Pour les autres couples q_i^0, q_i^2, on construit des arcs H_i et des ensembles V_i d'une manière analogue. On voit que dans la projection de V_i est contenue uniquement la projection de H_i. Il résulte des considérations antérieures que la projection de H_i n'a pas de points communs avec les projections des arcs $\widehat{q'_{j+1}q''_j}$. Donc la réunion des arcs H_i et des arcs $\widehat{q'_{j+1}q''_j}$ forme un arc à projection simple sur X entre a et b, c. q. f. d.

Ce théorème ne résout pas le cas de M — *non-connexe sur* X (comparer l'exemple 12⁰).

Si la dimension de X est plus grande que 1, il peut arriver que dans un ensemble non-connexe sur X, tous les couples de points peuvent être joints par des arcs à projection simple (voir par exemple la fig. 7).

Si X est une droite, on peut résoudre le problème partiellement — pourvu que M soit un domaine. Soit W l'ensemble de tous les x tels que $S[M,x]$ soit non-connexe (voir la fig. 9). Si W ne contient pas un intervalle, on peut montrer que la construction de la démonstration du Théorème 6 peut être faite.

Du Théorème 1, du Théorème 2, du Théorème 6 et du lemme du début de ce paragraphe résulte le

THÉORÈME 7. *Si la transformation continue et biunivoque Φ est définie dans le domaine M connexe sur X et transforme tous les arcs $\subset M$ à projection simple sur X en des arcs à projection simple sur X^*, il faut et il suffit que la condition $\boldsymbol{W}$ soit vérifiée.*

Lorsqu'il s'agit de transformations dans des domaines quelconques, nous pouvons seulement affirmer que la *condition* $\boldsymbol{W}$ est suffisante et qu'il est nécessaire que, dans chaque sous-ensemble connexe et connexe sur X, la transformation g vérifie la *condition* $\boldsymbol{W}$.

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On a problem of P. Turan concerning graphs

by

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The purpose of this paper is the solution of a problem put forward by P. Turan. The problem is to define the smallest number of intersection points of the sides of a graph, defined in Theorem I. The problem was derived from the following question. In a brickworks the bricks are made in burning-ovens. When they are burnt out, they are carried away to storerooms by workers on small trucks rolling on rails. The trucks move easily and fast except when they pass a crossing of the rails. Here the trucks are usually derailed a great loss of time and bricks occurs and the traffic is hindered on all rails crossing that point. This loss will be reduced to minimum when the number of intersections of the rails is as small as possible and no three rails intersect each other at an inner point. Theorem I [1]) gives the solution of this problem. Theorem II gives the minimum number of regions into which the above graph cuts the plane.

THEOREM I. *If*

(α) *in the Euclidean plane two sets of points, A and B, are given, A consisting of p points $a_1, a_2, a_3, \ldots, a_p$, and B consisting of q points $b_1, b_2, b_3, \ldots, b_q$, ($p$ and q are natural numbers);*

(β) *for each pair of points a_i, b_j, where $i=1,2,3,\ldots,p$, $j=1,2,3,\ldots,q$, there exists a simple arc lying in the plane and having the points a_i, b_j as its end points;*

(γ) *the arcs lie in such a way that no three arcs have an interior point (i. e. a point that is not an end point) in common;*

(δ) *$K(p,q)$ denotes the smallest number of intersection points of arcs; then the following formulas hold:*

$$K(2k,2n)=(k^2-k)(n^2-n), \tag{1}$$

$$K(2k,2n+1)=(k^2-k)n^2, \tag{2}$$

$$K(2k+1,2n+1)=k^2n^2. \tag{3}$$

[1]) This theorem was proved at the same time, quite independently, by K. Urbanik (Wrocław).

Before proceeding to prove the theorem let us introduce certain terms and notations and prove two lemmata.

A simple arc having x and y as its end points will be denoted by xy.

The sum of all the arcs a_ib_j, where $i=1,2,3,...,p$ and $j=1,2,3,...,q$, will be called the *graph* $G(p,q)$; so that we can write

$$G(p,q)=\sum_{i,j} a_ib_j.$$

The set of interior points of the arc a_ib_j (*i. e.* the points of the arc that are not its end points) we shall call a *side* of the graph. The points $a_1,a_2,a_3,...,a_p$ and $b_1,b_2,b_3,...,b_q$ will be called *vertices* of the graph.

