As N is dense in $\beta(N)$, we have $\beta(M)=\beta(G)$, where $M=G\cap N\subset N$. Therefore $\beta(M)\cup\beta(M-N)=\beta(N)$ and $\beta(M)\cap\beta(N-M)=0$, N being a normal [1] space. From this it follows that the set $\beta(G)$ is ambiguous in $\beta(N)$. Now, we can put $U(x)=\beta(M)\cap R$.

For any infinite subset $K\subset N$ we have $\overline{K}-K\neq 0$. Indeed, there is a point $y\in\beta(K)-K$. Then we have $a_k\in\overline{K}-K$ for $K\subset N_k$ and $y\in\overline{K}-K$ otherwise.

The space R is completely regular.

This follows instantly from the fact that the open basis of R consists of ambiguous neighbourhoods.

The space R has the property a.

Suppose, on the contrary, that $g(x)$, $x\in R$, is a continuous function and $X=\bigcup_{k=1}^{\infty}(x_k)$ a set of points $x_k\in R$ such that $g(x_k)>k$ for $k=1,2,\ldots$ The set X is isolated and closed in R.

Consequently, there is a disjoint system of ambiguous neighbourhoods $U(x_k)$ such that $g(x)>k$ for any $x\in U(x_k)$, $g(x)$ being continuous on R. Let us choose points $n_k\in N\cap U(x_k)$. Since $g(n_k)>k$, we have $\overline{K}-K=0$ where $K=\bigcup_{k=1}^{\infty}(n_k)$; this is a contradiction.

The space R fails to be compact.

Evidently, the set $\bigcup_{k=1}^{\infty}(a_k)$ has no point of accumulation in R.

Note. Since the property a implies the compacticity of any normal space, the space R constructed above cannot be normal. As a matter of fact the sets $\bigcup_{k=1}^{\infty}(a_k)$ and $R-\bigcup_{k=1}^{\infty}\overline{N}_k$ are both closed and disjoint, but they cannot be separated by any two disjoint open sets in R.

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[1] See E. Čech, *On bicompact spaces*, Annals of Mathematics 38 (1937), p. 833-844.

On completely regular spaces

by

S. Mrówka (Warszawa)

In the preceding paper J. Novák [1] has shown the existence of a non-compact, completely regular space X, on which all continuous real functions are bounded. Our purpose is to obtain that result in a more direct way.

LEMMA. *Let N be the set of all natural numbers, and $\mathfrak{N}$ the family of all its infinite subsets. There exists a family $\mathfrak{N}_1\subset\mathfrak{N}$ such that:*

(1) *The family $\mathfrak{N}_1$ is infinite,*

(2) *for every $N_1,N_2\in\mathfrak{N}_1$ the product N_1N_2 is finite,*

(3) *for every $N'\in\mathfrak{N}$ there exists a $N''\in\mathfrak{N}_1$ such that the product $N'N''$ is infinite.*

Proof. Let $N=N_1+N_2+\ldots+N_k+\ldots$, where N_k are infinite and disjoint sets. Let us put the family $\mathfrak{N}-\{N_1,N_2,\ldots,N_k,\ldots\}$ in a transfinite sequence

$$N_\omega,N_{\omega+1},\ldots,N_\alpha,\ldots$$

Hence

$$\mathfrak{N}=\{N_1,N_2,\ldots,N_\omega,N_{\omega+1},\ldots,N_\alpha,\ldots\}.$$

We define the family $\mathfrak{N}_1$ by transfinite induction:

1) $N_1\in\mathfrak{N}_1$,

2) $N_\alpha\in\mathfrak{N}_1$ if and only if for every $N_\beta\in\mathfrak{N}_1$ $(\beta<\alpha)$ the product $N_\alpha N_\beta$ is finite.

It is obvious that the family $\mathfrak{N}_1$, so defined, satisfies the conditions (1)-(3).

