Algebraic Treatment of the Notion of Satisfiability *)

By

H. Rasiowa (Warszawa) and R. Sikorski (Warszawa)

Tarski's [1]) definition of satisfiability can be formulated in the language of the theory of Boolean algebras as follows.

Let $J \neq 0$ be any set, and let B_0 be the two-element Boolean algebra. Each formula α from the classical functional calculus may be treated as an algebraic functional Φ_α defined in J if we interpret the individual variables as variables running over J, the functional variables — as variables running over some sets of mappings of J into B_0, and the logical signs — as the signs of the corresponding Boolean operations.

A formula α from the classical functional calculus is satisfiable in J in the sense defined by Tarski if the functional Φ_α assumes as its value the unit element of B_0. A formula α is valid in J in the sense defined by Tarski if $\Phi_\alpha =$ the unit element of B_0 identically.

The alors statements are not theorems, but another formulation of the original definition of Tarski. The algebraization of the notion of satisfiability and validity has already proved to be useful. *E. g.* it has enabled us to give a simple proof of the theorems of Gödel and of Skolem and Löwenheim for the classical functional calculus [2]).

The above algebraization of the notion of satisfiability and validity permits us to extend these notions to the case of other functional calculi, *e. g.* to the functional calculi of Heyting and of Lewis [3]).

The subject of this paper is the systematic study of the notion of satisfiability and validity in the general case, *i. e.* for a functional calculus $\mathcal{S}^*$ which is not exactly specified. The main idea is as follows.

Let $\mathcal{S}$ be a sentential calculus which contains the signs of disjunction, of conjunction and of implication, and possibly some other sentential operators. We suppose that all theorems of the positive logic are theorems in $\mathcal{S}$. The system $\mathcal{S}$ determines uniquely a functional calculus $\mathcal{S}^*$. On the other hand, the system $\mathcal{S}$ determines a type of abstract algebras called here $\mathcal{S}$-algebras. Each $\mathcal{S}$-algebra A is a relatively pseudocomplemented

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[1]) See Tarski [1]. See also Rasiowa-Sikorski [2], § 2.

[2]) See Rasiowa-Sikorski [2] and [3].

[3]) Rasiowa [1].

lattice with the unit element e. If A is a complete lattice, A is said to be an $\mathcal{S}^*$-algebra. Let $J \neq 0$ be any set and let A be an $\mathcal{S}^*$-algebra. Each formula α from $\mathcal{S}^*$ may be treated as an algebraical functional Φ_α defined in J if we interpret the individual variables as variables running over J, the functional variables — as variables running over some sets of mapping of J into A, and the logical signs — as the signs of the corresponding algebraic operations in A [4]).

A formula α from $\mathcal{S}^*$ is said to be satisfiable in J if, for an $\mathcal{S}^*$-algebra A, the functional Φ_α assumes as its value the unit element e of A. The formula α is said to be valid in J if $\Phi_\alpha = e \in A$ identically for every $\mathcal{S}^*$-algebra A.

We shall demonstrate that the Gödel and Skolem-Löwenheim theorems hold also for the functional calculus $\mathcal{S}^*$ (Theorems 6.1, 6.2, 7.2, 7.3) under an additional hypothesis. This hypothesis has a purely algebraic form: we must only require that the class of all $\mathcal{S}^*$-algebras should be sufficiently rich; more exactly, that some $\mathcal{S}$-algebras could be extended in a special way to $\mathcal{S}^*$-algebras. For the proof of some variants of the Skolem-Löwenheim theorem it is necessary to suppose that the system under consideration contains the negation sign.

The proof of the Gödel and Skolem-Löwenheim theorems for the general system $\mathcal{S}^*$ is very simple. The whole difficulty lies in showing that the algebraical conditions are fulfilled if we specialize the logical systems $\mathcal{S}$ and $\mathcal{S}^*$.

The first part of this paper contains an exact analysis of the proof of the Gödel and Skolem-Löwenheim theorems in the case of a general functional calculus $\mathcal{S}^*$. In the second part we apply the results obtained in the preceding general part to the case of the special functional calculus: of the two-valued logic $\mathcal{S}_\varkappa^*$, of Heyting $\mathcal{S}_\chi^*$, of Lewis $\mathcal{S}_\lambda^*$, of the positive logic $\mathcal{S}_\pi^*$, and of the minimal logic $\mathcal{S}_\mu^*$ [5]).

Clearly the main theorems for these special systems can be expressed in a stronger form than that used in the general part. For instance, the $\mathcal{S}_\varkappa^*$-algebras are complete Boolean algebras. However, the two-element Boolean algebra B_0 has a special importance for the classical functional calculus $\mathcal{S}_\varkappa^*$. In the case of a general system $\mathcal{S}^*$ there is no analogue to B_0.

[4]) The idea of the algebraic interpretation of logical formulas is due to A. Mostowski. See Mostowski [1].

[5]) Some results from the second part were published earlier by Rasiowa [1] and by Rasiowa-Sikorski [2] and [3].

The difference between the paper of Rasiowa [1] and the analogues questions in §§ 10-11 of this paper is this: in the present paper we reduce the problem of satisfiability to the domain of topological spaces.

The subject of the first part is similar to that in Henkin [2]. Henkin's results are weaker than our results. However, his hypotheses about systems are also weaker. He accepts systems containing only the implication signs $\rightarrow$ and quantifiers.

The $\mathcal{S}_\lambda^*$-algebras are complete closure algebras. It can be shown that, in all the definitions of satisfiability and validity, we may restrict ourself to closure algebras $C(\mathfrak{X})$ of all subsets of topological spaces $\mathfrak{X}\neq 0$. Moreover, there is a topological space $\mathfrak{X}_\lambda$ such that the closure algebra $C(\mathfrak{X}_\lambda)$ has the same importance for the Lewis functional calculus $\mathcal{S}_\lambda^*$ as B_0 in $\mathcal{S}_\varkappa^*$.

The $\mathcal{S}_\chi^*$-algebras, $\mathcal{S}_\pi^*$-algebras, and $\mathcal{S}_\mu^*$-algebras are complete pseudo-complemented lattices called here complete Heyting algebras. It can be shown that we may restrict ourself to Heyting algebras $H(\mathfrak{X})$ of all open subsets of topological spaces $\mathfrak{X}\neq 0$. The content of the notion of satisfiability and of validity remains unchanged. Moreover there is a topological space $\mathfrak{X}_\chi$ such that the Heyting algebra $H(\mathfrak{X}_\chi)$ plays the same part in the Heyting functional calculus $\mathcal{S}_\chi^*$ as B_0 in $\mathcal{S}_\varkappa^*$. Analogous spaces can be constructed for $\mathcal{S}_\pi^*$ and $\mathcal{S}_\mu^*$.

The last section of this paper contains the following two applications.

Each formula α from the positive calculus $\mathcal{S}_\pi^*$ may be interpreted as a formula from the Heyting functional calculus $\mathcal{S}_\chi^*$. We shall prove that α is provable in $\mathcal{S}_\pi^*$ if and only if it is provable in $\mathcal{S}_\chi^*$.

Every formula α from the Heyting functional calculus $\mathcal{S}_\chi^*$ can be translated (in a very natural way) into a formula $\psi(\alpha)$ from the Lewis functional calculus $\mathcal{S}_\lambda^*$. We shall prove that α is provable in $\mathcal{S}_\chi^*$ if and only if $\psi(\alpha)$ is provable in $\mathcal{S}_\lambda^*$.

The proof of the above two theorems is based on the Gödel completeness theorems from the systems $\mathcal{S}_\pi^*$, $\mathcal{S}_\chi^*$ and $\mathcal{S}_\lambda^*$.

An application to the problem of decidability in non-classical functional calculi will be discussed in a separate paper.

Part I

§ 1. The system $\mathcal{S}$. In the first part of this paper we shall consider a fixed system $\mathcal{S}$ of a sentential calculus described as follows:

The primitive symbols of $\mathcal{S}$ are the *sentential variables* $a_1, a_2, a_3, \ldots$, parentheses and the following constants:

(a) the *disjunction sign* $+$, the *conjunction sign* $\cdot$, *the implication sign* $\rightarrow$;

(b) some other *binary sentential operators* $o_1, \ldots, o_r$;

(c) some *unary sentential operators* $o^1, \ldots, o^s$.

The set of operators mentioned in (b) or (c) may be empty.

The set S of all formulas in $\mathcal{S}$ is the smallest set such that

1) $a_i \in S$ $(i=1,2,\ldots)$;

2) if $\alpha, \beta \in S$, then $(\alpha+\beta) \in S$, $(\alpha\cdot\beta) \in S$, $(\alpha\rightarrow\beta) \in S$, $(\alpha\, o_i\, \beta) \in S$ $(i=1,\ldots,r)$, $(o^i \alpha) \in S$ $(i=1,\ldots,s)$.

Instead of $\big((\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)\big)$ we shall write, for brevity, $\alpha\equiv\beta$.

In writing formulas we shall practice the omission of the parentheses, the rule being that

1° each of the operators $\cdot$, $+$, $\rightarrow$, $\equiv$ binds less strongly than the previous one;

2° each of the operators o^i binds an expression more strongly than any of the binary operators.