The set consisting of the *interior* points of each of the arcs xy_j for $j=1,2,3,...,k$, will be called a *fan* with the vertex x and the end points $y_1,y_2,y_3,...,y_k$, which we shall denote by $W_k(x)$. Consequently, neither any of the points $y_1,y_2,y_3,...,y_k$ nor the point x are included in the fan $W_k(x)$.

It follows from the definition of the number $K(p,q)$ that

$$K(p,q)=K(q,p).$$

LEMMA 1. *If three fans with different vertices b_1,b_2,b_3 have the same three points a_1,a_2,a_3 as their end points, at least two of those fans have a point in common.*

Proof. Let us assume that the fans $W_3(b_1)$, $W_3(b_2)$ and $W_3(b_3)$ are such that

(4) $\qquad W_3(b_1)W_3(b_2)=0, \qquad W_3(b_1)W_3(b_3)=0, \qquad W_3(b_2)W_3(b_3)=0.$

Let us draw a circle H_i with the centre a, and a radius so small that:

1° The circles H_i have no common points with the set

$$\sum_{j,k} a_jb_1+b_1a_k+a_jb_2+b_2a_k+a_jb_3+b_3a_k$$

where j and k take all the values 1,2,3 except the value i,

2° the circles H_i, for $i=1,2,3$ are disjoint from one another — which is of course possible.

Let e_{rs} be the first point of the simple arc b_ra_s, going from b_r to a_s, which lies on the circle H_s; then b_re_{rs} is a simple arc which has only one point in common with the circle H_s, namely its end point e_{rs}. Let R_{rs} be a segment of the radius of the circle H_s, with e_{rs} as one end point, while the other end point $\neq a_s$; let that segment be so small that it has no points in common with the set $\sum_j a_sb_j$, where j takes all the values 1, 2, 3 except the value s.

The continua

$$T_s=b_se_{s1}+b_se_{s2}+b_se_{s3}+R_{s1}+R_{s2}+R_{s3}$$

for $s=1,2,3$ are disjoint, according to the assumption (4), each of them has points in common with each circle H_i for $i=1,2,3$, and none of them *cuts* any of the circles. This is impossible on the strength of a theorem of mine [2]). Consequently, (4) cannot hold and at least two of the fans have a point in common, q. e. d.

LEMMA 2. *In the graph $G(p,3)$, which is the sum of three fans with the vertices b_1,b_2,b_3, each of which has the same end points $a_1,a_2,a_3,...,a_p$, the intersection points of the sides amount to at least*

(5) $\qquad k^2-k \quad$ *when* $\quad p=2k,$

(6) $\qquad k^2 \quad$ *when* $\quad p=2k+1.$

Proof by induction. For $k=1$ Lemma 2 is true on the strength of Lemma 1. Let is be assumed that Lemma 2 is true for the number k: we shall prove that the intersection points of the sides of the graph $G(p,3)$ will amount to at least

$$k^2+k \quad \text{when} \quad p=2k+2$$

and

$$(k+1)^2 \quad \text{when} \quad p=2k+3.$$

Let us consider the continua

$$C_i=W_3(a_i)+a_i+b_1+b_2+b_3 \quad \text{for} \quad i=1,2,3,...,p.$$

If each pair from among the continua C_i had one point in common, distinct from b_1,b_2,b_3, then, for different pairs of the continua C_i, those points would be distinct from one another according to the assumption (γ); therefore, the number of intersection points, distinct from one another, of the sides on the graph $G(p,3)$ would be at least as high as the number of different pairs of the continua C_i, *i. e.* it would be equal to at least $p(p-1)/2=m$. But in the case of $p=2k+2$ we have $m=2k^2+3k+1$, which is greater than k^2+k, while in the case of $p=2k+3$ we have $m=2k^2+5k+3$, which is greater than $(k+1)^2$; thus the induction premise would be fulfilled and the Lemma would be proved.

