On some Contractible Continua without Fixed Point Property

By

Shin'ichi Kinoshita (Osaka)

K. Borsuk[1]) has proposed the following problem: *Étant donnés dans l'espace de Hilbert un ensemble compact E et un point a, soit $C(a,E)$ le cône formé de tous les segments rectilignes $\overline{ax}$ où x parcourt E. Est-ce que toute fonction continue f qui transforme $C(a,E)$ en sous-ensemble de lui-même admet un point invariant, c'est-à-dire satisfaisant à l'equation $p=f(p)$?*

In this note this problem will be answered in the negative.

1. To this purpose we prove first the following proposition due to T. Shirota.

(*) *If there exists a contractible continuum A and a continuous mapping $f \in A^A$ which has no fixed point*[2]), *then our problem is answered in the negative.*

Proof. Assume that A is contained in the Hilbert cube I_ω and that for each $x \in A$ the first coordinate of x is equal to zero. Set $a=(1,0,0,\ldots)$. Let $C(a,A)$ be the cone with base A and with vertex a. Let x be a point of A and let $y \in C(a,A)$ be the point which divides the segment $\overline{ax}$ in ratio $(1-t)/t$ $(0 \leqslant t \leqslant 1)$. Set $y=[x,t]$. Let $c(x,t)$ be a contraction of A. For each $y=[x,t] \in C(a,A)$ let

$$\bar{f}(y)=f(c(x,t)).$$

Now if $0<t\leqslant 1$, then there exists $x' \in A$ such that $\bar{f}(y)=\bar{f}([x,t])=[x',0]\neq y$ and if $t=0$, then $\bar{f}(y)=\bar{f}([x,0])=[f(x),0]\neq[x,0]=y$. Therefore the continuous mapping $\bar{f}$ which maps $C(a,A)$ into itself has no fixed point. Hence the proposition (*) is proved.

2. Now we shall construct a contractible continuum A and a continuous mapping $f \in A^A$ which has no fixed point.

[1]) Coll. Math. I (1947-48), p. 332.

[2]) Whether or not such a continuum A exists is a problem due to K. Borsuk, Fund. Math. **19** (1932), p. 230. By the construction of the continuum A below, this problem is also solved.

Using the cylindrical coordinate system (r,φ,z)[3]) as a coordinate system in E^3, let us construct the contractible continuum A as follows: set

$$A_1=\underset{(r,\varphi,z)}{E}[0\leqslant r<1,\quad z=0],$$

$$A_2=\underset{(r,\varphi,z)}{E}\left[r=\frac{2}{\pi}\tan^{-1}\varphi,\quad 0\leqslant\varphi<\infty,\quad 0\leqslant z\leqslant 1\right]^{4)},$$

$$A_3=\underset{(r,\varphi,z)}{E}[r=1,\quad 0\leqslant z\leqslant 1],$$

$$A = A_1+A_2+A_3.$$

Clearly A is a contractible continuum.

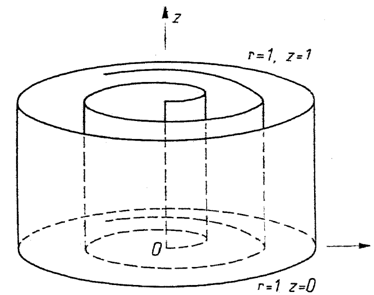

Fig. 1.

Let us now construct a continuous mapping $f \in A^A$ which has no fixed point. Set

$$\theta(r)=\tan\frac{\pi r}{2}\quad\text{ for }\quad 0\leqslant r<1.$$

For each $x \in (r,\varphi,0) \in A_1$ let

$$f(x)=\begin{cases}\left(\frac{2}{\pi}\tan^{-1}(\theta(r)-\pi),\varphi-\pi,0\right) & \text{for}\quad \theta(r)\geqslant\pi,\\ \left(0,0,1-\frac{\theta(r)}{\pi}\right) & \text{for}\quad 0\leqslant\theta(r)\leqslant\pi.\end{cases}$$

Set

$$g(x,y)=(-x)(1-y)+y\quad\text{ for }\quad 0\leqslant y\leqslant 1,$$

$$h(x,y)=\begin{cases}(1-y)(1-x)+\frac{y}{2}x & \text{for}\quad 0\leqslant y\leqslant\frac{1}{2},\\ (1-y)(1-x)+\left(\frac{1}{2}-\frac{y}{2}\right)x & \text{for}\quad \frac{1}{2}\leqslant y\leqslant 1.\end{cases}$$

[3]) $0\leqslant r<\infty$, $(r,\varphi,z)=(r,\varphi',z)$ for $\varphi=\varphi' \bmod 2\pi$ and $(0,\varphi,z)=(0,\varphi',z)$ for every φ and φ'.