In the set S of all formulas we distinguish a subset $S^0 \subset S$ of all *theorems*. We assume that the set S^0 of all theorems fulfils the following conditions:

(i) if α and $\alpha\rightarrow\beta$ are in S^0, then β is in S^0 (modus ponens);

(ii) if γ is a part of α $(\alpha,\gamma \in S)$, if $\gamma\equiv\delta$ is in S^0, and if β is the formula obtained from α by the replacement of the part γ by δ, then the formula $\alpha\equiv\beta$ is in S^0 (the rule of replacement);

(iii) if $\alpha,\beta,\gamma \in S$, then each of the formulas T_1-T_8, given below, is in S^0:

T_1 $\alpha\rightarrow(\beta\rightarrow\alpha)$

T_2 $\big(\alpha\rightarrow(\beta\rightarrow\gamma)\big)\rightarrow\big((\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma)\big)$

T_3 $\alpha\cdot\beta\rightarrow\alpha$

T_4 $\alpha\cdot\beta\rightarrow\beta$

T_5 $(\gamma\rightarrow\alpha)\rightarrow\big((\gamma\rightarrow\beta)\rightarrow(\gamma\rightarrow\alpha\cdot\beta)\big)$

T_6 $\alpha\rightarrow\alpha+\beta$

T_7 $\beta\rightarrow\alpha+\beta$

T_8 $(\alpha\rightarrow\gamma)\rightarrow\big((\beta\rightarrow\gamma)\rightarrow(\alpha+\beta\rightarrow\gamma)\big)$.

The formulas T_1-T_8 are the axioms of the positive logic [6]). Consequently all formulas, which are substitution of theorems of the positive logic, are also in S^0. In particular the following formulas are in S^0:

T_1' $\alpha\rightarrow\alpha$

T_2' $(\alpha\rightarrow\beta)\rightarrow\big((\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\gamma)\big)$

T_3' $\big(\alpha\rightarrow(\beta\rightarrow\gamma)\big)\rightarrow\big(\beta\rightarrow(\alpha\rightarrow\gamma)\big)$

T_4' $\alpha\cdot\beta\rightarrow\gamma\equiv\alpha\rightarrow(\beta\rightarrow\gamma)$.

We shall suppose that the system $\mathcal{S}$ is consistent, *i. e.* that the set $S-S^0$ is non-empty.

Note that the condition (ii) implies the extensionality of all sentential operators mentioned in (a), (b) and (c).

§ 2. The system $\mathcal{S}^*$. The letter I_0 will always denote the set of all positive integers.

Let I be a fixed set of integers, such that $I_0 \subset I$.

The system $\mathcal{S}$ and the set I determine uniquely a system $\mathcal{S}^*$ of a functional calculus described as follows:

[6]) See Hilbert-Bernays [1], p. 422-450.

The primitive symbols of $\mathcal{S}^*$ are the parentheses and:

(a) the *individual variables* x_i where $i \in I_0$;

(b) the *individual constants* x_i where $i \in I - I_0$ [7]);

(c) the *functional variables* with k arguments F_m^k where $k, m \in I_0$;

(d) the *binary* and *unary sentential operators* mentioned in § 1 (a), (b) and (c);

(e) the *quantifiers* $\sum_{x_i}$ and $\prod_{x_i}$ where $i \in I_0$.

The set S^* of all formulas in $\mathcal{S}^*$ is the smallest set such that

1) $F_m^k(x_{i_1}, \ldots, x_{i_k}) \in S^*$ where $i_1, \ldots, i_k \in I$ and $k, m \in I_0$;

2) if $\alpha, \beta \in S^*$ and $i \in I_0$, then $(\alpha + \beta) \in S^*$, $(\alpha \cdot \beta) \in S^*$, $(\alpha \to \beta) \in S^*$, $(\alpha\, o_k\, \beta) \in S^*$ $(k = 1, \ldots, r)$, $(o^k \alpha) \in S^*$ $(k = 1, \ldots, s)$, $\sum_{x_i} \alpha \in S^*$ and $\prod_{x_i} \alpha \in S^*$.

According to § 1 we shall write, for brevity, $\alpha \equiv \beta$ instead of $((\alpha \to \beta) \cdot (\beta \to \alpha))$.

In writing formulas we shall practice the omission of the parentheses. We assume the rules 1° and 2° from § 1 and the following rule

3° the quantifiers bind more strongly than any of the operators in § 1, (a), (b) and (c).

We assume that the notion of *free* and *bound* occurence of an individual variable is familiar. A formula $\alpha \in S^*$ is said to be *closed* if it contains no free occurence of an individual variable x_i $(i \in I_0)$.

The axioms of $\mathcal{S}^*$ are all substitutions of all theorems in S^0 (see § 1).

Let $R \subset S^*$ be any set of formulas. The set $K(R)$ of all *consequences* of R is the least set of formulas such that

(i) $K(R)$ contains all axioms and all formulas belonging to R;

(ii) if $\alpha \in K(R)$, $(\alpha \to \beta) \in K(R)$, then $\beta \in K(R)$;

(iii) if $\alpha \in K(R)$ and if β is obtained from α by the admissible replacement [8]) of all free occurrence of x_i $(i \in I_0)$ by x_k $(k \in I)$, then $\beta \in K(R)$;

(iv) if $(\alpha \to \prod_{x_i} \beta) \in K(R)$, then $(\alpha \to \beta) \in K(R)$; if $(\sum_{x_i} \alpha \to \beta) \in K(R)$, then $(\alpha \to \beta) \in K(R)$; if there is no free occurrence of x_i $(i \in I_0)$ in α, and if $(\alpha \to \beta) \in K(R)$, then $(\alpha \to \prod_{x_i} \beta) \in K(R)$; if there is no free occurrence of x_i $(i \in I_0)$ in β, and if $(\alpha \to \beta) \in K(R)$, then $(\sum_{x_i} \alpha \to \beta) \in K(R)$;

(v) if $\alpha \in K(R)$ and $(\gamma \equiv \delta) \in K(R)$ and if γ is a part of α, then the formula β obtained from α by the replacement of the part γ by δ, is also in $K(R)$.

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[7]) The case $I = I_0$ is admissible. Then the system $\mathcal{S}^*$ has no individual constants.

[8]) The definition of an admissible replacement is the same as for the calssical functional calculus.

Clearly the rules (ii)-(v) are respectively modus ponens, the rule of substitution for free individual variables, the four quantifier rules and the rule of replacement [9]).

Clearly the rule (v) implies the extensionality of all logical operators $+, \cdot, \to, o_1, \ldots, o_r, o^1, \ldots, o^s, \sum_{x_i}, \prod_{x_i}$.

If $\alpha \in K(R)$, we write also $R \vdash \alpha$ (read: α is a consequence of R). If $R = 0$ is the empty set, instead of $(0) \vdash \alpha$ we write simply $\vdash \alpha$. If $\vdash \alpha$, the formula α is said to be *provable* in $\mathcal{S}^*$.

A set $R \subset S^*$ is said to be *consistent* if the set $S^* - K(R)$ is not empty, *i. e.* if there is a formula $\alpha \in S^*$ which is not a consequence of R.

A set $R \subset S^*$ is said to have the *property* (D) if either $R = 0$ or if, for every $\alpha \in K(R)$, there is a sequence $\alpha_1, \ldots, \alpha_n \in R$ such that $\vdash \alpha_1 \cdot \ldots \cdot \alpha_n \to \alpha$. In other words, R has *the property* (D) whenever the deduction theorem holds in the formalized system which we obtain from $\mathcal{S}^*$ by addmitting the set R as the set of additional axioms [10]).

§ 3. $\mathcal{S}$-algebras. With the system $\mathcal{S}$ described in § 1 we shall associate a type of abstract algebras. Each algebra of this type is an ordered set $\langle A; e; +, \cdot, \to, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$ where

(a) e is a distinguished element of A;

(b) $+, \cdot, \to, o_1, \ldots, o_r$ are binary operations defined over A and class-closing on A;

(c) $o^1, \ldots, o^s$ are unary operations defined over A and class-closing on A.

For convenience, we shall denote such an algebra by the same letter as the set of its elements, *i. e.* we shall write "the algebra A" instead of "the algebra $\langle A; e; +, \ldots, o^s \rangle$".

Every formula $\alpha \in S$ may be interpreted as an algebraical polynomial Φ_α defined in A if we treat the logical constants (mentioned in § 1 (a), (b), (c)) as the corresponding algebraical operations in A, and the sentential variables $a_1, a_2, \ldots$ as the variables running through the set A.

The algebra $\langle A; e; +, \cdot, \to, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$ is said to be an *$\mathcal{S}$-algebra* if the following conditions are fulfilled (for every $a, b \in A$):

(i) if $a \to b = e$ and $b \to a = e$, then $a = b$;

(ii) if $e \to a = e$, then $a = e$;

(iii) if $\alpha \in S^0$ (*i. e.* is a theorem), then $\Phi_\alpha = e$ identically;

(iv) A contains at least two elements.

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[9]) The rule of replacement may clearly be omitted in the case of some special systems since it is often a consequence of the remaining rules of inference. In the case of the Lewis sentential or functional calculus (see § 10), the rule of replacement is independent of other rules of inference.

[10]) *E. g.* each set of closed formulas of the classical functional calculus has the property (D). In other systems, *e. g.* in the Lewis functional calculus, the assumption that all formulas $\alpha \in R$ are closed, does not imply the property (D) (see 10.2).

In the rest of § 3 the letter A will always denote an $\mathcal{S}$-algebra. Let $a, b \in A$. We shall write $a \subset b$ whenever $a \to b = e$.