Let us assume, therefore, that there exists a pair of continua C_{i_1} and C_{i_2} which have no points in common except the points b_1,b_2,b_3. Then, on the strength of Lemma 1, each continuum C_i for $i_1\neq i\neq i_2$ must

[2]) C. Zarankiewicz [1]. Cf. also C. Kuratowski et C. Zarankiewicz [2].

have a point, distinct from b_1, b_2, b_3, in common with the set $C_{i_1}+C_{i_2}$, different C_i having with $C_{i_1}+C_{i_2}$ common points, distinct from one another — according to the assumption (γ). Therefore, in the set $C_{i_1}+C_{i_2}$, the number of intersection points, distinct from one another, of the sides of the graph $G(p,3)$ is at least as high as the number of continua C_i for $i_1\neq i\neq i_2$; thus, those intersection points of the sides amount to at least $p-2$.

The graph $G(p,3)$ can be represented as the sum

$$G(p,3)=C_{i_1}+C_{i_2}+\sum_{i_1\neq i\neq i_2} C_i=C_{i_1}+C_{i_2}+G(p-2,3);$$

the number of intersection points of the sides on the graph $G(p,3)$ will be at least equal to the number of intersection points of the sides of the graph $G(p-2,3)$ with the continuum $C_{i_1}+C_{i_2}$ plus the number of intersection points of the sides on the graph $G(p-2,3)$. Let us assume that the number of intersection points of the sides on the graph $G(p-2,3)$ is expressed by formulas (5) and (6); in that case, the number of intersection points of the sides in the graph $G(p,3)$, when $p-2=2k$, will be at least $2k+k^2-k=k^2+k$, and when $p-2=2k+1$, that number will be at least $2k+1+k^2=(k+1)^2$. Our induction premise has been proved, and, consequently, Lemma 2 has been proved.

Proof of the Theorem 1. For $k=1$ and $n=1$ the Theorem 1 is true on the strength of Lemma 2. Let us assume that formulas (1), (2), (3) remain valid for any graph $G(p,q)$; we shall prove, that they will remain valid for the graphes $G(p,q+1)$, $G(p+1,q)$, $G(p+1,q+1)$.

Let any graph $G(p,q+1)$ be given. That graph may be regarded as the sum of the graph $G(p,q)$ and one continuum V, which is a fan with the vertex b_{q+1} and the end points $a_1, a_2, a_3, \ldots, a_p$ completed by its vertex and its end points; so that

$$V=W_p(b_{q+1})+b_{q+1}+a_1+a_2+a_3+\ldots+a_p.$$

On the other hand the graph $G(p,q)$ can be regarded as a sum of his vertices and of q fans with vertices $b_1, b_2, b_3, \ldots, b_q$, having the same end points $a_1, a_2, a_3, \ldots, a_p$. But the same graph can also be treated as the sum of *pairs of fans* (plus one fan when q is an odd number), because if $q=2n$ we can write

$$G(p,q)=\sum_{j=1}^{n}[W_p(b_{2j})+W_p(b_{2j-1})]+\sum_{i=1}^{p}a_i+\sum_{r=1}^{q}b_r \tag{7}$$

and if $q=2n+1$

$$G(p,q)=\sum_{j=1}^{n}[W_p(b_{2j})+W_p(b_{2j-1})]+W_p(b_{2n+1})+\sum_{i=1}^{p}a_i+\sum_{r=1}^{q}b_r. \tag{8}$$

The number of intersection points of the sides on the graph $G(p,q+1)$ will be equal to the number of intersection points of the sides on the graph $G(p,q)$ plus the number of intersection points of the fan $W_p(b_{q+1})$ with the graph $G(p,q)$.

On the strength of Lemma 2 the fan $W_p(b_{q+1})$ must have with each pair of fans appearing in the graph $G(p,q)$ at least k^2-k points in common when $p=2k$, and at least k^2 points in common when $p=2k+1$. Since n different pairs of fans appear in the graph $G(p,q)$ (both when $q=2n$ and when $q=2n+1$), those intersection points are distinct from one another and amount to at least

$$n(k^2-k) \quad \text{when} \quad p=2k$$

and

$$nk^2 \quad \text{when} \quad p=2k+1.$$

If we assume, therefore, that on the graph $G(p,q)$ the minimum number of intersection points of the sides is expressed by the formulas (1), (2), (3), then, for the graph $G(p,q+1)$ that minimum number of intersection points of the sides will be expressed by

$$\begin{aligned}K(2k,2n+1)&=K(2k,2n)+n(k^2-k)=(k^2-k)(n^2-n)+n(k^2-k)\\&=(k^2-k)n^2,\\K(2k,2n+2)&=K(2k,2n+1)+n(k^2-k)=(k^2-k)n^2+n(k^2-k)\\&=(k^2-k)[(n+1)^2-(n+1)],\\K(2k+1,2n+1)&=K(2k+1,2n)+nk^2=K(2n,2k+1)+nk^2\\&=(n^2-n)k^2+nk^2=k^2n^2,\\K(2k+1,2n+2)&=K(2k+1,2n+1)+nk^2=k^2n^2+nk^2\\&=[(n+1)^2-(n+1)]k^2=K[2(n+1),2k+1].\end{aligned}$$

