pour obtenir l'existence d'une droite $D_A$ telle que $P(D_A)=O$; on en résulte que le plan mené par $D_A$ et $L$ coupe $V$ suivant une section dont le centre de gravité se trouve sur $L$, c. q. l. d.

Il est évident que le théorème reste vrai quand on entend par centre de gravité d'une section le centre de gravité du contour suivant lequel elle tranchée $S$.

Envisageons maintenant un corps convexe dont la surface $S$ est assujettie à des conditions de régularité garantissant en tout point $P$ de $S$ l'existence de rayons principaux de courbure, $E_1(P)$ et $E_2(P)$ ($E_1 > E_2$), et leur continuité comme fonctions de $P$ sur $S$. En écrivant $x = E_1(P)$, $y = E_2(P)$, on définit une représentation de $S$ sur le plan cartésien $(x, y)$.

D'après (11), il y a sur $S$ deux points antipodiques $P$ et $Q$, ayant une image commune $(x_0, y_0)$. Il s'ensuit que les deux courbure principales en $P$ et en $Q$ sont respectivement égales. On peut rendre ce fait plus intuitif au dépens de la précision du langage en disant:

(13) Il y a sur $S$ deux antipodes telles que leurs voisinsages infiniment proches sont congruents.

Remarquons maintenant que l'on peut définir l'antipodie sur $S$ par la condition que la corde $PQ$ passe par un point fixe $O$ donné d'avance à l'extérieur de $V$. Considérons une suite $(O_n)$ de tels points convergents vers un point $O$ situé sur $S$. D'après (13), on trouve sur $S$ une suite de points $(P_n)$ et une autre $(Q_n)$ tels que leurs voisinsages respectifs sont congruents et que la corde $P_nQ_n$ contient $O_n$. Comme $O_n$ tend vers $O$, on peut extraire de $(P_n)$ ou de $(Q_n)$ une suite partielle $(P_n)$ tendant vers $P$. Appelons $T_n$ l'antipode de $P_n$ par rapport à $O_n$ — on peut extraire de $(T_n)$ une suite partielle $(T_n)$ tendant vers un point $T$ de $S$, leurs antipodes respectifs $E_1(T)$ (par rapport aux $O_n$) tendent toujours vers $P$. Il faut distinguer deux cas: 1° $T \neq P$, 2° $T = P$.

Dans le premier cas il existe sur $S$ un point différent de $P_n$ dont le voisinsage est congru à celui de $P_n$, dans le second cas il y a dans tout voisinsage de $P_n$ deux points dont les voisinsages respectifs sont congruents. Comme $P$ est arbitraire, on peut dire:

(14) Les points de $S$ peuvent être divisés en deux catégories: la première consiste de paires de points aux voisinsages respectivement congruents, la deuxième de limites de telles paires.

Exemple. Supposons que la Terre est un ellipsoïde de rotation et que le Pôle Nord soit plus aplati que le Pôle Sud. Alors les Pôles appartiennent à la deuxième catégorie, tous les autres points à la première.

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Algebraic models of axiomatic theories *)

by

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Let $S$ be a sentential calculus which contains the signs of disjunction, of conjunction and of implication, and perhaps some other sentential operators. We suppose that all theorems of the positive logic are theorems in $S$. The system $S$ determines a first order 1) functional calculus $S^*$.

The subject of the papers [10], [12] was the general algebraic method of the examination of a non-specified functional calculus $S^*$, with applications to the special functional calculi: of the two-valued logic $S_2^*$, of Heyting $S_H^*$, of Lewis $S_L^*$, of the positive logic $S^*$ and of the minimal logic $S^*_0$.

In this paper I shall apply the method mentioned above to the study of theories formalized on the basis of a logical calculus, which may be either a functional calculus $S^*$, or a functional calculus $S^*$ with equality. The theories with functions are included.

The first part of this paper contains the general definition of the model of a formalized theory based on a functional calculus $S^*$ (or $S^*$ with equality) where $S^*$ is not exactly specified. This definition is closely related to the general notion of satisfiability introduced in [10].

If $S^*$ is the classical functional calculus, this definition is a generalization of the definition of the model in the usual sense, called here the "semantic model". Known theorems on models of elementary axiomatic theories based on the classical logic hold also in the general case.

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1) For the exact description of the systems $S$ and $S^*$ see [10], §§ 1, 2.

2) See [8], p. 538.

3) The results of the first part establish an easy generalization of the results of [10] (see also L. Henkin [1]). They can also be regarded as generalizations of some investigations of J. Łoś [4]. The majority of them are necessary for the second part, containing the essential results of this paper.

Fundamenta Mathematicae, T. XXI. 19
The second part of the paper treats of axiomatic theories based on the Heyting functional calculus $G$. A theory $G$ is said to be constructive provided that:

1. If $\Sigma \xi$ is a theorem in $G$, then there exists a term $\xi$ such that:

the formula $\xi \left( \frac{\alpha}{\alpha} \right)$ which results from $\alpha$ by the substitution of $\xi$ for $\alpha$.\footnote{\textsuperscript{7}}

is a theorem in $G$;

2. If $\alpha + \beta$ is a theorem in $G$, then either $\alpha$ or $\beta$ is a theorem in $G$.

If a theory $G$ is constructive, one can associate with each formula $\alpha = \exists \beta \left( \theta \right)$ (where $\exists$ is a sequence of quantifiers and $\beta$ contains no quantifiers) a sequence $\alpha_0, \alpha_1, \ldots$ of formulas without quantifiers such that $\alpha$ is a theorem if and only if at least one of the formulas $\alpha_0, \alpha_1, \ldots$ is a theorem. This remark may be regarded as an analogue of Herbrand's\footnote{\textsuperscript{7}} theorem.

I shall formulate a necessary and sufficient condition for a theory $G$ to be constructive. This condition has a purely algebraic form. I shall prove that the theory $G$ is constructive whenever all axioms of $G$ (except the axioms of equality) belong to the least set $\mathcal{E}$ of formulas of $G$ such that:

1. $\mathcal{E}$ contains all elementary formulas of $G$;

2. If $\beta \rightarrow \gamma \in \mathcal{E}$, then $\beta \in \mathcal{E}$;

3. If $\gamma \in \mathcal{E}$, then $\beta \rightarrow \gamma \in \mathcal{E}$.

In particular, every theory whose axioms are equalities is constructive. For instance, the theories of groups, of rings, of Boolean algebras are constructive. More generally, if we eliminate the sign $+$ and $\Sigma$ from all axioms of a general theory $G$, that do not belong to $\mathcal{E}$, using de Morgan's laws, we obtain a weaker theory $G$ which is constructive. In this way one can obtain a constructive fragment of arithmetic.

The method used in the second part is similar to that used in [12], and is due essentially to Tarski and McKinsey\footnote{\textsuperscript{7}}.

As an application I obtain the theorem 4.11, which is stronger than the fundamental theorem (12) of [12] about $G$.