[4]) $-\frac{\pi}{2}<\tan^{-1}\varphi<\frac{\pi}{2}$.

where $0 \leqslant x \leqslant 1$. For each $x=(r,\varphi,z) \in A_2$ let

$$f(x)=\begin{cases}\left(\frac{2}{\pi}\tan^{-1}\left(\varphi+\pi g\left(\frac{\varphi}{\pi},z\right)\right),\ \varphi+\pi g\left(\frac{\varphi}{\pi},z\right),\ z+h\left(\frac{\varphi}{\pi},z\right)\right) & \text{for } 0\leqslant\varphi\leqslant\pi,\\ \left(\frac{2}{\pi}\tan^{-1}(\varphi-\pi+2\pi z),\ \varphi-\pi+2\pi z,\ z+\frac{z}{2}\right) & \text{for } \pi\leqslant\varphi<\infty,\ 0\leqslant z\leqslant\frac{1}{2},\\ \left(\frac{2}{\pi}\tan^{-1}(\varphi-\pi+2\pi z),\ \varphi-\pi+2\pi z,\frac{1}{2}+\frac{z}{2}\right) & \text{for } \pi\leqslant\varphi<\infty,\ \frac{1}{2}\leqslant z\leqslant 1.\end{cases}$$

Finally for each $x=(1,\varphi,z) \in A_3$ let

$$f(x)=\begin{cases}\left(1,\varphi-\pi+2\pi z, z+\frac{z}{2}\right) & \text{for } 0\leqslant z\leqslant\frac{1}{2}\\ \left(1,\varphi-\pi+2\pi z,\frac{1}{2}+\frac{z}{2}\right) & \text{for } \frac{1}{2}\leqslant z\leqslant 1.\end{cases}$$

It is easy to see that $f(x)$ is a continuous mapping of $A=A_1+A_2+A_3$ into itself and that $f(x)$ has no fixed point. Thus we have constructed a contractible continuum A and a continuous mapping $f \in A^A$ which has no fixed point.

Thus our problem is solved in the negative in virtue of Proposition (*).

Department of Mathematics, Osaka University

Sur la dérivée algébrique

Par

J. G.-Mikusiński (Wrocław)

1. Soit A un anneau commutatif. Supposons qu'une opération fasse correspondre un élément $a' \in A$ à tout élément $a \in A$ de manière que

(α) et (β) $\qquad (a+b)'=a'+b' \qquad$ et $\qquad (ab)'=a'b+ab'.$

Cette opération sera dite *dérivation*.

Dans cet article, nous étudierons quelques propriétés d'une telle dérivation et en montrerons quelques interprétations.

La dérivation dans un corps a été étudié par A. Weil[1]); son objet de recherches diffère d'ailleurs du nôtre.

2. On a

$$0'=0, \qquad (-a)'=-a', \qquad (a-b)'=a'-b'.$$

En effet, il vient de (α), $0'=0'+0'$, d'où $0'=0$; $a'+(-a)' =[a+(-a)]'=0'=0$, d'où $(-a)'=-a'$; $(a-b)'=[a+(-b)]'=a'+(-b)' =a'-b'$.

Il est aussi facile de démontrer par induction que

$$\Big(\sum_{i=1}^{n} a_i\Big)'=\sum_{i=1}^{n} a_i', \qquad \Big(\prod_{i=1}^{n} a_i\Big)'=\sum_{i=1}^{n} a_1\ldots a_{i-1}a_i'a_{i+1}\ldots a_n.$$

3. Posons par recurrence

$$a^{(n)}=(a^{(n-1)})' \qquad (n=1,2,\ldots;\ a^{(0)}=a).$$

(I) *Si l'un au moins des éléments a, b n'est pas un diviseur de zéro, la relation $a'b-ab'=0$ entraîne $a^{(m)}b^{(n)}-a^{(n)}b^{(m)}=0$.*

En effet, supposons que a ne soit pas un diviseur de zéro. Si $a'b-ab'=0$ et $a^{(p)}b-ab^{(p)}=0$, on a

$$a(a^{(p)}b'-a'b^{(p)})=a'(a^{(p)}b-ab^{(p)})-a^{(p)}(a'b-ab')=0,$$

d'où

$$a^{(p)}b'-a'b^{(p)}=0.$$

[1]) A. Weil, *Foundations of Algebraic Geometry*, American Mathematical Society Colloquium Publications, 1946, p. 11-14. Voir aussi N. Bourbaki, *Algèbre*, (Actualités Sc. Ind. 1102, Paris 1950), Chap. IV, § 4, p. 37-52.