3.1. *Every $\mathcal{S}$-algebra A is a relatively pseudocomplemented lattice* [11]) *with respect to the operations "$+$" (join), "$\cdot$" (meet), and "$\to$". The relation $a \subset b$ is the ordering relation in the lattice A (i. e. $a \subset b$ if and only if $a + b = b$). The element e is the unit of the lattice A.*

In fact, (i) implies that the relation $\subset$ is antisymmetric (*i. e.* $a \subset b$ and $b \subset a$ imply $a = b$). Theorem T_1' implies the reflexivity of $\subset$ (*i. e.* $a \subset a$). Theorem T_2' implies the transitivity (*i. e.* $a \subset b$ and $b \subset c$ imply $a \subset c$). Hence $\subset$ is a partial ordering. It follows from T_3, T_4, T_5 that $a \cdot b$ is the meet (product) of a and b. Analogously, Theorems T_6, T_7, T_8 imply that $a + b$ is the join (sum) of a and b. Hence A is a lattice. Theorem T_1 implies that $a \subset e$ for every $a \in A$, *i. e.* that e is the unit element of A. It follows from T_4' that $d \subset a \to b$ if and only if $ad = b$. Hence A is a relatively pseudocomplemented lattice, and the operation $\to$ in A coincides with the lattice operation [11]) $\to$.

Let $a, a_u \in A$, where u runs over a non-empty set U. We say that a is the *sum (product)* in A of all elements a_u if simultaneously

1° $a_u \subset a$ $(a \subset a_u)$ for all $u \in U$;

2° if $a_u \subset b$ $(b \subset a_u)$ for all $u \in U$, then $a \subset b$ $(b \subset a)$.

We then write

$$(*) \qquad a = (A) \sum_{u \in U} a_u \qquad \Big(a = (A) \prod_{u \in U} a_u\Big).$$

If an $\mathcal{S}$-algebra A is a complete lattice, *i. e.* if the sums and products (*) exist for every family a_u $(u \in U)$, then A is said to be an *$\mathcal{S}^*$-algebra.*

Let $\langle A; e; +, \ldots, o^s \rangle$ and $\langle A'; e'; +, \ldots, o^s \rangle$ be two $\mathcal{S}$-algebras. A mapping h of A into A' is said to be an *$\mathcal{S}$-homomorphism* if it preserves all (finite) algebraic operations, *i. e.* if

$h(aob) = h(a) o h(b)$ when o is one of the signs in § 1, (a) and (b),
$h(oa) = oh(a)$ if o is one of the signs in § 1 (c).

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[11]) A lattice A (with join $+$, meet $\cdot$, and the lattice order $\subset$) is said to be *relatively pseudocomplemented* if, for every $a, b \in A$, there is an element $c \in A$ such that, for every $d \in A$, $ad \subset b$ if and only if $d \subset c$. The element c, determined uniquely by a and b, will always be denoted by $a \to b$. Each relatively pseudocomplemented lattice has the unit element e. In fact, $e = a \to a$ is the unit element. Moreover, $e \to a = a$ for every a. Each relatively pseudocomplemented lattice is distributive. See Birkhoff [1], p. 147-148 and 195.

A *pseudocomplemented lattice* is a relatively pseudocomplemented lattice with the zero element 0 (the pseudocomplement of a is the element $\neg a = a \to 0$).

If A is a lattice under consideration, then 0 and e always denote the zero element and the unit element of A respectively, whenever they exist. By definition, $0 \subset a \subset e$ for every $a \in A$. Each relatively pseudocomplemented lattice A (with the unit e) will always be considered as an abstract algebra $\langle A; e; +, \cdot, \to \rangle$.

An $\mathcal{S}$-homomorphism h maps e onto e' since, by T_1',

$$e' = h(a) \to h(a) = h(a \to a) = h(e).$$

An $\mathcal{S}$-homomorphism is called an *$\mathcal{S}$-isomorphism* if it is one-to-one. An $\mathcal{S}$-homomorphism h is said to *preserve* the sum (product) (*) if

$$h(a) = (A') \sum_{u \in U} h(a_u) \qquad \Big(h(a) = (A') \prod_{u \in U} h(a_u)\Big).$$

The system $\mathcal{S}$ is said to have *the property* (E) if, for every $\mathcal{S}$-algebra A and for arbitrary enumerable sequences of equations

$$(**) \qquad a_n = (A) \sum_{u \in U_n} a_{nu}, \qquad b_n = (A) \prod_{v \in V_n} b_{nv} \qquad [(n = 1, 2, \ldots)$$

there is an $\mathcal{S}$-isomorphism h of A into an $\mathcal{S}^*$-algebra A' which preserves all the sums and products (**).

§ 4. The Lindenbaum algebra $L(R)$. Let $R \subset S^*$ be a consistent set. For every $\alpha \in S^*$, let $|\alpha|$ denote the class of all $\beta \in S^*$ such that $R \vdash \alpha \equiv \beta$. Let $L(R)$ be the set of all cosets $|\alpha|$ where $\alpha \in S^*$. We define in $L(R)$ the algebraical operations $+, \cdot, \to, o_1, \ldots, o_r, o^1, \ldots, o^s$ as follows:

$$|\alpha| o |\beta| = |\alpha o \beta|$$

if o is one of the binary operations from § 1 (a) and (b), and

$$o|\alpha| = |o\alpha|$$

if o is one of the unary operations from § 1 (c).

This definition is correct. In fact, it follows easily from the rule of replacement (§ 2 I (v)) that $R \vdash \alpha o \beta \equiv \gamma o \delta$ whenever $R \vdash \alpha \equiv \gamma$ and $R \vdash \beta \equiv \delta$; and analogously $R \vdash o\alpha \equiv o\gamma$ whenever $R \vdash \alpha \equiv \gamma$. Therefore the result of the operation o on cosets $\in L(R)$ does not depend on the choice of their representants α, β.

If $R \vdash \alpha$, then $|\alpha|$ is the class of all $\beta \in S^*$ such that $R \vdash \beta$. The element $|\alpha|$, where $R \vdash \alpha$, will be denoted by e.

4.1. *The algebra $\langle L(R); e; +, \cdot, \to, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$ is an $\mathcal{S}$-algebra.*

According to § 3, this algebra will be denoted, for brevity, by $L(R)$.

We have, in particular,

4.2. *$L(R)$ is a lattice with respect to the operations $+$ (join) and $\cdot$ (meet). The inclusion $|\alpha| \subset |\beta|$* [12]) *holds if and only if $R \vdash \alpha \to \beta$. $|\alpha| = e$ if and only if $R \vdash \alpha$.*

The easy proof of 4.1 and 4.2 is omitted. Note that the condition (iv) from § 3 is fulfilled since R is consistent.

For every $\alpha \in S^*$ and for $p \in I$, $k \in I_0$, let $\alpha\binom{x_p}{x_k}$ be the formula which we obtain from α in the following way:

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[12]) Clearly $\subset$ denotes here the lattice inclusion in $L(R)$ (not the set-theoretical inclusion between cosets $|\alpha|$ and $|\beta|$).

We choose an $l \in I_0$ such that $l \neq p$ and α contains neither x_l nor $\sum\limits_{x_l}$ nor $\prod\limits_{x_l}$. We replace every bound occurrence of x_p by x_l, and every quantifier $\sum\limits_{x_p}$ $(\prod\limits_{x_p})$ by $\sum\limits_{x_l}$ $(\prod\limits_{x_l})$. Further we replace every free occurrence of x_k by x_p.

The formula $\alpha\binom{x_p}{x_k}$ defined in such a way is not uniquely determined. However the element $\left|\alpha\binom{x_p}{x_k}\right| \in L(R)$ is uniquely determined since it does not depend on the choice of l.

Using the above notation we obtain

4.3. *For every* $\alpha \in S^*$

$$(*) \qquad (L(R))\sum_{p \in I}\left|\alpha\binom{x_p}{x_k}\right| = \left|\sum_{x_k}\alpha\right|$$

$$(**) \qquad (L(R))\prod_{p \in I}\left|\alpha\binom{x_p}{x_k}\right| = \left|\prod_{x_k}\alpha\right|.$$

Since $\vdash \alpha\binom{x_p}{x_k} \to \sum\limits_{x_k}\alpha$, we have $R \vdash \alpha\binom{x_p}{x_k} \to \sum\limits_{x_k}\alpha$ and consequently, by 4.2,

$$\left|\alpha\binom{x_p}{x_k}\right| \subset \left|\sum_{x_k}\alpha\right| \qquad \text{for each } p \in I.$$

Suppose a formula $\beta \in S^*$ satisfies the relation

$$\left|\alpha\binom{x_p}{x_k}\right| \subset |\beta| \qquad \text{for each } p \in I.$$

By 4.2, we have $R \vdash \alpha\binom{x_p}{x_k} \to \beta$. Let q be a positive integer such that neither β nor γ contains a free occurrence of x_q. Then

$$R \vdash \sum_{x_q}\alpha\binom{x_q}{x_k} \to \beta.$$

Hence

$$\left|\sum_{x_k}\alpha\right| = \left|\sum_{x_q}\alpha\binom{x_q}{x_k}\right| \subset |\beta|,$$

which completes the proof of (*). The proof of (**) is analogous.