§ 1. Elementary axiomatic theories

Let $\mathcal{S}$ be a fixed consistent system of sentential calculus containing:

(a) the disjunction sign $+$, the conjunction sign $\cdot$, the implication sign $\rightarrow$;

(b) some other binary sentential operators $\alpha_1, \ldots, \alpha_n$;

(c) some unary sentential operators $\alpha_1' \ldots, \alpha_n'$.

The set of operators mentioned in (b) or (c) may be empty. The rules of inference in $\mathcal{S}$ are modus ponens, and the rule of replacement of equivalent parts. We suppose that all theorems of the positive sentential calculus are theorems of $\mathcal{S}$.

The system $\mathcal{S}$ determines in an obvious way a system $\mathcal{S}^*$ of the first order functional calculus with the following rules of inference: modus ponens, the rule of replacement of equivalent parts, the rule of substitution for individual variables, and the four known rules for quantifiers. The theorems in $\mathcal{S}^*$ are all substitutions of theorems of $\mathcal{S}$ and all their consequences.

The system $\mathcal{S}$ determines also a kind of abstract algebras (called $\mathcal{S}$-algebras\footnote{\textsuperscript{8}}) with algebraical operations corresponding to the logical sentential operators $+$, $\cdot$, $\rightarrow$, $\alpha_1, \ldots, \alpha_n$. The $\mathcal{S}$-algebras which are the matrices of the system $\mathcal{S}$, are relatively pseudocomplemented lattices (with the sum (join) $a+b$ and the product (meet) $a \cdot b$) having the unit element $e$, which is the distinguished element corresponding to the logical value of truth. If an $\mathcal{S}$-algebra is a complete lattice, it is called an $\mathcal{S}^*$-algebra. We shall suppose that the system $\mathcal{S}$ has the following property (B):\footnote{\textsuperscript{9}} given a denumerable set of infinite sums and products in an $\mathcal{S}$-algebra $A$, $a_0 = \sum a_0$, $b_0 = \prod b_0$, there is an isomorphism with respect to all the operations $+$, $\cdot$, $\rightarrow$, $\alpha_1, \ldots, \alpha_n$ of $A$ into an $\mathcal{S}^*$-algebra which preserves all these sums and products.

Assume the following notations. Let $\mathcal{I}$ always denote the set of all positive integers $\mathcal{I} = \emptyset$; $\mathcal{I}$ is a fixed set of integers, such that $\mathcal{I} \subseteq \mathcal{I}$; $\mathcal{I}$ (for every $\mathcal{I} \subseteq \mathcal{I}$) some fixed sets of positive integers which can be empty. We always suppose that there exists $k \mathcal{I} \subseteq \mathcal{I}$ such that $\mathcal{I} \neq \emptyset$. We suppose also, that the condition $\mathcal{I} \neq \emptyset$, for some $k \mathcal{I} \subseteq \mathcal{I}$.

An elementary theory $G(\mathcal{S})$ based on the system of logic $\mathcal{S}$ and on the set of axioms $\mathcal{S}$ can briefly be described as follows:

The primitive symbols of $G(\mathcal{S})$ are parentheses and individual constants $x_0$, where $x_0 \subseteq \mathcal{I}$; individual constants $x_0$, where $x_0 \subseteq \mathcal{I} \subseteq \mathcal{I}$;

functions with $1$ arguments $f_1$, (i.e. symbols for functions from individuals to individuals) where $f_1 \subseteq \mathcal{I} \subseteq \mathcal{I}$;

predicates (i.e. symbols for relations) with $k$ arguments $P_k$, where $k \mathcal{I} \subseteq \mathcal{I}$

\footnote{\textsuperscript{10}}.
sentential operators of the system \(S\), mentioned in (a), (b), (c); quantifiers \(\Sigma\) and \(\Pi\) where \(i \in I_0\).

The relation \(F_T^i\) will play an outstanding part in our consideration.

More exactly, \(F_T^i\) is the sign of equality of the system \(G(\mathcal{H})\) (see the axioms (*)) below.

Since for \(i \in I_0\), \(I_i\) may be the empty set, it is possible that the theory \(G(\mathcal{H})\) contains no functions.

Among the expressions which can be constructed from these signs we distinguish terms and formulas.

The set \(J_0\) of all terms is the least set such that:

(i) \(x_i \in J_0\) for \(i \in I_1\);
(ii) \(\xi_1, \ldots, \xi_n \in J_0\), then \(\xi(\xi_1, \ldots, \xi_n) \in J_0\) for \(n \in I_0\) and \(n \in I_1\).

In the case of \(I_1 = \varnothing\) for each \(i \in I_0\) the set \(J_0\) is the set of all \(x_i\) where \(i \in I_1\).

If \(\xi \in J_0\) and \(x_1, \ldots, x_n\) are all individual variables which appear in \(\xi\) and \(i_1 < i_2 < \ldots < i_t\), then we shall write also \(\xi(x_1, \ldots, x_n)\).

The set \(T\) of all formulas in \(G(\mathcal{H})\) is the least set fulfilling the following conditions:

(i) \(F_T(\xi_1, \ldots, \xi_n) \in T\) where \(\xi_1, \ldots, \xi_n \in J_0\) and \(n \in I_1\);\n(ii) \(\alpha, \beta \in T\) and \(i \in I_0\), then \((\alpha \Rightarrow \beta) \in T\), \((\alpha \Rightarrow \beta) \in T\), \((\alpha \& \beta) \in T\), \((\alpha \& \beta) \in T\), \((\alpha \& \beta) \in T\), \((\alpha \& \beta) \in T\), \((\alpha \& \beta) \in T\), \((\alpha \& \beta) \in T\).

We shall write, for brevity, \((\alpha \Rightarrow \beta)\) instead of \((\alpha \Rightarrow \beta)\).

In writing formulas we shall practice the omission of the parentheses, the rules being that:

1\(^{st}\) each of the operators \(\neg, +, \Rightarrow, \&\) binds less strongly than the previous one;
2\(^{st}\) each of the operators \(\exists, \forall\) binds an expression more strongly than any of the binary operators;
3\(^{rd}\) the quantifiers bind more strongly than any of the operators mentioned in 1\(^{st}\) and 2\(^{nd}\).

We assume that the notion of free and bound occurrence of an individual variable is familiar. A formula \(\alpha \in T\) is said to be closed if it contains no free occurrence of an individual variable \(x_i \in I_0\).

We assume that the set \(\mathcal{A}\) of all axioms of \(G(\mathcal{H})\) consists of some closed formulas belonging to \(T\). If the sign of equality \(F_T^1\) appears among the primitive signs of \(G(\mathcal{H})\) then the set \(\mathcal{A}\) contains the following set \(E\) of the axioms of equality:

\[ E = \left\{ F_T^1(x_1, x_2) \implies (a \Rightarrow a) \right\} \]

for \(a \in T\), where \(a \Rightarrow a\) results from \(a\) by the substitution \(\alpha\) of \(x_2\) for \(x_1\), and where \(E\) is written instead of the sequence of the quantifiers \(\exists\) binding all free occurrences of individual variables in the next formula.

The set of all axioms of \(G(\mathcal{H})\) except the axioms of equality will be denoted by \(\mathcal{A}\).