We can see that the validity of formulas (1), (2), (3) for the numbers k and n involves their validity for the numbers $k+1$ and $n+1$; thus, the theorem is proved, q. e. d.

For each pair of natural number p, q it is possible to construct a minimum graph in which the number of intersection points of all its sides is expressed exactly by formulas (1), (2), (3); that means that $K(p,q)$ has been reached.

Minimum graph. If $p=2k$, the set A consists of the points on the x axis whose abscissas are

$$-k, -(k-1), \ldots, -2, -1, 1, 2, \ldots, k;$$

if $p=2k+1$, the set A consists of the points on the x axis whose abscissas are

$$-k,-(k-1),\ldots,-2,-1,1,2,\ldots,k,k+1;$$

if $q=2n$, the set B consists of the points on the y axis whose ordinates are

$$-n,-(n-1),\ldots,-2,-1,1,2,\ldots,n;$$

if $q=2n+1$, the set B consists of the points on the y axis whose ordinates are

$$-n,-(n-1),\ldots,-2,-1,1,2,\ldots,n,n+1.$$

By joining with a line segment each point of the set A with each point of the set B, we obtain a minimum graph, in which the number of intersection points of the sides is expressed exactly by formulas (1), (2), (3), which can easily be checked.

Let us observe, by the way, that the assumption (γ) is essential. If we reject it, $K(p,q)=1$ for any pair of number $p>3$ and $q>3$. Indeed, taking any point x of the plane, we can join it by means of simple arcs with each of the points of the set $A+B$ in such a way that those arcs should have only one point in common, namely the point x; then the assumption (β) is observed and the number of intersection points of all the arcs is 1.

THEOREM II. *If the conditions* (α), (β), (γ) *of Theorem* I *are observed, and if*

(ε) $L(p,q)$ *denotes the smallest number of regions into which the plane is cut by all the simple arcs, then the following formulas hold*:

$$L(2k,2n)=(k^2-k)(n^2-n)+4nk-2(n+k)+2, \tag{9}$$

$$L(2k,2n+1)=(k^2-k)n^2+4nk-2n+1, \tag{10}$$

$$L(2k+1,2n+1)=k^2n^2+4nk+1. \tag{11}$$

The proof by induction will be based on a Theorem of Straszewicz [3]). Let $k=1$ and $n=1$.

(i) The graph $G(2,2)$ may be regarded as the sum of two continua, each of which is a fan completed by its vertex and its two end points. Those continua have two points in common, consequently, according to Theorem III, $G(2,2)$ cuts the plane into at least two regions — thus, formula (9) is valid.

[3]) This theorem can be formulated as follows:

THEOREM III. *If the continua M and N lie in the plane, the continuum M cuts the plane into m regions, and $M\cdot N$ consists of n points, then the continuum $M+N$ cuts the plane into at least $m+n-1$ regions.*

See S. Straszewicz [3].

(ii) The graph $G(2,3)$ may be regarded as the sum of two continua: the graph $G(2,2)$ and the continuum which is a fan completed by its vertex and its two end points. Those two continua have at least two points (the end points of the fan) in common; consequently, their sum, according to Theorem III, cuts the plane into at least $2+2-1=3$ regions; thus formula (10) is valid.

(iii) The graph $G(3,3)$ may be regarded as the sum of three continua W_1, W_2, W_3, each of which is a fan completed by its vertex and its three end points. On the strength of Lemma 1, at least two of those fans have a point in common, distinct from their end points; let them be called continua W_1 and W_2. The graph $G(3,3)$ may be regarded as the sum of W_1 and (W_2+W_3); the continuum W_2+W_3 cuts the plane into at least 3 regions (as W_2W_3 consists of at least 3 fan end points), according to Theorem III; while W_1 has at least four points in common with W_2+W_3. Therefore, in view of Theorem III, $W_1+W_2+W_3=G(3,3)$ cuts the plane into at least $3+4-1=6$ regions; thus formula (11) is valid.