An $\mathcal{S}$-homomorphism ($\mathcal{S}$-isomorphism) h of $L(R)$ into an $\mathcal{S}$-algebra A is said to be an *$\mathcal{S}^*$-homomorphism ($\mathcal{S}^*$-isomorphism)* if h preserves all the sums (*) and all the products (**).

An $\mathcal{S}^*$-algebra A is said to be an *$\mathcal{S}^*$-extension* of $L(R)$ if there is an $\mathcal{S}^*$-isomorphism of $L(R)$ into A. Clearly if $\mathcal{S}$ has the property (E) (see § 3), such an $\mathcal{S}^*$-extension of $L(R)$ exists.

If R is the empty set, we simply write L instead of $L(0)$.

§ 5. (J,A) functionals. Let J and A be non-empty sets. The class of all mappings of the Cartesian product $\underbrace{J \times J \times \ldots \times J}_{k\text{-times}}$ into A will be denoted by $F^k(J,A)$. Every mapping of a Cartesian product

$$\underbrace{J \times J \times \ldots \times J}_{l\text{-times}} \times F^{k_1} \times F^{k_2} \times \ldots \times F^{k_m}$$

into A is called an (J,A) *functional*. The case $l=0$ is admissible.

In the rest of § 5 the letter A will always denote an $\mathcal{S}^*$-algebra. J is always a non-empty set.

Every formula $\alpha \in S^*$ may be interpreted as an (J,A) functional, denoted by $(J,A)\Phi_\alpha$, by treating

(a) all the individual variables x_i $(i \in I_0)$ and all individual constants x_i $(i \in I - I_0)$ as variables running over J;

(b) all k-arguments functional variables F_m^k as variables running over $F^k(J,A)$ $(k, m \in I_0)$;

(c) each of the logical signs mentioned in § 1 (a), (b), (c) as the corresponding algebraical operation in A;

(d) the logical quantifiers $\sum\limits_{x_i}$ and $\prod\limits_{x_i}$ as the signs of (infinite) sums $(A)\sum\limits_{x_i \in J}$ and products $(A)\prod\limits_{x_i \in J}$ respectively $(i \in I_0)$.

The symbol $(J,A)\Phi_\alpha(\{j_i\}, \{\varphi_m^k\})$ will denote the value of the functional $(J,A)\Phi_\alpha$ for the following values of its arguments x_i and F_m^k

$$(s) \qquad x_i = j_i \in J \quad \text{and} \quad F_m^k = \varphi_m^k \in F^k(J,A).$$

Clearly it is sufficient to define the substitution (s) only for such i, m, k that F_m^k appears in α, x_i appears in α and $i \leqslant 0$, or $i > 0$ and there is a free occurrence of x_i in α. For simplicity, we shall assume (except in the proof of 9.2) that the substitution (s) is determined for all $i \in I$ and for all $k, m \in I_0$.

The following equations may be considered as the inductive definition of $(J,A)\Phi_\alpha$.

5.1. *If* $\alpha, \beta \in S^*$ *and* o *is binary operation* (§ 1, (a), (b)), *then*

$$(J,A)\Phi_{\alpha o \beta}(\{j_i\}, \{\varphi_m^k\}) = ((J,A)\Phi_\alpha(\{j_i\}, \{\varphi_m^k\}))\, o\, ((J,A)\Phi_\beta(\{j_i\}, \{\varphi_m^k\})).$$

If o *is a unary operation* (§ 1, (c)), *then*

$$(J,A)\Phi_{o\alpha}(\{j_i\}, \{\varphi_m^k\}) = o\,((J,A)\Phi_\alpha(\{j_i\}, \{\varphi_m^k\})).$$

Analogously, for $p \in I_0$,

$$(*)\quad \begin{aligned} (J,A)\Phi_{\underset{x_p}{\Sigma}\alpha}\big(\{j_i\},\{\varphi_m^k\}\big) &= (A)\sum_{x_p \in J}(J,A)\Phi_\alpha\big(\{j_i\}',\{\varphi_m^k\}\big) \\ (J,A)\Phi_{\underset{x_p}{\Pi}\alpha}\big(\{j_i\},\{\varphi_m^k\}\big) &= (A)\prod_{x_p \in J}(J,A)\Phi_\alpha\big(\{j_i\}',\{\varphi_m^k\}\big) \end{aligned}$$

where $\{j_i\}'$ *denotes the sequence* $\{j_i\}$ *where the p-th term* j_p *is replaced by* x_p.

5.2. *Let* $R \subset S^*$ *be a consistent set, and let h be an* $\mathcal{S}^*$*-homomorphism of* $L(R)$ *into an* $\mathcal{S}^*$*-algebra* L^*. *Then*

$$(I,L^*)\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big) = h(|\alpha|)$$

where $j_i = i$ *for* $i \in I$, *and* φ_m^k *is defined as follows*

$$(**)\qquad \varphi(i_1,i_2,\ldots,i_k) = h\big(|F_m^k(x_{i_1},x_{i_2},\ldots,x_{i_k})|\big) \quad \textit{for} \quad i_1,i_2,\ldots,i_k \in I.$$

Lemma 5.2 follows from 5.1, 4.3 and from the definition of the algebraical operations in $L(R)$. The easy proof by induction on the length of α is omitted.

If the functional $(J,A)\Phi_\alpha$ assumes only the unit element $e \in A$ as its value, we write $(J,A)\Phi_\alpha = e$.

5.3. *If* $\vdash \alpha$, *then* $(J,A)\Phi_\alpha = e$ *for every set* $J \neq 0$ *and for every* $\mathcal{S}^*$*-algebra* A.

Theorem 5.3 has been proved by Mostowski [1] in the case of $\mathcal{S}^*=$ the functional calculus of Heyting, and by Rasiowa [1] in the case of $\mathcal{S}^*=$ the functional calculus of Lewis. The proof in the general case is similar.

Notice moreover that

5.4. *If* $\beta = \alpha\binom{x_q}{x_p}$, *then*

$$(J,A)\Phi_\beta\big(\{j_i\},\{\varphi_m^k\}\big) = (J,A)\Phi_\alpha\big(\{\bar{j}_i\},\{\varphi_m^k\}\big)$$

where $\bar{j}_i = j_i$ *if* $i \neq p$, *and* $\bar{j}_p = j_q$.

5.5. *If an* $\mathcal{S}$*-homomorphism g of* A *into another* $\mathcal{S}^*$*-algebra* A' *preserves all infinite sums and products, then*

$$g\big((J,A)\Phi_\alpha(\{j_i\},\{\varphi_m^k\})\big) = (J,A')\Phi_\alpha\big(\{j_i\},\{g\varphi_m^k\}\big).$$

§ 6. Satisfiability and validity. A formula $\alpha \in S^*$ is said to be *satisfiable in a set* $J \neq 0$ *and in an* $\mathcal{S}^*$*-algebra* A if there is a substitution

$$\text{(a)}\qquad x_i = j_i \in J \quad \text{and} \quad F_m^k = \varphi_m^k \in \boldsymbol{F}^k(J,A) \qquad (i \in I,\ k,m \in I_0)$$

such that

$$\text{(b)}\qquad (J,A)\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big) = e \in A.$$

More generally, a set $R \subset S^*$ is said to be *satisfiable in a set* $J \neq 0$ *and in an* $\mathcal{S}^*$*-algebra* A if there is a (common) substitution (a) such that the equation (b) holds for all $\alpha \in R$.

A formula $\alpha \in S^*$ (or: a set $R \subset S^*$) is said to be *satisfiable in a set* $J \neq 0$ if there is an $\mathcal{S}^*$-algebra A such that α (or: R) is satisfiable in J and in A.

A formula $\alpha \in S^*$ (or: a set $R \subset S^*$) is said to be *satisfiable*, if there is a set $J \neq 0$ such that α (or: R) is satisfiable in J.

A formula $\alpha \in S^*$ is said to be *valid in a set* $J \neq 0$ if $(J,A)\Phi_\alpha = e$ for every $\mathcal{S}^*$-algebra A.

A formula $\alpha \in S^*$ is said to be *valid* if it is valid in each set $J \neq 0$.

It is obvious that, in all the above definitions, only the cardinal of J plays an essential part.

6.1. *Let* $\alpha \in S^*$. *If the Lindenbaum algebra L has an* $\mathcal{S}^*$*-extension* L^* *(in particular, if the system* $\mathcal{S}$ *has the property* (E)), *then the following conditions are equivalent*:

(i) $\vdash \alpha$;
(ii) α *is valid*;
(iii) α *is valid in the enumerable set* I;
(iv) $(I,L^*)\Phi_\alpha = e^* =$ *the unit of* L^*.

The implication (i)→(ii) is another formulation of 5.3. The implications (ii)→(iii) and (iii)→(iv) are trivial.

Suppose that (iv) holds. Let h be an $\mathcal{S}^*$-isomorphism of L into L^*. By (iv), $(I,L^*)\Phi_\alpha = e^*$. Hence, by 5.2, $h(|\alpha|) = e^*$. Since h is an isomorphism, we infer that $|\alpha|$ is the unit element of L. Consequently $\vdash \alpha$ by 4.2.

6.2. *Let* $R \subset S^*$ *be a consistent set. If there is an* $\mathcal{S}^*$*-homomorphism h of* $L(R)$ *into an* $\mathcal{S}^*$*-algebra* L^* *(in particular if* $\mathcal{S}$ *has the property* (E)), *then the set R is satisfiable in the enumerable set I.*

More exactly: *R is satisfiable in I and in* L^*.