The set \(\mathcal{K}(\mathcal{H})\) of theorems of \(G(\mathcal{H})\) is the least set such that:

1\(^{st}\) \(\mathcal{K}(\mathcal{H})\) contains all axioms of \(G(\mathcal{H})\); \n2\(^{nd}\) \(\mathcal{K}(\mathcal{H})\) contains all substitutions of all theorems of \(S\); \n3\(^{rd}\) if \(\alpha \in \mathcal{K}(\mathcal{H})\) and \(\beta\) is obtained from \(\alpha\) by the admissible replacement of all free occurrences of \(x_i \in I_0\) by some \(\xi\), then \(\beta \in \mathcal{K}(\mathcal{H})\);

we shall always assume that the necessary changes in the bound occurrences of variables of \(a\) were performed before the operation of substitution;

4\(^{th}\) if \(\alpha \in \mathcal{K}(\mathcal{H})\), \((a \Rightarrow \beta) \in \mathcal{K}(\mathcal{H})\) then \(\beta \in \mathcal{K}(\mathcal{H})\);

5\(^{th}\) if \(\alpha \in \mathcal{K}(\mathcal{H})\) then \((a \Rightarrow \beta) \in \mathcal{K}(\mathcal{H})\);

6\(^{th}\) if there is no free occurrence of \(x_i\) in \(a\) in the \(\beta\), \(i \in I_0\), and \((a \Rightarrow \beta) \in \mathcal{K}(\mathcal{H})\), then \((a \Rightarrow \beta) \in \mathcal{K}(\mathcal{H})\);

7\(^{th}\) if \(\alpha \in \mathcal{K}(\mathcal{H})\) and \((\gamma \Rightarrow \delta) \in \mathcal{K}(\mathcal{H})\) if \(\gamma\) is a part of \(a\), then the formula \(\beta\) obtained from \(a\) by replacing the part \(\gamma\) by \(\delta\) is also in \(\mathcal{K}(\mathcal{H})\).

If \(\alpha \in \mathcal{K}(\mathcal{H})\) we shall write \(\mathcal{H} \vdash \alpha\).

The theory \(G(\mathcal{H})\) is consistent if there is a formula \(\alpha \in T\) such that \(\alpha \not\in \mathcal{K}(\mathcal{H})\).

§ 2. The Lindenbaum algebra \(L(\mathcal{H})\)

Let \(G(\mathcal{H})\) be a consistent theory, based on a fixed system of logic \(S\). For every \(\alpha \in T\) let \([\alpha]\) denote the class of all \(\beta \in T\) such that \(\mathcal{H} \vdash \alpha = \beta\).

Let \(L(\mathcal{H})\) be the set of all classes \([\alpha]\) where \(\alpha \in T\). We define in \(L(\mathcal{H})\) the algebraical operations \(+, \cdot, \vdash, \alpha, \ldots, \alpha, \vdash\) as follows:

\[ [\alpha][\beta] = [\alpha \& \beta] \]

if \(\alpha\) is one of the binary logical operations of \(S\) and \([\alpha] = [\alpha]\) if \(\alpha\) is one of the unary operations of \(S\). Since the relation \([\alpha] \vdash [\beta] = [\alpha \Rightarrow [\beta]]\) between \(\alpha\) and \(\beta\) is a congruence relation in the sense of modern algebra, the definition of operations in \(L(\mathcal{H})\) is correct. The element \([\alpha]\) where \(\alpha \in \mathcal{K}(\mathcal{H})\) will be denoted by \(e\).

2.1. The algebra \(L(\mathcal{H})\); \(\vdash, +, \cdot, \alpha, \ldots, \alpha\) is an \(S\)-algebra; more precisely, it is a relatively pseudocomplemented lattice with the unit element.
The algebra $L(\mathfrak{M})$ is analogous to the algebra $L(R)$ of [10].

2.2. $|a| \subseteq |\beta|$ if and only if $\mathfrak{M} \vdash a \Rightarrow \beta$. $|a| = |\beta|$ if and only if $\mathfrak{M} \vdash a \land \beta$.

2.3. For every $a \in T$

\[\sum_{x_i} a \left( \frac{\xi}{x_i} \right) = \sum_{x_i} a,\]

\[\prod_{x_i} a \left( \frac{\xi}{x_i} \right) = \prod_{x_i} a.\]

The proof, similar to that of 4.3 in [10], is omitted. Obviously, $\sum$ and $\prod$ on the left side of the equalities (* and **) are the signs of sum and product in $L(\mathfrak{M})$, respectively, and the signs $\sum$, $\prod$ on the right side of these equalities are quantifiers.

By an analogous reasoning to that used in the proof of 2.3 we obtain also

\[\sum_{x_i} a \left( \frac{\xi}{x_i} \right) = \sum_{x_i} a,\]

\[\prod_{x_i} a \left( \frac{\xi}{x_i} \right) = \prod_{x_i} a.\]

If no function symbols occur among the primitive symbols of $L(\mathfrak{M})$, the equalities (*) and (**) are reducible to the equalities (i) and (ii) respectively.

According to [10] (1) an $\mathcal{S}$-homomorphism (or $\mathcal{S}$-isomorphism) $h$ of $L(\mathfrak{M})$ into an $\mathcal{S}$-algebra $A$ is said to be an $\mathcal{S}$-homomorphism (or $\mathcal{S}$-isomorphism) if $h$ preserves all the sums (*) and all the products (**). An $\mathcal{S}$-algebra $A$ is said to be an $\mathcal{S}$-extension of $L(\mathfrak{M})$ if there is an $\mathcal{S}$-isomorphism of $L(\mathfrak{M})$ into $A$. Obviously such an $\mathcal{S}$-extension exists since $\mathcal{S}$ has the property (E).

§ 3. Algebraic models of elementary axiomatic theories

We shall consider a theory $\mathcal{L}(\mathfrak{M})$ based on a fixed logical calculus $\mathcal{S}$. Let $J$ be a non-empty set. Let $A$ be an $\mathcal{S}$-algebra. The class of all mappings of the Cartesian product $J^a$ into $A$ will be denoted by $F^a(J, A)$. $\varphi^a, \psi^a$ will always denote functions belonging to $F^a(J, A)$. Let $f^a(J)$ be the class of all mappings of the Cartesian product $J^a$ into $J$. The letters $\xi^a$ will always denote elements of $f^a(J)$.

Every formula $\alpha \in T$ may be interpreted as a $(J, A)$-functional $\mathcal{B}$ denoted by $(J, A)\Phi_\alpha$, by regarding

(a) all individual variables $x_i (\forall i \in I_a)$ (individual constants $x_i$, where $i \in I - I_a$) as variables running over $J$ (as fixed elements of $J$);
(b) all functions $f^a_i (\forall i \in I_a, \forall i \in I_1)$ as fixed elements of $F^a_i (J)$;
(c) all $k$-ary predicate symbols $P^a_k$ as fixed elements of $F^a_i (J, A)$;
(d) each of the logical operations of $\mathcal{S}$ mentioned in § 1 ((a), (b), (c)) as a corresponding $\mathcal{S}$-extension of $A$;
(e) the logical quantifiers $\sum$ and $\prod$ on the signs of infinite sums.