Let us assume that formulas (9), (10), (11) remain valid for any graph $G(p,q)$; we shall prove, that they will remain valid for the graphes $G(p,q+1)$, $G(p+1,q)$, $G(p+1,q+1)$.

Let any graph $G(p,q+1)$ be given. That graph may be regarded as the sum of the graph $G(p,q)$ and one continuum V which is a fan $W_p(b_{q+1})$ with the vertex b_{q+1} and the end points $a_1,a_2,a_3,\ldots,a_p$, completed by its vertex and its end points.

Let us calculate how many points in common have the continuum V and the graph $G(p,q)$. The graph $G(p,q)$ may be represented as the sum of pairs of fans (plus one fan when q is an odd number) according to formulas (7) and (8). According to Lemma 2, the fan $W_p(b_{q+1})$ has with each pair of fans at least

k^2-k points in common when $p=2k$

and

k^2 points in common when $p=2k+1$.

As there are n pairs of fans in the decompositions (7), (8) (both when $q=2n$ and when $q=2n+1$), therefore, the fan $W_p(b_{q+1})$ and the graph $G(p,q)$ have at least

$n(k^2-k)$ points in common when $p=2k$

and

nk^2 points in common when $p=2k+1$.

In that case the continuum V and the graph $G(p,q)$ have at least

$n(k^2-k)+2k$ points in common when $p=2k$

and

nk^2+2k+1 points in common when $p=2k+1$.

Therefore, according to Theorem III, if we join the continuum V to the graph $G(p,q)$, the number of regions into which the plane will be cut by the graph $G(p,q+1)$ will be increased by

$$n(k^2-k)+2k-1 \quad \text{when} \quad p=2k$$

and

$$nk^2+2k+1-1 \quad \text{when} \quad p=2k+1$$

in relation to the number of regions into which the plane is cut by the graph $G(p,q)$.

Thus we can write:

$$\begin{aligned} L(2k,2n+1) &= L(2k,2n)+n(k^2-k)+2k-1 \\ &= (n^2-n)(k^2-k)+4nk-2(k+n)+2+n(k^2-k)+2k-1 \\ &= n^2(k^2-k)+4nk-2n+1, \end{aligned}$$

$$\begin{aligned} L(2k,2n+2) &= L(2k,2n+1)+n(k^2-k)+2k-1 \\ &= [(n+1)^2-(n+1)](k^2-k)+4(n+1)k-2(k+n+1)+2 \end{aligned}$$

$$\begin{aligned} L(2k+1,2n+1) &= L(2k+1,2n)+nk^2+2k+1-1 \\ &= L(2n,2k+1)+nk^2+2k \\ &= k^2(n^2-n)+4nk-2k+1+nk^2+2k \\ &= k^2n^2+4nk+1, \end{aligned}$$

$$\begin{aligned} L(2k+1,2n+2) &= L(2k+1,2n+1)+nk^2+2k \\ &= k^2n^2+4nk+1+nk^2+2k \\ &= k^2[(n+1)^2-(n+1)]+4k(n+1)-2k+1. \end{aligned}$$

We can see that the validity of formulas (9), (10), (11) for the number k and n involves their validity for the number $k+1$ and $n+1$; thus the theorem is proved.

Note 1. Let us observe that all the theorems will remain true when, instead of simple arcs joining the points a_i and b_j, we take any continua M_{ij} joining those points, provided $M_{ij}-a_i-b_j$ is a connected set. This assumption enters into the proof of Lemma 1.

Note 2. As has been found by K. Urbanik and noticed by A. Rényi and P. Turan, independently of one another, formulas (1), (2) and (3) can be written in the form of a single formula

$$K(p,q)=\left(p-1-\mathrm{E}\left(\frac{p}{2}\right)\right)\mathrm{E}\left(\frac{p}{2}\right)\left(q-1-\mathrm{E}\left(\frac{q}{2}\right)\right)\mathrm{E}\left(\frac{q}{2}\right),$$

where $\mathrm{E}(x)$ denotes the greatest integer $\leqslant x$. In this case formulas (9), (10) and (11) can be written in the form of a single formula

$$L(p,q)=K(p,q)+(p-1)(q-1)+1.$$

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