We have $h(|\alpha|) = e^* =$ the unit of L^* for every $\alpha \in R$ since $|\alpha| = e \in L(R)$ (see 4.2). Consequently, by 5.2, we have for each $\alpha \in R$

$$(I,L^*)\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big) = h(|\alpha|) = e^*$$

where the substitution $x_i = j_i$ and $F_m^k = \varphi_m^k$ is defined as in 5.2.

An $\mathcal{S}^*$-algebra A is said to be *functionally free* if, for every $\alpha \in S^*$, the condition

$$(J,A)\Phi_\alpha = e \text{ for every set } J \neq 0$$

implies that α is valid.

An $\mathcal{S}^*$-algebra A is said to be *functionally* σ*-free* if, for every $\alpha \in S^*$, the condition

$$(J,A)\Phi_\alpha = e \text{ for an enumerable set } J$$

implies that α is valid.

Clearly, if A is functionally σ-free, then A is functionally free. The implication (iv)$\to$(i) in 6.1 may be formulated as follows.

6.3. *Every $\mathcal{S}^*$-extension L^* of L is functionally σ-free.*

The following lemmas will be useful later.

6.4. *Let A' be a complete subalgebra of an $\mathcal{S}^*$-algebra A, let $J\neq 0$ be any set, let $\alpha \in S^*$ and $R\subset S^*$.*

(a) *If $(J,A)\Phi_\alpha = e$, then $(J,A')\Phi_\alpha = e$ also.*

(b) *If R is satisfiable in J and A', then R is satisfiable in J and A.*

This follows immediately from the definitions of satisfiability and of validity since $F^k(J,A')\subset F^k(J,A)$.

6.5. *Let A be an $\mathcal{S}^*$-algebra, and let J and J' be two non-empty sets.*

(A) *If $\overline{\overline{J'}}\leqslant\overline{\overline{J}}$, $\alpha\in S^*$, and if $(J,A)\Phi_\alpha=e$, then $(J',A)\Phi_\alpha=e$.*

(B) *If $\overline{\overline{J'}}\leqslant\overline{\overline{J}}$, and if a set $R\subset S^*$ is satisfiable in J' and in A, then R is also satisfiable in J and in A.*

Let $\imath$ be a mapping of J onto J'. If $\varphi\in F^k(J',A)$, let $\varphi_\imath$ be defined as follows

$$\varphi_\imath(j_1,\ldots,j_k)=\varphi(\imath(j_1),\ldots,\imath(j_k)) \quad \text{for} \quad j_1,\ldots,j_k\in J.$$

Clearly $\varphi_\imath\in F^k(J,A)$.

The statements (A) and (B) follow from the obvious equation, which holds for every $\alpha\in S^*$

$$(J',A)\Phi_\alpha\big(\{\imath(j_i)\},\{\varphi_m^k\}\big)=(J,A)\Phi_\alpha\big(\{j_i\},\{(\varphi_m^k)_\imath\}\big).$$

§ 7. Systems with negations. In this section we suppose that the sequence (c) from § 1 contains a sign called the *negation sign* and denoted by $-$. We suppose moreover that the set T of theorems of $\mathcal{S}$ contains every formula of the form

$$\text{(N)} \qquad -(\beta\to\beta)\to\alpha.$$

Our new assumption implies that we have an operation $-a$ (called sometimes the *complementation*) in each $\mathcal{S}$-algebra A. The axiom (N) implies that, for every $a,b\in A$,

$$-(b\to b)\to a=e.$$

Since $b\to b=e$ by T_1', we obtain

$$(*) \qquad -e\to a=e \quad \text{for every} \quad a\in A.$$

Hence

7.1. *The element $-e\in A$ is the zero element of the $\mathcal{S}$-algebra A, i. e. if $-e\subset a$ for every $a\in A$. Consequently* (*see* § 3 (iv))

$$e\neq -e.$$

Note that only the inequality $e\neq -e$ will play an essential part in the proof of the theorems 7.2 and 7.3.

The following theorem is converse to 6.2.

7.2. *If a set $R\subset S^*$ is satisfiable and has the property* (D), *then R is consistent.*

By the hypothesis, there are a set $J\neq 0$, an $\mathcal{S}^*$-algebra A and a substitution

$$x_i=j_i\in J \qquad F_m^k=\varphi_m^k\in F^k(J,A) \qquad (i\in I, k,m\in I_0)$$

such that

$$(*) \qquad (J,A)\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big)=e\in A \quad \text{for every} \quad \alpha\in R.$$

Let $\vdash\gamma$ $(\gamma\in S^*)$. Suppose that R is not consistent. Then $R\vdash -\gamma$. The property (D) implies that there is a sequence $\beta_1,\ldots,\beta_n\in R$ such that $\vdash\beta\to-\gamma$ where $\beta=\beta_1\cdot\ldots\cdot\beta_n$. Consequently $(J,A)\Phi_{\beta\to-\gamma}=e$.

Let $a=(J,A)\Phi_\gamma\big(\{j_i\},\{\varphi_m^k\}\big)$. Since by (*)

$$(J,A)\Phi_\beta\big(\{j_i\},\{\varphi_m^k\}\big)=(J,A)\Phi_{\beta_1}\big(\{j_i\},\{\varphi_m^k\}\big)\cdot\ldots\cdot(J,A)\Phi_{\beta_n}\big(\{j_i\},\{\varphi_m^k\}\big)=e$$

we obtain by § 3 (ii) and 5.1 that

$$-a=(J,A)\Phi_{-\gamma}\big(\{j_i\},\{\varphi_m^k\}\big)=e.$$

On the other hand, $(J,A)\Phi_\gamma=e$ by 5.3. Hence $a=e$. This implies that $-e=-a=e$ which is impossible by 7.1.

7.3. *Let $R\subset S^*$ be a satisfiable set having the property* (D). *If $\mathcal{S}$ has the property* (E), *then R is satisfiable in the enumerable set I.*

This is a direct consequence of 6.2 and 7.2.

§ 8. A lemma. The purpose of this section is to prove the lemma 8.2 which will be useful later.

Let $\mathcal{S}$ be the system described in § 1 (with or without negation). Let $\mathcal{S}^*$ be the functional calculus determined by $\mathcal{S}$ and the set $I=$ the set I_0 of all positive integers by the method described in § 2. Further, let $\bar{\mathcal{S}}^*$ be the functional calculus determined by $\mathcal{S}$ and the set $I=$ the set $\bar{I}_0$ of all integers by the method of § 2. The system $\mathcal{S}^*$ has no individual constants. The system $\bar{\mathcal{S}}^*$ has individual constants $x_0, x_{-1}, x_{-2},\ldots$

The sets of formulas of $\mathcal{S}^*$ and $\bar{\mathcal{S}}^*$ are denoted by S^* and $\bar{S}^*$ respectively.

If $\alpha\in S^*$, then $\bar{\alpha}$ will denote the formula which arises from α when we replace each free occurrence of x_i $(i=1,2,\ldots)$ by x_{-i}. Clearly $\bar{\alpha}\in\bar{S}^*$ and $\bar{\alpha}$ is closed.

If $R\subset S^*$, then $\bar{R}$ denotes the set of all $\bar{\alpha}$ where $\alpha\in R$. Clearly $\bar{R}\subset\bar{S}^*$.

Let J be a non-empty set and let A be an $\bar{\mathcal{S}}^*$-algebra. Clearly A is also an $\mathcal{S}^*$-algebra.

Under the above assumptions

8.1. *For every* $\alpha \in S^*$, *the equation*

$$(J,A)\,\Phi_{\alpha}\big(\{j_i\},\{\varphi_m^k\}\big) = (J,A)\,\Phi_{\bar{\alpha}}\big(\{\bar{j}_i\},\{\varphi_m^k\}\big)$$

holds whenever $j_i = \bar{j}_{-i}$ *for every* $i \in I_0$.

This follows immediately from the definition of the functionals $(J,A)\Phi_\alpha$ and $(J,A)\Phi_{\bar{\alpha}}$ (see 5.1). The exact proof by induction on the length of α is omitted.

8.2. *A set* $R \subset S^*$ *is satisfiable in* J *and* A *if and only if the set* $\bar{R} \subset \bar{S}^*$ *is satisfiable in* J *and* A.

Lemma 8.2 follows immediately from 8.1.

Part II

§ 9. The classical calculus. We shall now specialize the system S described in § 1.

First let us consider the case where $\mathcal{S}$ is the classical sentential calculus which will be denoted here by $\mathcal{S}_\varkappa$.

Besides the signs $+, \cdot,$ and $\to$ the system $\mathcal{S}_\varkappa$ contains also the negation sign $-$. The axioms of $\mathcal{S}_\varkappa$ are the formulas T_1-T_8 and the following (see Łukasiewicz [1], p. 86)

T_9 $(\alpha \to -\beta) \to (\beta \to -\alpha)$

T_{10} $-\alpha \to (\alpha \to \beta)$

T_{11} $--\alpha \to \alpha$.

Clearly the formulas (N) are theorems in $\mathcal{S}_\varkappa$.

Set $I = I_0$ in § 2. The functional calculus determined by $\mathcal{S} = \mathcal{S}_\varkappa$ and $I = I_0$ by the method described in § 2 is the classical functional calculus which will be denoted here by $\mathcal{S}_\varkappa^*$.