Let $(A)\mathcal{X}$ and products $(A)\prod_{x_i}$ in the algebra $(A)$, respectively $(i \in I_a)$.

Let

\[x_i = j_i \in J \quad (i \in I)\]

\[f^a_i = \varphi^a_i \in F^a_i (J) \quad (i \in I_a, \forall i \in I_1)\]

\[\mathcal{B} = \varphi^a_i \in F^a_i (J, A) \quad (k \in I_a, \forall i \in I_k)\]

be arbitrary but fixed system of values of individual signs, functions and predicate symbols of $\mathcal{L}(\mathfrak{M})$. This system of valuations will also be denoted by $(\mathcal{B}, \varphi^a_i, \varphi^a_i)$. Let $(\mathcal{B}, \varphi^a_i, \varphi^a_i)$ always denote substitution $(\mathcal{B})$ for individual signs of $\mathcal{L}(\mathfrak{M})$ reduced to $i \in I - I_a$ and to $i \in I_a$ respectively.

The symbol $(J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i))$ will denote the value of the functional $(J, A)\Phi_\alpha$ for the values of its arguments fixed above by $(\mathcal{B}, \varphi^a_i, \varphi^a_i)$. If $\mathfrak{A}$ is a closed formula of $T$, then the value of the functional $(J, A)\Phi_\alpha$ does not depend on the values of $x_i$, where $i \in I_a$. Therefore we write $(J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i))$ instead of $(J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i))$.

It is easy to verify that

3.1. $\mathcal{B} = \alpha \in F^a(J)\]

\[(J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i)) = (J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i))\]

where $\mathcal{B} = \mathcal{B}_p \in J$, if $i \neq p$, and $j_i = \mathcal{B}_i$ is the value of $i$ by the substitutions $(\mathcal{B}, \varphi^a_i)$.\]

3.2. $\mathcal{B} \in F^a(J)\]

\[(J, A)\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i)) = (J, A')\Phi_\alpha(j_i, (\mathcal{B}, \varphi^a_i))\]

where $\mathcal{B} \in F^a(J)$.

See [10], 64, p. 72.\]

12] See [10], 54, p. 72.

13] See [10], 54, p. 72.
The system $\mathcal{M} = \{ (i_j)^{-1}, (t_i^j), (p_{i_j}) \}$ is said to be a generalized model of the theory $\mathcal{G}(\mathfrak{F})$ in the algebra $A$ and in the domain $J$ if, for every $a \in \mathfrak{F}$, 

$$(J, A) \phi_a ((i_j)^{-1}, (t_i^j), (p_{i_j})) = a.$$ 

If among the primitive symbols of $\mathcal{G}(\mathfrak{F})$ the sign of equality $E$ does not occur, then a generalized model of $\mathcal{G}(\mathfrak{F})$ will be called a model of $\mathcal{G}(\mathfrak{F})$.

Suppose that $E$ is the primitive symbol of $\mathcal{G}(\mathfrak{F})$. Then the function $\phi$ establishes an interpretation of the equality sign $E$ in the generalized model $\mathcal{M}$, by the relation $\equiv$, defined as follows:

$$j_k \equiv j_i \text{ if and only if } \phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = a.$$ 

### 3.3. The relation $\equiv$ is a congruence relation.

Obviously, it is possible that $\phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = a_{j_k}$. 

A $(J, A)$ generalized model $\mathcal{M}$ of a theory $\mathcal{G}(\mathfrak{F})$ with equality is said to be the $(J, A)$ model $M$ of $\mathcal{G}(\mathfrak{F})$ if, for every $j_k, j_i \in J$, $\phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = a_{j_k}$ if and only if $j_k = j_i$. 

### 3.4. Let $\mathcal{M} = \{ (i_j)^{-1}, (t_i^j), (p_{i_j}) \}$ be a generalized model of a theory $\mathcal{G}(\mathfrak{F})$ with equality. For every $j \in J$, let $\mathfrak{L}_j$ be the class of all $i \in I$ such that $\phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = a_j$. Let $\mathfrak{F}$ be the set of all classes $[j_i]$, where $j \in J$. Obviously $\mathfrak{F} \subset J$.

Further let

$$\mathfrak{L}_k((i_j)^{-1}, (t_i^j), (p_{i_j})) = \mathfrak{L}_k((i_j)^{-1}, (t_i^j), (p_{i_j}))$$ 

for any $j_k, j_i \in J$, 

$$\phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = \phi^E((i_j)^{-1}, (t_i^j), (p_{i_j}))$$ 

for any $j_k, j_i \in J$. 

Then the generalized model $\mathcal{M} = \{ (i_j)^{-1}, (t_i^j), (p_{i_j}) \}$ is a generalized model of the theory $\mathcal{G}(\mathfrak{F})$.

The easy proof is omitted.

Consider a system $\{ (i_j), (t_i^j), (p_{i_j}) \}$ of valuations for primitive symbols of $\mathcal{G}(\mathfrak{F})$. Suppose that $\mathcal{M} = \{ (i_j)^{-1}, (t_i^j), (p_{i_j}) \}$ is a generalized model of $\mathcal{G}(\mathfrak{F})$ in $J$ and $A$. Then, instead of $(J, A) \phi_a ((i_j)^{-1}, (t_i^j), (p_{i_j}))$, we shall write $(J, A, \mathcal{M}, \phi_a ((i_j)^{-1}))$.

We shall say that a theory $\mathcal{G}(\mathfrak{F})$ has a model in a domain $J$, if there is an $\mathfrak{F}$-algebra $A$ such that $\mathcal{M}$ has a model in $J$ and $A$. More generally, we shall say that a theory $\mathcal{G}(\mathfrak{F})$ has a model, if there is a domain $J$ and an $\mathfrak{F}$-algebra $A$ such that $\mathcal{M}$ has a model in $J$ and $A$.

### 3.5. Let $\mathcal{G}(\mathfrak{F})$ be an arbitrary theory and let $\mathcal{M}$ be a generalized model of $\mathcal{G}(\mathfrak{F})$ in a domain $J$ and an $\mathfrak{F}$-algebra $A$. Then, given arbitrary a $\mathfrak{F}$-algebra and arbitrary system of valuations $(i_j)^{-1}$ for individual variables in $\mathcal{G}(\mathfrak{F})$, we have $(J, A, \mathcal{M}, \phi_a (i_j)^{-1}) = a$. In other words, if $a$ is a theorem of $\mathcal{G}(\mathfrak{F})$, then $(J, A, \mathcal{M}, \phi_a (i_j)^{-1}) = a$ identically in every generalized model $\mathcal{M}$ of $\mathcal{G}(\mathfrak{F})$.

This follows from the definition of a generalized model and the fact that the class of formulas $\beta \in \mathcal{T}$, having the property $\phi_a (i_j)^{-1}$ identically in a fixed generalized model $\mathcal{M}$ of $\mathcal{G}(\mathfrak{F})$, is closed under the rules of inference.

