots,l)$ by x_{j_q} and every quantifier $\sum_{x_{i_q}}(\prod_{x_{i_q}})$ by $\sum_{x_{j_q}}(\prod_{x_{j_q}})$. Further, we replace every free variable x_p by the term $\xi(x_{i_1},\dots,x_{i_l})$. If $\beta=\sum_{x_p}a$, then we shall denote by $Z(\beta)$ the set of all formulas $a\binom{\xi}{x_p}$ where either $\xi=x_p$, or $\xi=\xi(x_{i_1},\dots,x_{i_l})$ and a contains at least one free occurrence of every $x_{i_1},\dots,x_{i_l}$, or ξ contains no individual variables. If $\beta=\prod_{x_p}a$, then $Z(\beta)$ is the set containing only one element: the formula a. More generally, if $R\subset T_\chi(\mathfrak{A})$ is a set of formulas β of the form $\beta=\sum_{x_p}a$ or $\beta=\prod_{x_p}a$, then $Z(R)$ is the union of all sets $Z(\beta)$ where $\beta\in R$. Suppose now that β is a formula of the form $(*)$. Let $R_1=Z(\beta)$ and, by induction, $R_k=Z(R_{k-1})$, $k=2,\dots,n$. It follows from the assumption of $T_\chi(\mathfrak{A})$ being constructive that β is provable if and only if R_1 contains at least one theorem of $\mathfrak{T}_\chi(\mathfrak{A})$. By induction with respect to k we find that β is a theorem of $T_\chi(\mathfrak{A})$ if and only if R_k contains at least one provable formula. Consequently β is provable if and only if R_n contains at least one provable formula. However, the set R_n is denumerable, which completes the proof.

In the case of a theory $T_\chi(\mathfrak{A})$ without functors, and with a finite set of individual constants, the set R_n is finite. Hence

4.10. *Let $\mathfrak{T}_\chi(\mathfrak{A})$ be a constructive theory without functors, let $\overline{\overline{I-I_0}}<\aleph_0$, and let β be an arbitrary formula of $\mathfrak{T}_\chi(\mathfrak{A})$ of the form $(*)$. Then there exists an effectively determined finite sequence $a_1,\dots,a_m$ of formulas without quantifiers, such that β is a theorem of $\mathfrak{T}_\chi(\mathfrak{A})$ if and only if at least one the formulas $a_1,\dots,a_m$ is a theorem of $\mathfrak{T}_\chi(\mathfrak{A})$.*

Interesting generalization of the theorem (χ) of [12] arises by applying the above-mentioned results to the functional calculus of Heyting. Let a be an arbitrary closed formula of this system. Such formula can be treated as the single axiom of some theory. If the theory based on a is constructive, we shall say that a is a constructive closed formula.

4.11. *If a is a constructive closed formula, then for every formula β, $a\to\sum_{x_p}\beta$ is provable if and only if there exists $q\in I$, such that $a\to\beta\binom{x_q}{x_p}$ is provable.*

This follows immediately from the definition of constructivity of a theory and from the deduction theorem for the functional calculus of Heyting.

Moreover

4.12. *If α is a constructive closed formula, then for every formula β of the form $\beta = \overset{(1)}{\underset{x_{p_1}}{\Xi}} \ldots \overset{(n)}{\underset{x_{p_n}}{\Xi}} \gamma$ where γ contains no quantifiers and $\overset{(i)}{\Xi}$ is either the sign Σ or Π, there exists a finite sequence $\gamma_1, \ldots, \gamma_m$ of the formulas without quantifiers such that $\alpha \to \beta$ is provable if and only if at least one of the formulas $\alpha \to \gamma_1, \ldots, \alpha \to \gamma_m$ is provable.*

This follows immediately from the deduction theorem for the functional calculus of Heyting and from 4.10.

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Elementarily definable analysis

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The purpose of this paper is to give a strict mathematical shape to some ideas expressed by H. Weyl in "Das Kontinuum" [4]. Weyl proposes a restriction of the logical methods of analysis to the elementarily definable ones. A *notion* is *elementarily definable* if it is definable by means of the quantifiers bounding the integral variables only. A strict definition will be given later. It is very interesting to note how many theorems of the classical analysis can be obtained by means of elementary methods. It is shown in this paper that the classical analysis of continuous functions can be reproduced in an elementary manner. The problem of how many theorems from the theory of non continuous real functions can be obtained in an elementary way remains open. Some counter examples are given in the sequel.

To begin with the problem arrises how to define elementary definability. There are at least two answers:

1. A mathematical *notion A* is *elementarily definable* if it is definable by means of an elementary definition

$$A(f, \ldots x, \ldots) \equiv \Xi(\ldots f, \ldots x, \ldots).$$

2. A *notion A* is *elementarily definable* if there exists a finite set of elementary conditions such that A is the unique object which satisfies those conditions.

We shall call the first the *narrower*, the second the *broader concept* of elementary definability. In this paper we shall consider the narrower notion.

1. Elementary definability in the arithmetic of integers

We shall introduce the notion of elementary definability in the arithmetic of integers. Let $\boldsymbol{I}$ be the set of all integers (positive, negative and zero). Let $\boldsymbol{N}$ be the set of non negative integers (natural numbers). The variables x, y, z, p, q will stand for the integers, the variables n, k, l, m will represent natural numbers. The letters f, g, h, will be used to denote the functions defined over the set $\boldsymbol{I}$ and assuming the integral values.