The set of all formulas in $\mathcal{S}_\varkappa^*$ will be denoted by $S_\varkappa^*$. $L_\varkappa$ will denote the Lindenbaum algebra constructed from formulas $\alpha \in S_\varkappa^*$ by the method described in § 4 where R = the empty set.

It is well known that $\mathcal{S}_\varkappa$-algebras are Boolean algebras [13]) and conversely. Consequently $\mathcal{S}_\varkappa^*$-algebras are complete Boolean algebras and conversely.

The letter B will exclusively denote a Boolean algebra. The letter B_0 will denote the two-element Boolean algebra. The Boolean algebra of all subsets of a set $\mathfrak{X} \neq 0$ will be denoted by $\boldsymbol{B}(\mathfrak{X})$.

According to § 3, a Boolean algebra B will be considered as an algebra with respect to the four operations: join (sum) $a+b$, complementation $-a$, meet (product) $a \cdot b$ $\big(= -((-a)+(-b))\big)$, and the operation corresponding to the implication $a \to b = (-a)+b$ $(a, b \in B)$.

13) We assume in this paper that each Boolean algebra contains, ex definitione, at least two elements: the unit e and the zero 0 (*i. e.* $0 \subset a \subset e$ for every $a \in B$, and $0 \neq e$).

The system $\mathcal{S}_\varkappa$ has the property (E) *i. e.* for each Boolean algebra B there is an $\mathcal{S}_\varkappa$-isomorphism h of B into a complete Boolean algebra B^* which preserves a given enumerable set of infinite sums and products (see § 3). For instance, if B^* is the minimal extension [14]) of B, then the imbedding mapping preserves all infinite sums and products [15]). However, for our purpose, it is convenient to construct another $\mathcal{S}_\varkappa^*$-extension of B. This construction is given by the following lemma which clearly implies the property (E).

9.1. *Let* $a_n, b_n, a_{nu}, b_{nv} \in B$ (*where* $u \in U_n$, $v \in V_n$, $n = 1,2,\ldots$). *Suppose that*

$$(+) \qquad a_n = (B)\sum_{u \in U_n} a_{nu} \quad \text{and} \quad b_n = (B)\prod_{v \in V_n} b_{nv}$$

for $n = 1,2,\ldots$ *Then there is an* $\mathcal{S}_\varkappa$-*isomorphism* h *of* B *into the complete Boolean algebra* $\boldsymbol{B}(\mathfrak{X})$ *of all subsets of a set* $\mathfrak{X} \neq 0$ *such that* h *preserves all the sums and product* (+), *i. e.*

$$(++)\,[16]) \qquad h(a_n) = \sum_{u \in U_n} h(a_{nu}) \quad \textit{and} \quad h(b_n) = \sum_{v \in V_n} h(b_{nv_n}).$$

Let $\mathcal{Y}$ be the set of all prime ideals [17]) of B. For every $a \in B$, let $Y(a)$ be the set of all prime ideals p such that a non $\in p$.

Consider $\mathcal{Y}$ as a topological space with the sets $Y(a)$ $(a \in B)$ as the class of neighbourhoods. Stone [18]) has proved that $\mathcal{Y}$ is a totally disconnected bicompact Hausdorff space and the mapping $Y = Y(a)$ is an isomorphism of B into $\boldsymbol{B}(\mathcal{Y})$.

The sets

$$Y_n = Y(a_n) - \sum_{u \in U_n} Y(a_{nu}),$$

$$Z_n = \prod_{v \in V_n} Y(b_{nv}) - Y(b_n)$$

$(n = 1,2,\ldots)$ are nowhere dense [19]) in $\mathcal{Y}$. The set $Z = \sum_{n=1}^{\infty} Y_n + \sum_{n=1}^{\infty} Z_n$ being of the first category in $\mathcal{Y}$, we infer that the set $\mathfrak{X} = \mathcal{Y} - Z$ is dense [20]) in $\mathcal{Y}$. Consequently, if $a \neq 0$, then the set

$$h(a) = \mathfrak{X}\,Y(a)$$

is not empty, *i. e.* h is an isomorphism of B into $\boldsymbol{B}(\mathfrak{X})$.

14) MacNeille [1], p. 437.

15) See *e. g.* Sikorski [2], th. 3.6.

16) If X_u (Y_v) are sets, then ΣX_u (ΠY_v) denotes the set-theoretical union (intersection) of all X_u (Y_v).

17) A set $p \subset B$ is said to be a *prime ideal* provided that 1° if $a, b \in p$, and $c \subset a$, then $a+b \in p$ and $c \in p$; 2° for every $a \in B$, either $a \in p$ or $-a \in p$; 3° e non $\in p$.

18) Stone [1], p. 378.

19) For the proof, see Rasiowa-Sikorski [2], lemma (iii).

20) See Sikorski [1], p. 257, footnote [21]).

Since

$$Y(a_n) = Y_n + \sum_{u \in U_n}^{\infty} Y(a_{nu}),$$

$$Y(b_n) + Z_n = \prod_{v \in V_n}^{\infty} Y(a_{nv}),$$

we find by multiplying this equations by $\mathfrak{X}$ that the equations (++) holds.

9.2. *Let $J \neq 0$ be an at most enumerable set, let B be a complete Boolean algebra, and let $\varphi_m^k \in F^k(J,B)$ be given mappings ($k, m \in I_0$). Then there is an $\mathcal{S}_\varkappa$-homomorphism h_0 of B into B_0 such that* [21])

$$(*) \qquad (J,B_0)\,\Phi_\alpha(\{j_i\},\{h_0\varphi_m^k\}) = h_0\Big((J,B)\,\Phi_\alpha(\{j_i\},\{\varphi_m^k\})\Big)$$

for every $\alpha \in S^*$ and for every substitution $x_i = j_i \in J$.

Moreover, if $b \neq e$ is a given element in B, we may suppose that $h_0(b)$ = the zero of B_0.

The class of all sums and products 5.1 (*) (where $A = B$, $I = I_0$, $\alpha \in S^*$, $p = 1, 2, \ldots$, $\{j_i\}$ is any sequence of elements in J) is at most enumerable since we may suppose that the sequences $\{j_i\}$ are finite (see the remark before lemma 5.1). By 9.1 there is a set $\mathfrak{X} \neq 0$ and an $\mathcal{S}_\varkappa$-isomorphism h of B into $\boldsymbol{B}(\mathfrak{X})$ which preserves all the sums 5.1 (*). Let $x_0 \in \mathfrak{X}$ and let

$$h_0(a) = h(a)\cdot(x_0) \quad \text{for} \quad a \in B.$$

h_0 is an $\mathcal{S}_\varkappa$-homomorphism of B into the two-element Boolean algebra $B_0 = \boldsymbol{B}((x_0))$ which preserves all sums and products 5.1 (*). Consequently the equation 9.2 (*) holds. The easy proof by induction on the length of α is omitted.

If $b \neq e$, then $h(b) \neq \mathfrak{X}$. If we choose x_0 so that x_0 non $\in h(b)$ we obtain $h_0(b) = 0 \in B_0$.

Since the system $\mathcal{S}_\varkappa$ has the property (E) and the formula (N) is a theorem, theorems 6.1, 6.2, 7.2 and 7.3 hold for the system $\mathcal{S}_\varkappa^*$. These theorems are the classical Gödel and Skolem-Löwenheim theorems [22]). This remark results from the following theorems 9.3, 9.4, and 9.5.

9.3. *A set $R \subset S^*$ is satisfiable in a set $J \neq 0$ (in the sense defined in § 6) if and only if R is satisfiable in J in the sense of* Tarski [1].

Consequently, R is satisfiable in the sense defined in § 6 if and only if it is satisfiable in the sense defined by Tarski [1].

For brevity we shall say that R is T-satisfiable (in J) if it is satisfiable (in J) in the sense defined by Tarski [1].

[21]) Clearly $h_0\varphi$ is the superposition of h_0 and φ.

[22]) Löwenheim [1]; Skolem [1], [2], [3]; K. Gödel [1]; Hilbert-Bernays [1]; Henkin [1]; Beth [1]; Rieger [1]; Rasiowa-Sikorski [2] and [3].

Tarski's original definition of satisfiability can be translated in the algebraical language from § 6: a set $R \subset S^*$ is T-satisfiable in J if and only if R is satisfiable in J and B_0 [23]). Consequently, if R is T-satisfiable in J, then R is satisfiable in J in the sense of § 6. Therefore it is sufficient to prove that

9.4. *If a set $R \subset S_\varkappa^*$ is satisfiable in $J \neq 0$ and in a complete Boolean algebra B, then*

(a) *R is satisfiable in J and B_0;*

(b) *R is satisfiable in I_0 and B_0.*

Consider first the case where $\overline{\overline{J}} \leqslant \aleph_0$. By hypothesis, there is a substitution

$$x_i = j_i \in J \quad \text{and} \quad F_m^k = \varphi_m^k \in F^k(J,B)$$

such that

$$(J,B)\,\Phi_\alpha(\{j_i\},\{\varphi_m^k\}) = e \in B \quad \text{for every} \quad \alpha \in R.$$

Apply the first part of 9.2. We find that

$$(J,B_0)\,\Phi_\alpha(\{j_i\},\{h_0\varphi_m^k\}) = h_0(e) = \text{the unit of } B_0$$

for every $\alpha \in R$, which proves (a). The part (b) follows from (a) and 6.5 (B).