### 3.6. Let $\mathcal{G}(\mathfrak{F})$ be a consistent theory. Let $h$ be an $\mathfrak{F}$-homomorphism of $L(\mathfrak{F})$ into an $\mathfrak{F}$-algebra $L'$. Then, for every $a \in \mathcal{T}$,

$$(J, A, L') \phi_a (i_j), (t_i^j), (p_{i_j}) = h | a,$$

where

$$\phi^E((i_j)^{-1}, (t_i^j), (p_{i_j})) = h | a,$$

Consequently, the system $\{ (i_j)^{-1}, (t_i^j), (p_{i_j}) \}$ is the generalized model of $\mathcal{G}(\mathfrak{F})$ in the domain $J$ and the algebra $L$. This generalized model will be called the natural generalized model and will always be denoted by $\mathcal{G}(\mathfrak{F}, h, L')$ or briefly by $\mathcal{G}(\mathfrak{F})$.

The easy proof by induction with respect to the length of $a$, based on the definitions of $(J, A)$-functional $\phi_a$, of $L(\mathfrak{F})$ and of $\mathfrak{F}$-homomorphism, and on the lemma 2.3, is omitted.

### 3.7. If $\mathcal{G}(\mathfrak{F})$ is a consistent theory with equality (without equality), then $\mathcal{G}(\mathfrak{F})$ has a model in a domain whose cardinal number is not greater than $\kappa$ (is $\kappa$).

Indeed, if $\mathcal{G}(\mathfrak{F})$ is a theory without equality, then the natural generalized model $\mathcal{G}(\mathfrak{F}, h, L')$ is a model in the domain $J = \kappa$. In this case $\mathcal{G}(\mathfrak{F})$ will be called the natural model of $\mathcal{G}(\mathfrak{F})$. If $\mathcal{G}(\mathfrak{F})$ is a theory with equality, then 3.7 follows from 3.6 and 3.4. The model $\mathcal{G}(\mathfrak{F})$ obtained by $\mathcal{G}(\mathfrak{F})$ by the method mentioned in the formulation of 3.4, will also be called the natural model of $\mathcal{G}(\mathfrak{F})$.

### 3.8. Let $\mathfrak{F}$ be a logical system with the negation sign $\neg$, such that the formulas $\neg (\beta \rightarrow \beta) \rightarrow \alpha$ are theorems of $\mathfrak{F}$. If a theory $\mathcal{G}(\mathfrak{F})$, based on the system $\mathfrak{F}$, has a model, then $\mathcal{G}(\mathfrak{F})$ is consistent.

Suppose that $\mathcal{G}(\mathfrak{F})$ is not consistent. Let $a$ be an axiom of $\mathcal{G}(\mathfrak{F})$, and let $\mathcal{M}$ be a model of $\mathcal{G}(\mathfrak{F})$ in an algebra $A$ and a domain $J$. Thus

\[ \neg (\beta \rightarrow \beta) \rightarrow \alpha \rightarrow \neg \alpha \]
we have \( (J, A, M) \vDash \varphi \) since \( \mathcal{G}(K) \) is not consistent, \( K \vdash -\varphi \). Hence, by 3.5, \( (J, A, M) \vDash -\varphi \). On the other hand, \( (J, A, M) \vDash \varphi \). Consequently, \( \varphi = -\varphi \), which is impossible, since in \( \mathcal{F} \)-algebras of the logical systems such that \( -\varphi \rightarrow \varphi \) are theories, we have \( \vDash \varphi = -\varphi \).

3.9. Let \( \mathcal{S} \) be the system with the negation sign \( \neg \), satisfying the following condition: \( \neg (\beta \rightarrow \neg \beta) \rightarrow \alpha \) in a theorem of \( \mathcal{S} \). Then, if a theory \( \mathcal{G}(K) \) with equality (without equality), based on the system \( \mathcal{S} \), has a model, it has a model in a domain whose cardinal number is not greater than \( \kappa_\alpha \) (is equal to \( \kappa_\alpha \)).

This follows immediately from 3.8 and 3.7.

Notice that theorems 3.6, 3.7, and 3.8 are analogous to 5.2, 5.2, 7.2 and 7.3 of [10] respectively.

Theorems 3.7 and 3.9 for the special case where \( \mathcal{S}_\alpha \) is the classical functional calculus, are well known Gödel and Skolem-Löwenheim results. Indeed, Tarski's original definition of satisfiability may be translated into the algebraic language \(^{30}\). Consequently, a theory \( \mathcal{G}(K) \) based on \( \mathcal{S}_\alpha \) has a semantic model in a domain \( J \) if and only if it has a model in \( J \) and the two-element Boolean algebra \( B_\alpha \). Thus the statement mentioned above results from the following theorem:

3.10. If a theory \( \mathcal{G}(K) \) with equality (without equality) based on the classical functional calculus \( \mathcal{S}_\alpha \) has a model in a domain \( J \neq \emptyset \) and in a complete Boolean algebra \( B_\alpha \), then:

(i) \( \mathcal{G}(K) \) has a model in a domain whose cardinal number is not greater than \( J \) (in the domain \( J \)) and in the two-element Boolean algebra \( B_\alpha \).

(ii) \( \mathcal{G}(K) \) has a model in a domain whose cardinal number is not greater than \( \kappa_\alpha \) (is equal to \( \kappa_\alpha \)) and in \( B_\alpha \).

The proof is analogous to the proof of 9.4 in [10].

Notice moreover that there are theories, based on \( \mathcal{S}_\alpha \), with no semantic models in a domain \( J \), having algebraic models in that domain. For instance, the theory \( \mathcal{G}(\mathcal{W}_\theta + \mathcal{E}) \) with equality and the single axiom

\[
\sum_{z_1} \sum_{z_2} \left[ -P_{z_1}(z_1, z_2) : \left( P_{z_2}(x, z_1) + P_{z_2}(x, z_1) \right) \right],
\]

Indeed, \( \mathcal{G}(\mathcal{W}_\theta + \mathcal{E}) \) has no infinite semantic models; on the other hand, the domain of the natural model is denumerable.

\(^{30}\) See [10], 5.1, p. 74.

\(^{31}\) Cf. for instance, L. Henkin (1).

\(^{32}\) Cf. [9], p. 196.
In the case of $\mathcal{V}_3 \neq \mathcal{V}_3$, the proof is similar. Suppose now that $a = b + \gamma$. Then $\mathcal{V}_3 = \mathcal{V}_3 \iff \mathcal{V}_3$, then $\mathcal{V}_3 = \mathcal{V}_3$, then $h[a] = h[\gamma] = h[a]$. Consequently $h[a] = h[\beta + \gamma] = h[\gamma] = h[\beta]$. If $\mathcal{V}_3 = \mathcal{V}_3$, then $\mathcal{V}_3 = \mathcal{V}_3$. Hence $\mathcal{V}_3 = \mathcal{V}_3$. Thus $h[a] = h[\beta]$. Consider now the case of $\mathcal{V}_3 \neq \mathcal{V}_3$. Then $\mathcal{V}_3 = \mathcal{V}_3$. Moreover, since $\mathcal{V}_3 = \mathcal{V}_3$, we have $\mathcal{V}_3 = h[\gamma]$. Suppose that $\mathcal{V}_3 = \mathcal{V}_3$. Then $h[\beta] = h[\gamma]$. Hence $\mathcal{V}_3 = \mathcal{V}_3 = \mathcal{V}_3 = h[\gamma] = h[a]$.