Now, let us put $X=N+\mathfrak{N}_1$. The neighbourhoods in X are defined as follows:

1⁰ If $x\in N$ then $O(x)=\{x\}$,

2⁰ if $x\in\mathfrak{N}_1$, i. e. $x=N'\subset N$, then $O(x)=\{x\}+N'-S$ where S is an arbitrary finite subset of N'.

[1] J. Novák, On a problem concerning completely regular sets, this volume, p. 103-104.

Let us notice the following properties of the space X:

(4) *$O(x)$ is open and closed for every x,*

(5) *$Y=\overline{Y}$ for every $Y\subset\mathfrak{R}_1$,*

(6) *N is dense in X,*

(7) *X satisfies the first axiom of countability.*

The complete regularity of X follows from (4). It follows from (1) and (5) that X is non-compact.

(8) *N is compact relatively to X.*

Let f be a real-valued continuous mapping of X. As N is dense in X and compact relatively to X, it follows at once that the set $f(X)$ is bounded.

Remark. The following statements are true:

1. Let T be a topological space satisfying the first axiom of countability and let $f\in T^X$ (*i. e.* f is a continuous mapping of X into T). Then $f(X)$ is closed in T.

2. If T satisfies the first axiom of countability and is compact and $f\in T^X$, then $f(X)$ is compact.

3. If T is a metric space and $f\in T^X$, then $f(X)$ is compact.

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Fonctions rationnelles qui sont homotopes à des fonctions biunivoques sur certains sous-ensembles du plan

par

K. Kuratowski (Warszawa)

Désignons par $\mathcal{S}_2$ le plan des nombres complexes, le point à l'infini y compris, et par $\mathcal{P}$ le plan des nombres complexes finis diminué du point 0. Étant donnés un ensemble $A\subset\mathcal{S}_2$ et une fonction continue f définie sur A, à valeurs complexes et telle que

(1) $$0\neq f(z)\neq\infty,$$

nous écrirons, comme d'habitude, $f\in\mathcal{P}^A$.

Soit, en outre, $g\in\mathcal{P}^A$. On dit que les fonctions f et g sont homotopes (relativement à $\mathcal{P}$), en symbole

(2) $$f\sim g,$$

lorsqu'il existe une déformation continue $h(z,t)$, où $z\in A$ et $0\leqslant t\leqslant 1$, telle que

(3) $$h(z,0)=f(z),\qquad h(z,1)=g(z),\qquad h(z,t)\in\mathcal{P},$$

quels que soient z et t.

D'après un théorème de S. Eilenberg[1]), la relation (2) équivaut à l'existence d'une fonction continue $u(z)$ définie sur A et telle que

(4) $$f(z)=g(z)e^{u(z)}\quad\text{pour}\quad z\in A.$$

Dans le N°2 nous allons établir le théorème suivant:

THÉORÈME I. *Soient A un sous-ensemble localement connexe*[2]) *du plan $\mathcal{S}_2$ et $p_0,\ldots,p_n$ un système de points situés en dehors de A et tels que A coupe*[3]) *le plan entre tout couple p_i,p_j pour $i\neq j$.*

Si la fonction rationnelle[4])

(5) $$r(z)=(z-p_0)^{k_0}\cdot\ldots\cdot(z-p_n)^{k_n},$$

[1]) Voir [1], p. 68 ou [3], p. 388.

[2]) Un espace est dit *localement connexe* lorsqu'à tout point correspond un entourage connexe aussi petit qu'on le veut. Rappelons que la connexité locale d'un continu C équivaut à l'existence d'une représentation paramétrique continue de C sur l'intervalle.

[3]) Un ensemble *coupe* l'espace entre les points p et q lorsque tout continu qui contient ces points admet au moins un point en commun avec cet ensemble.

[4]) Il peut arriver, bien entendu, que l'un des points p_j soit point à l'infini. Nous convenons que $z-\infty\equiv 1$.