Suppose now that $\overline{\overline{J}} > \aleph_0$. Apply all the notations of § 8 to the case $\mathcal{S} = \mathcal{S}_\varkappa$. Then $\mathcal{S}^* = \mathcal{S}_\varkappa^*$ and $\overline{\mathcal{S}}_\varkappa^*$ is the classical functional calculus with individual constants $x_0, x_{-1}, x_{-2}, \ldots$ Since R is satisfiable in J and B, we find from 8.2 that $\bar{R}$ is also satisfiable in J and B. However, each formula $\bar{\alpha} \in \bar{R}$ is closed. Consequently $\bar{R}$ has the property (D) in $\bar{S}_\varkappa^*$. By 7.2 the set $\bar{R}$ is consistent. Hence, by 6.2, $\bar{R}$ is satisfiable in $\bar{I}_0$ and in a complete Boolean algebra L^*. By 8.2 and 6.5 (B), R is satisfiable in I_0 and L^*. Since $\overline{\overline{I}}_0 = \aleph_0$, we may apply the already proved part of 9.4 (b). We infer that R is satisfiable in I_0 and B_0, *i. e.* (b) is completely proved. The statement (a) in the case of $\overline{\overline{J}} > \aleph_0$ follows immediately from (b) and 6.5 (B).

Theorem 6.1 may be expressed in the following, more precise form:

9.5. *For every $\alpha \in S_\varkappa^*$, the following conditions are equivalent:*

(i) *α is provable in $\mathcal{S}_\varkappa^*$;*

(ii) *α is valid, i. e. $(J,B)\,\Phi_\alpha = e \in B$ for every set $J \neq 0$ and for every complete Boolean algebra B;*

(iii) *α is valid in the set I_0, i. e. $(I_0,B)\,\Phi_\alpha = e \in B$ for every complete Boolean algebra B;*

(iv) *if L^* is an $\mathcal{S}^*$-extension of $L_\varkappa$, then $(I_0,L^*)\,\Phi_\alpha = e \in L^*$;*

(v) *$(J,B_0)\,\Phi_\alpha = e \in B_0$ for every set $J \neq 0$;*

[23]) See Rasiowa-Sikorski [2], § 2 and the proof of (i).

(vi) *there is an infinite set J and a complete Boolean algebra B such that* $(J,B)\,\Phi_\alpha = e \in B$;

(vii) *there is a complete Boolean algebra B such that* $(I_0,B)\,\Phi_\alpha = e \in B$;

(viii) $(I_0,B_0)\,\Phi_\alpha = e \in B_0$.

Notice that the condition $(J,B_0)\,\Phi_\alpha = e$ is the algebraical formulation of the validity of α in J in the sense defined by Tarski [1] [24]). The condition (v) is thus the algebraical formulation of the validity of α in the sense defined by Tarski. The implications (i)→(v), (v)→(viii), (viii)→(i) form the classical completeness theorem of Gödel.

By 6.2 it suffices to prove the following implications:

(ii)→(v). This is trivial.

(v)→(vi). This is trivial.

(vi)→(vii). This follows from 6.5 (A).

(vii)→(viii). This follows from 6.4 (a) since B_0 may be interpreted as the subalgebra of B composed of the unit and the zero of B.

(viii)→(iv). Suppose that there is a substitution

$$x_i = j_i \in I_0, \quad F_m^k = \varphi_m^k \in F^k(I_0, L^*)$$

such that

$$(I_0, L^*)\,\Phi_\alpha(\{j_i\}, \{\varphi_m^k\}) = b \neq e \in L^*.$$

By 9.2 there is an $\mathcal{S}_\varkappa$-homomorphism h_0 of L^* onto B_0 such that $h_0(b) = 0 \in B_0$ and that 9.2 (∗) holds. Hence

$$(I_0, B_0)\,\Phi_\alpha(\{j_i\}, \{h_0\varphi_m^k\}) = h_0(b) = 0 \in B_0$$

which contradicts (viii).

The equivalence (i)≡(vi) implies that

9.6. *Every complete Boolean algebra is functionally free and functionally σ-free.*

§ 10. The Lewis calculus. Consider now the case where $\mathcal{S}$ is the sentential calculus of Lewis designated here by $\mathcal{S}_\lambda$ [25]). Besides the sign $+, \cdot, \rightarrow$ the system $\mathcal{S}_\lambda$ contains also the two unary sentential operators: the negation sign $-$ and the possibility sign $\mathbf{C}$. The axioms of $\mathcal{S}_\lambda$ are the formulas T_1-T_{11} and the following

T_{12} $\mathbf{C}(\alpha+\beta) \equiv \mathbf{C}\alpha + \mathbf{C}\beta$

T_{13} $\alpha \rightarrow \mathbf{C}\alpha$

T_{14} $\mathbf{C}\mathbf{C}\alpha \rightarrow \mathbf{C}\alpha$

T_{15} $-\mathbf{C}\big(\alpha\cdot(-\alpha)\big)$.

Clearly the formulas (N) of § 7 are theorems in $\mathcal{S}_\lambda$.

[24]) See also Rasiowa-Sikorski [2], (ii).

[25]) This system is referred to as the system S_4 in Lewis-Langford [1].

The functional calculus obtained by the method described in § 2 where $\mathcal{S} = \mathcal{S}_\lambda$ and $I = I_0$ will be denoted by $\mathcal{S}_\lambda^*$. Clearly $\mathcal{S}_\lambda^*$ is the Lewis functional calculus [26]).

By S_λ and S_λ^* we shall denote the sets of all formulas is $\mathcal{S}_\lambda$ and $\mathcal{S}_\lambda^*$ respectively. Analogously L_λ will denote the Lindenbaum algebra described in § 4 where $\mathcal{S}^* = \mathcal{S}_\lambda^*$ and $R =$ the empty set.

For brevity we shall write $\mathbf{I}\alpha$ instead of $-\mathbf{C}-\alpha$ for every $\alpha \in S_\lambda^*$ or S_λ.

It is well known [27]) that $\mathcal{S}_\lambda$-algebras are closure algebras and conversely. A *closure algebra* [28]) is an algebra $\langle C; e; +, \cdot, \rightarrow, -, \mathbf{C}\rangle$ such that $\langle C; e; +, \cdot, \rightarrow, -\rangle$ is a Boolean algebra (with the unit e, join $a+b$, meet $a\cdot b$, complement $-a$, and the operation $a \rightarrow b = (-a)+b$) in which a closure operation $\mathbf{C}a$ is defined, such that, for every $a, b \in C$,

I. $\mathbf{C}(a+b) = \mathbf{C}a + \mathbf{C}b$ III. $a \subset \mathbf{C}a$

II. $\mathbf{C}\mathbf{C}a \subset \mathbf{C}a$ IV. $\mathbf{C}0 = 0$.

Clearly the operation $\mathbf{I}a = -\mathbf{C}-a$ is the algebraical analogue to the sentential operator $\mathbf{I}$.

An element $a \in C$ is closed if $a = \mathbf{C}a$; a is open if $a = \mathbf{I}a$.

An $\mathcal{S}_\lambda^*$-algebra is a complete closure algebra and conversely.

In the sequel the letter C will exclusively denote a closure algebra.

If $\mathfrak{X} \neq 0$ is a topological space [29]) with the closure operation $\mathbf{C}X$ defined for all $X \subset \mathfrak{X}$, then the class of all subsets of $\mathfrak{X}$ is a complete closure algebra denoted by $C(\mathfrak{X})$.

The system $\mathcal{S}_\lambda$ has the property (E), *i. e.* for every closure algebra C there is a complete closure algebra C^* and an $\mathcal{S}_\lambda$-isomorphism which preserves a given enumerable set of infinite sums and products (see § 3). For instance, one can construct h and C^* (by means of MacNeille's minimal extensions [14]) of Boolean algebras) so that h preserves all infinite sums and products in C [30]). However, for our purpose it is convenient to construct the $\mathcal{S}_\lambda^*$-extension C^* in another way. This construction is given by the following lemma, which clearly implies the property (E).

10.1. *Let* $a_n, b_n, a_{nu}, b_{nv} \in C$ (*where* $u \in U_n, v \in V_n$, $n = 1, 2, \ldots$). *Suppose that*

$$(+) \qquad a_n = (C)\sum_{u \in U_n} a_{nu}, \qquad b_n = (C)\prod_{v \in V_n} b_{nv} \qquad (n = 1, 2, \ldots).$$

[26]) This system is equivalent to that in Rasiowa [1].

[27]) McKinsey-Tarski [3], p. 4.

[28]) McKinsey-Tarski [1], p. 145.

[29]) A topological space is a set $\mathfrak{X}$ with a closure operation defined for all $X \subset \mathfrak{X}$ and satisfying the above axioms I-IV of Kuratowski [1], p. 145.

[30]) See Rasiowa [1], p. 111.

Then there is an $\mathcal{S}_\lambda$-isomorphism h of C into the closure algebra $\boldsymbol{C}(\mathfrak{X})$ of all subsets of a topological space $\mathfrak{X}\neq 0$ such that h preserves all the sums and products (+), *i. e.* [16])

$$h(a_n)=\sum_{u\in U_n} h(a_{nu}),\qquad h(b_n)=\prod_{v\in V_n} h(b_{nv}).$$

By 9.1 there is a set $\mathfrak{X}$ and $\mathcal{S}_\lambda$-isomorphism h of $\langle C; e; +, \cdot, \rightarrow, - \rangle$ into the Boolean algebra of all subsets of a set $\mathfrak{X}$ which preserves all the sums and products (+). We define the closure operation in $\mathfrak{X}$ as follows:

for every $X\subset\mathfrak{X}$, $\mathbf{C}X$ is the intersection of all sets $h(a)$ where $X\subset h(a)$, $a\in C$ and $a=\mathbf{C}a$.