If $\mathcal{V}_3 = \mathcal{V}_3$, then $\mathcal{V}_3 = h[\beta]$. Hence $\mathcal{V}_3 = h[\beta] = h[\gamma] = h[a]$. Suppose that $\mathcal{V}_3 = \mathcal{V}_3$. Then $\mathcal{V}_3 = \mathcal{V}_3$. Hence $\mathcal{V}_3 = \mathcal{V}_3$. That is $h[a] = h[\beta] = h[\gamma] = h[a]$. Let $\mathcal{V}_3 = \mathcal{V}_3$, then $h[\beta] = h[\gamma]$. Thus $\mathcal{V}_3 = \mathcal{V}_3$, then $\mathcal{V}_3 = h[\beta] = h[a]$. Hence $\mathcal{V}_3 = \mathcal{V}_3$. But $\mathcal{V}_3 = \mathcal{V}_3$. Thus $\mathcal{V}_3 = \mathcal{V}_3 = \mathcal{V}_3 = h[\beta] = h[a]$.

Consider now the case of $a = \beta$.

\[ \mathcal{V}_3 = \mathcal{V}_3 = \mathcal{V}_3 = h[\beta] = h[a] \]

where $\mathcal{V}_3 = \mathcal{V}_3$. If $h[\beta] = h[a]$, then $\mathcal{V}_3 = \mathcal{V}_3$. Consequently making use of 3.3 and of the property of $\mathcal{V}_3$-isomorphism $h$ we infer that

\[ h[a] = h[\beta] \]

If $\mathcal{V}_3 = \mathcal{V}_3$, then for every $\mathcal{V}_3 \neq J$, $\mathcal{V}_3 = h[\beta]$. Hence, by (**) $\mathcal{V}_3 = \mathcal{V}_3$. In the case of $a = \beta$ the proof is similar.

4.2. Let $\mathcal{V}_3 = h[\beta]$. Then for arbitrary $a \in T(\mathcal{V}_3)$, involving the term $\mathcal{V}_3$ and such that every individual variable occurring in $\mathcal{V}_3$ is free in $a$, we have

\[ \mathcal{V}_3 = \mathcal{V}_3 \]

where $a(\mathcal{V}_3)$ denote the formula obtained from $a$ by the replacement of the term $\mathcal{V}_3$ by $\mathcal{V}_3$.

We recall that the necessary changes in the bound variables of $a$ performed before the operation of substitution.

4.3. Let $\mathcal{V}_3 = h[\beta]$. Then for arbitrary $a \in T(\mathcal{V}_3)$, involving the term $\mathcal{V}_3$ and such that every individual variable occurring in $\mathcal{V}_3$ is free in $a$, we have

\[ \mathcal{V}_3 = \mathcal{V}_3 \]

where $a(\mathcal{V}_3)$ denote the formula obtained from $a$ by the replacement of the term $\mathcal{V}_3$ by $\mathcal{V}_3$.
Indeed, consider a formula $a$ as a formula which is obtained from some formula $\beta$ by substituting the term $\xi$ for an individual variable $x_\gamma$. Obviously $\beta$ is a formula with one free occurrence of $x_\gamma$, and we can suppose that the quantifiers binding the individual variables which appear in $\xi$ and in $x_\gamma$ do not occur in $\beta$. It follows from the axioms of equality and $\forall \gamma \vdash \text{F}_1^1(\xi, x_\gamma)$ that $\forall \gamma \vdash \beta(\xi) \leftrightarrow \beta(x_\gamma)$. Thus $\forall \gamma \vdash a(\xi) \leftrightarrow a(x_\gamma)$.

Hence, by a simple inducive argument, $\forall \gamma a = \forall \gamma a^\ast$.

4.3. If $\Theta \subseteq \Psi$ and $a \in \Theta$ (i. e. if $a$ is one of the axioms of equality of the theory $\Theta$), then $\forall \gamma a = \forall \gamma a^\ast$.

Suppose that $a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$. For every $x \in J_a$ we have $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$.

Hence $\text{F}_1^1(\xi, x) = \forall \gamma a$. Consequently, $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(\xi, x) = \forall \gamma a^\ast$.

Suppose now that $a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$, where $\xi_1, \xi_2$ are arbitrary terms of $\Theta$. To show that $\forall \gamma a = \forall \gamma a^\ast$, it suffices to prove that $\forall \gamma a = \forall \gamma a^\ast$.

Suppose that $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$. Then, on account of 4.2, $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$.

Hence $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$ = $\forall \gamma a^\ast$. Consequently, $\forall \gamma a = \forall \gamma a^\ast$.

Consider the case where non $\Psi_1 \vdash \forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$. Since $\Psi_1 \vdash \beta$, we obtain $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma) \in \Psi_1 \vdash \forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$.

Then by 4.1, $\forall \gamma a = \bigvee_{x_\gamma} \text{F}_1^1(x_\gamma, x_\gamma)$ = $\forall \gamma a^\ast$.

A theory $\Theta$ is said to be constructive if it fulfills the following conditions:

(I) $a \vee \beta \in \Theta$ implies that $a \in \Theta$ or $\beta \in \Theta$,

(II) if $\Sigma a \in \Theta$, then there exists $\xi \in J_a$ such that $a(\xi) \in \Theta$.

4.4. Let $\Theta$ be a theory such that $\Delta^0_1 = \{\langle \alpha \rangle, \langle \xi_1 \rangle, \langle \xi_2 \rangle\}$ is the generalized model of $\Theta$. Then $\Theta$ is a constructive theory.

The proof of 4.4 is similar to that of the fundamental theorem in [12].

Since $\exists \gamma a$ is the open subset of $\exists \gamma a$, the formula

$$\phi(\Delta) = a \cdot \exists \gamma a$$

defines an $\exists \gamma a$-homomorphism of the $\exists \gamma a$-algebra $H(\exists \gamma a)$ onto the $\exists \gamma a$-algebra $H(\exists \gamma a)$. This homomorphism preserves all infinite sums and products.

Suppose that $\beta = \sum_{\gamma} a(\xi) \in K(\Theta)$. Hence, by 3.5,

$$\exists \gamma a = \sum_{\gamma} |\alpha| \cdot H(\exists \gamma a) \cdot \psi(\alpha, \gamma)$$

where $\alpha$ denotes the sequence of the $p$-th term of which replaced by $\xi$.

By (4) there is a component of the sum equal to $\exists \gamma a$, i. e. there exists such $\xi \in J_\alpha$, that

$$\langle \xi, H(\exists \gamma a) \rangle \cdot \phi(\xi, \gamma, \langle \xi_1 \rangle, \langle \xi_2 \rangle, \langle \xi_3 \rangle) = \exists \gamma a$$

where $\xi = \xi_i$ for $i \neq \xi$ and $\xi = \xi_\gamma$.

Let $\chi(\xi) = a^\ast(\xi)$, by 3.1

Then $\gamma = a(\xi)$, by 3.1.

$$\phi(a^\ast(\xi), \gamma) = \langle \xi, H(\exists \gamma a) \rangle \cdot \phi(\xi, \gamma, \langle \xi_1 \rangle, \langle \xi_2 \rangle, \langle \xi_3 \rangle) = \exists \gamma a$$

Hence we obtain by 3.2 and 3.6, on account of $\gamma(\xi_1, \xi_2, \xi_3) = h(\sum_{\gamma} |\alpha| \cdot H(\exists \gamma a))$

$$\exists \gamma a = g(H(\exists \gamma a)) = g(\langle \xi, H(\exists \gamma a) \rangle \cdot \phi(\xi, \gamma, \langle \xi_1 \rangle, \langle \xi_2 \rangle, \langle \xi_3 \rangle))$$

$$= (\langle \xi, H(\exists \gamma a) \rangle \cdot \phi(\xi, \gamma, \langle \xi_1 \rangle, \langle \xi_2 \rangle, \langle \xi_3 \rangle)) \cdot \phi(\xi, \gamma)$$

Consequently, by 2.2, $\gamma = a(\xi) \in K(\Theta)$.

In a similar way we can prove the property (1) of the constructive theories.

It follows immediately from 4.3 and 4.4 that

4.5. The functional calculus of Heyting with equality is a constructive theory.

4.6. Let $\Theta$ be an arbitrary theory and let $\Delta$ be the set of all formulas $a \in T_\Delta(\Theta)$ satisfying the following condition

$\exists \gamma a$ implies that $\forall \gamma a = \forall \gamma a^\ast$.

Then the least set $\Delta$ of formulas of $\Theta$ such that:

1. if $\beta \in \Delta$, then $\beta \in \Delta$ and $\beta \in \Delta$

2. if $\gamma \in \Delta$ and $\beta \in \Delta$, then $\beta \gamma \in \Delta$

3. if $\gamma \in \Delta$ and $\beta \in \Delta$, then $\beta \gamma \in \Delta$ and $\beta \gamma \in \Delta$

is contained in $\Delta$. 

Indeed, if \( \beta = F_{\alpha}^j(x_1, \ldots, x_k) \) and \( \mathfrak{M} \models \beta \), then \( \mathfrak{M}[\beta] = \mathfrak{M}[F_{\alpha}^j(x_1, \ldots, x_k)] = \mathfrak{M}^\alpha_{x_k} \).

Thus \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\alpha_{x_k}(x_1, \ldots, x_k) = \mathfrak{P}^\alpha_{x_k} \).

Suppose that \( \beta \cdot \gamma \subseteq Z \) and \( \alpha = \beta \cdot \gamma \). If \( \mathfrak{M} \models \alpha \), then \( \mathfrak{M} \models \beta \) and \( \mathfrak{M} \models \gamma \).

Consequently, \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \cdot \mathfrak{P}^\gamma_{x_k} \). Hence \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Now suppose that \( \gamma \subseteq Z \) and \( \alpha = \beta \cdot \gamma \). Let \( \mathfrak{M} \models \alpha \).

Then \( \mathfrak{P}^\beta_{x_k} \subseteq \mathfrak{P}^\gamma_{x_k} \).

If \( \beta \cdot \gamma \subseteq \mathfrak{P}^\gamma_{x_k} \), then \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \cdot \mathfrak{P}^\gamma_{x_k} \).

Hence \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

In the case of \( \beta \cdot \gamma \not\subseteq \mathfrak{P}^\gamma_{x_k} \), then \( \mathfrak{P}^\alpha_{x_k} \not\subseteq \mathfrak{P}^\beta_{x_k} \).

Then by 4.1, \( \mathfrak{P}^\beta_{x_k} \not\subseteq \mathfrak{P}^\gamma_{x_k} \).

Thus \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Since \( \beta \cdot \gamma \subseteq \mathfrak{P}^\gamma_{x_k} \), we have \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Suppose that \( \gamma \subseteq Z \) and \( \mathfrak{M} \models \alpha \).

Hence \( \mathfrak{P}^\beta_{x_k} = \mathfrak{P}^\beta_{x_k} \).

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Thus \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

In the case of \( \gamma \subseteq Z \) and \( \mathfrak{M} \models \alpha \), we have \( \mathfrak{P}^\beta_{x_k} = \mathfrak{P}^\gamma_{x_k} \).

Thus \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

That \( T_{\mathfrak{M}}(\mathfrak{N}) = Z \). On account of the proof of the theorem 4.6 it suffices to show that

i) \( \beta \cdot \gamma \subseteq Z \) then \( \beta \cdot \gamma 

\[\text{(i) if } \beta \cdot \gamma \subseteq Z \text{ then } \beta \cdot \gamma \subseteq Z, \]

(ii) if \( \beta \cdot \gamma \subseteq Z \) then \( \beta \cdot \gamma \subseteq Z, \)

Suppose that \( \mathfrak{M} \models \alpha \) and \( \mathfrak{M} \models \beta \). Since \( \mathfrak{N}_{\mathfrak{M}} = \mathfrak{M} \), then \( \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M} \).

Hence \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Consequently, \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Now let \( \alpha = \beta \cdot \gamma \).

It follows from the constructivity of \( \mathfrak{N}_{\mathfrak{M}} \) that there exists such \( \xi \in \mathfrak{M} \) that \( \mathfrak{M} \models \beta \cdot \gamma \).

Hence \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Thus \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

It follows from 4.7 that every theory whose axioms are equalities is constructible. In particular, the theories of groups, of rings, of lattices of Boolean algebras, of closure algebras are constructive. More generally, every elementary theory \( \mathfrak{N}_{\mathfrak{M}} \) can be modified so as to be a constructible one. It suffices to remove the existential quantifiers and the sign of the alternative from the axioms \( \alpha \in \mathfrak{N}_{\mathfrak{M}} \) of \( \mathfrak{N}_{\mathfrak{M}} \), by joining the functors \( ^* \) or by the use of de Morgan’s laws. In the last case we obviously obtain a theory weaker than \( \mathfrak{N}_{\mathfrak{M}} \).

We intend to apply the results mentioned above to arithmetic in the following form. Consider the system \( \mathfrak{N}_{\mathfrak{M}}(\mathfrak{M} + \mathfrak{D}) \) of arithmetic, in the description of which we shall use, for convenience, the generally assumed notation. As specific constants of \( \mathfrak{N}_{\mathfrak{M}}(\mathfrak{M} + \mathfrak{D}) \) let us assume the individual constant \( \mathfrak{M} \), the one-argument functors \( \mathfrak{M} \) and \( \mathfrak{M} \), and the sign of equality \( \mathfrak{M} \).

The set of axioms consists of the axioms of equality and of the following formulae:

\[
\begin{align*}
\forall x_1 (x_1 = 1), \\
\forall x_1 (x_1 = 1), \\
\forall x_1 (x_1 = 1). \\
\end{align*}
\]

\[\begin{align*}
\forall x_1 (x_1 \neq 1), \\
\forall x_1 (x_1 \neq 1), \\
\forall x_1 (x_1 \neq 1). \\
\end{align*}
\]

Suppose that \( \alpha = \beta \cdot \gamma \) and \( \mathfrak{M} \models \beta \). Since \( \mathfrak{N}_{\mathfrak{M}} \) is constructive, then either \( \mathfrak{M} \models \beta \) or \( \mathfrak{M} \models \gamma \).

Hence \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Consequently, \( \mathfrak{P}^\alpha_{x_k} = \mathfrak{P}^\beta_{x_k} \).

Now let \( \alpha = \beta \cdot \gamma \).

It follows from 4.7 that every theory whose axioms are equalities is constructible. In particular, the theories of groups, of rings, of lattices of Boolean algebras, of closure algebras are constructive. More generally, every elementary theory \( \mathfrak{N}_{\mathfrak{M}} \) can be modified so as to be a constructible one. It suffices to remove the existential quantifiers and the sign of the alternative from the axioms \( \alpha \in \mathfrak{N}_{\mathfrak{M}} \) of \( \mathfrak{N}_{\mathfrak{M}} \), by joining the functors \( ^* \) or by the use of de Morgan’s laws. In the last case we obviously obtain a theory weaker than \( \mathfrak{N}_{\mathfrak{M}} \).

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The set of axioms consists of the axioms of equality and of the following formulae:

\[
\begin{align*}
\forall x_1 (x_1 = 1), \\
\forall x_1 (x_1 = 1), \\
\forall x_1 (x_1 = 1). \\
\end{align*}
\]

\[\begin{align*}
\forall x_1 (x_1 \neq 1), \\
\forall x_1 (x_1 \neq 1), \\
\forall x_1 (x_1 \neq 1). \\
\end{align*}
\]
The expression \( \gamma(\overline{a}_{1}, \ldots, \overline{a}_{n}) \) has not been uniquely determined. However, it will be uniquely determined as follows: let \( j_{1} \) be the least positive integer such that \( j_{1} \neq i_{k}, \) \( k = 1, \ldots, l \) and \( \gamma \) contains neither \( a_{j_{1}} \) nor \( \bigwedge \) nor \( \sum \). Let \( j_{k} (q = 2, \ldots, l) \) be the least positive integer satisfying \( a_{j_{k}} \) all conditions for \( j_{k} \) and also \( j_{k} \neq j_{r} (r = 1, \ldots, q - 1) \). We replace every occurrence of the bound variable \( z_{q} \) \((q = 1, \ldots, l)\) by \( x_{j_{k}} \) and every quantifier \( \Sigma(I) \) by \( \Sigma(I) \). Further, we replace every free variable \( z_{q} \) by the \( a_{j_{k}} \) term \( \xi_{q_{1}} \ldots, \xi_{q_{n}} \). If \( \beta = \sum a_{j_{k}} \), then we shall denote by \( Z(a_{j_{k}}) \) the set of all formulas \( a \) where either \( \xi = x_{j_{k}} \) or \( \xi = \xi_{q_{1}} \ldots, \xi_{q_{n}} \) and \( a \) contains at least one free occurrence of every \( x_{q_{1}} \ldots, x_{q_{n}} \), or \( \xi \) contains no individual variables. If \( \beta = \bigwedge a_{j_{k}} \), then \( Z(a_{j_{k}}) \) is the set containing only one element: the formula \( a \). More generally, if \( ECT_{\omega}(R) \) is a set of formulas \( \beta \) of the form \( \beta = \sum a \) or \( \beta = \bigwedge a \), then \( Z(\beta) \) is the union of all sets \( Z(\beta) \) where \( \beta \in R \). Suppose now that \( \beta \) is a formula of the form \( (\ast) \). Let \( E_{k} = Z(\beta) \) and, by induction, \( E_{k} = Z(E_{k-1}) \ldots, E_{2} = \bigwedge E_{1} \). It follows from the assumption of \( T_{\omega}(\mathbb{N}) \) being constructive that \( \beta \) is provable if and only if \( E_{k} \) contains at least one theorem of \( \Gamma_{\omega}(\mathbb{N}) \). By induction with respect to \( k \) we find that \( \beta \) is a theorem of \( T_{\omega}(\mathbb{N}) \) if and only if \( E_{k} \) contains at least one provable formula. Consequently \( \beta \) is provable if and only if \( E_{k} \) contains at least one provable formula. However, the set \( E_{k} \) is denumerable, which completes the proof.

In the case of a theory \( T_{\omega}(\mathbb{N}) \) without functors, and with a finite set of individual constants, the set \( R_{k} \) is finite. Hence

**4.10.** Let \( \Gamma_{\omega}(\mathbb{N}) \) be a constructive theory without functors, let \( \beta \) be an arbitrary formula of \( \Gamma_{\omega}(\mathbb{N}) \) of the form \( \ast \). Then there exists an effectively determined finite sequence \( \overline{a}_{1}, \ldots, \overline{a}_{n} \) of formulas without quantifiers such that \( \beta \) is a theorem of \( \Gamma_{\omega}(\mathbb{N}) \) if and only if at least one of the formulas \( a_{1}, \ldots, a_{n} \) is a theorem \( \ast \). The sequence \( a_{1}, \ldots, a_{n} \) can be determined effectively.

**4.11.** If \( a \) is a constructive closed formula, then for every formula \( \beta \),\( a \rightarrow \beta(\overline{a}_{1}) \) is provable if and only if there exists \( q \in I \), such that \( a \rightarrow \beta(\overline{a}_{1}) \) is provable.
Elementarily definable analysis

by

A. Grzegorczyk (Warszawa)

The purpose of this paper is to give a strict mathematical shape to some ideas expressed by H. Weyl in "Das Kontinuum" [4]. Weyl proposes a restriction of the logical methods of analysis to the elementarily definable ones. A notion is elementarily definable if it is definable by means of the quantifiers bounding the integral variables only. A strict definition will be given later. It is very interesting to note how many theorems of the classical analysis can be obtained by means of elementary methods. It is shown in this paper that the classical analysis of continuous functions can be reproduced in an elementary manner. The problem of how many theorems from the theory of non-continuous real functions can be obtained in an elementary way remains open. Some counter-examples are given in the sequel.

To begin with the problem arrives how to define elementary definability. There are at least two answers:

1. A mathematical notion \( A \) is elementarily definable if it is definable by means of an elementary definition

\[ A(x_1, \ldots, x_n) = \exists f \ldots g(x_1, \ldots, x_n). \]

2. A notion \( A \) is elementarily definable if there exists a finite set of elementary conditions such that \( A \) is the unique object which satisfies those conditions.

We shall call the first the narrower, the second the broader concept of elementary definability. In this paper we shall consider the narrower notion.

1. Elementary definability in the arithmetic of integers

We shall introduce the notion of elementary definability in the arithmetic of integers. Let \( I \) be the set of all integers (positive, negative and zero). Let \( N \) be the set of non-negative integers (natural numbers). The variables \( x, y, z, p, g \) will stand for the integers, the variables \( a, b, f \) will represent natural numbers. The letters \( f, g, h \) will be used to denote the functions defined over the set \( I \) and assuming the integral values.