The mapping $X=h(a)$ is the required S_λ-isomorphism of C into $\boldsymbol{C}(\mathfrak{X})$.

Since $\mathcal{S}_\lambda$ has the property (E) and since the formula (N) is a theorem in $\mathcal{S}_\lambda$, we may apply all the theorems 6.1, 6.2, 7.2, 7.3 to the case $\mathcal{S}^*=\mathcal{S}^*_\lambda$. Those theorems may be treated as analogous to the Gödel and Skolem-Löwenheim theorems, proved for the classical functional calculus. As we shall see later (10.5 and 10.6), in all definitions of satisfiability and validity (see § 6) we may restrict the domain of arbitrary complete closure algebras to the domain of closure algebras $\boldsymbol{C}(\mathfrak{X})$, formed from all subsets of topological spaces $\mathfrak{X}$. Moreover, we can construct a universal topological space $\mathfrak{X}_\lambda$ such that $\boldsymbol{C}(\mathfrak{X}_\lambda)$ plays the same part in $\mathcal{S}^*_\lambda$ as the two-element Boolean algebra B_0 in the classical functional calculus.

For this purpose, besides the system $\mathcal{S}^*_\lambda$ we shall also examine another system $\bar{\mathcal{S}}^*_\lambda$ constructed by the method described in § 2 where we set $\mathcal{S}=\mathcal{S}_\lambda$ and $I=$ the set $\bar{I}_0$ of all integers (see § 8). The set of all formulas in $\bar{\mathcal{S}}^*_\lambda$ is denoted by $\bar{S}^*_\lambda$.

10.2. (*Deduction Theorem*). *Every set $R\subset S^*_\lambda$ (or: $\subset\bar{S}^*_\lambda$) of closed formulas a, each of which has the form $a=\mathbf{I}\beta$, has the property* (D) *in $\mathcal{S}^*_\lambda$ (in $\bar{\mathcal{S}}^*_\lambda$).*

It is sufficient to prove that the set K of all formulas a with the property

there is a sequence $a_1,\ldots,a_n\in R$ such that $\vdash a_1\cdot\ldots\cdot a_n\rightarrow a$

fulfils the conditions (i)-(v) from § 2 where $K(R)$ is replaced by K.

The proof of this fact in the case of (i)-(iv) is the same as for the classical functional calculus.

The case (v) follows from the statements given below:

(1) If $\vdash\mathbf{I}\gamma\rightarrow a$, then $\vdash\mathbf{I}\gamma\rightarrow\mathbf{I}a$.

(2) If the formula β_1 is a part of a_1, and if a_2 is the formula obtained from a_1 by the replacement of β_1 by a formula β_2, then

$$\vdash\mathbf{I}\Big(\prod_{x_{i_1}}\prod_{x_{i_2}}\ldots\prod_{x_{i_n}}(\beta_1\equiv\beta_2)\Big)\rightarrow(a_1\equiv a_2)$$

where the sequence $x_{i_1},x_{i_2},\ldots,x_{i_n}$ contains all variables which are free in $\beta_1\equiv\beta_2$.

The proof of (2) is by induction on the length of a_1.

10.3. *A set $R\subset S^*_\lambda$ is satisfiable in a set $J\neq 0$ and in a complete closure algebra C if and only if the set of all formulas $\mathbf{I}a$, where $a\in R$, is satisfiable in J and in C.*

This follows immediately from 5.1 and the fact that the equation $a=e$ is equivalent to $\mathbf{I}a=e$.

Let $Q\subset\bar{S}^*_\lambda$ be a consistent (in $\bar{\mathcal{S}}^*_\lambda$) set of closed formulas a, each of which has the form $a=\mathbf{I}\beta$ ($\beta\in\bar{\mathcal{S}}^*_\lambda$). Let $\bar{L}_\lambda(Q)$ denote the Lindenbaum algebra constructed by the method of § 4 where we set $S^*=\bar{S}^*_\lambda$ and $R=Q$. By 10.1 there is a topological space $\mathfrak{X}_{\lambda Q}$ such that $\boldsymbol{C}(\mathfrak{X}_{\lambda Q})$ is an $\bar{\mathcal{S}}^*_\lambda$-extension of $\bar{L}_\lambda(Q)$. Analogously there is a topological space $\mathfrak{X}^0_\lambda$ such that $\boldsymbol{C}(\mathfrak{X}^0_\lambda)$ is an $\mathcal{S}^*_\lambda$-extension of L_λ. Let $\mathfrak{X}_\lambda$ be the Cartesian product of all the spaces $\mathfrak{X}^0_\lambda$ and $\mathfrak{X}_{\lambda Q}$ where Q fulfils the conditions mentioned above.

10.4. *Each closure algebra $\boldsymbol{C}(\mathfrak{X}^0_\lambda)$ and $\boldsymbol{C}(\mathfrak{X}_{\lambda Q})$ is $\mathcal{S}_\lambda$-isomorphic to a complete subalgebra of $\boldsymbol{C}(\mathfrak{X}_\lambda)$. The $\mathcal{S}_\lambda$-isomorphism mentioned above preserves all infinite sums and products.*

Consequently $\boldsymbol{C}(\mathfrak{X}_\lambda)$ is an $\mathcal{S}^*_\lambda$-extension of L_λ and an $\bar{S}^*_\lambda$-extension of $\bar{L}_\lambda(Q)$.

This is a particular case of the following general theorem: Let $\mathfrak{X}$ and $\mathcal{Y}$ be two non-empty topological spaces. The transformation $h(X)=X\times\mathcal{Y}$ is an $\mathcal{S}_\lambda$-isomorphism of $\boldsymbol{C}(\mathfrak{X})$ into $\boldsymbol{C}(\mathfrak{X}\times\mathcal{Y})$ which preserves all infinite sums and products.

Theorem 7.3 can be formulated in the following stronger form:

10.5. *A set $R\subset S^*_\lambda$ is satisfiable if and only if it is satisfiable in I_0 and $\boldsymbol{C}(\mathfrak{X}_\lambda)$.*

It suffices to prove that each satisfiable set $R\subset S^*_\lambda$ is satisfiable in I_0 and $\boldsymbol{C}(\mathfrak{X}_\lambda)$. By 10.3 we may restrict ourselves to the case where each formula $a\in R$ is of the form $a=\mathbf{I}\beta$ ($\beta\in S^*_\lambda$).

Apply the notations of § 8. By 10.2 the set $\bar{R}\subset\bar{S}^*_\lambda$ has the property (D). By 8.2 $\bar{R}$ is satisfiable. By 7.2 $\bar{R}$ is consistent. By 6.2 and 10.4 $\bar{R}$ is satisfiable in $\bar{I}_0$ and in $\boldsymbol{C}(\mathfrak{X}_\lambda)$. By 8.2 and 6.5 (B), R is satisfiable in I_0 and $\boldsymbol{C}(\mathfrak{X}_\lambda)$.

The following theorem is another formulation of the completeness theorem 6.1 for the system $\mathcal{S}^*_\lambda$.

10.6. *The following conditions are equivalent for every $a\in S^*_\lambda$:*

(i) *a is provable in S^*_λ;*

(ii) *a is valid;*

(iii) $\big(J,\boldsymbol{C}(\mathfrak{X})\big)\,\Phi_a=\mathfrak{X}$ [31]) *for every set $J\neq 0$ and for every topological space $\mathfrak{X}\neq 0$;*

(iv) $\big(I_0,\boldsymbol{C}(\mathfrak{X}_\lambda)\big)\,\Phi_a=\mathfrak{X}_\lambda$.

The implication (i)→(ii) follows from 5.3. The implication (ii)→(iii) and (iii)→(iv) are trivial. The implication (iv)→(i) follows from 6.1 (iv) and 10.4.

[31]) Clearly $\mathfrak{X}$ is the unit element of $\mathbf{C}(\mathfrak{X})$.

The equivalence (i)$\equiv$(iv) in 10.6 may be otherwise formulated as follows:

10.7. *The closure algebra* $C(\mathfrak{X}_\lambda)$ *is functionally* σ*-free.*

Clearly the space $\mathfrak{X}_\lambda$ may be replaced in 10.6 (iv) and in 10.7 by the topological space $\mathfrak{X}_\lambda^0$ which has an enumerable open basis.

On the other hand,

10.8. *If* $\mathfrak{X}$ *is a complete metric space, or a locally compact regular space, then* $C(\mathfrak{X})$ *is not functionally* σ*-free.*

Consider the formula

$$\alpha_0 = \mathbf{C} - \sum_{x_1}\Big(F_1^1(x_1)\cdot\big(\mathbf{C}-\mathbf{C}F_1^1(x_1)\big)\Big)$$

$F^1\big(I_0, C(\mathfrak{X})\big)$ is the class of all sequences $\{X_i\}$ of subsets of $\mathfrak{X}$. The value of the functional $\big(I_0, C(\mathfrak{X})\big)\Phi_{\alpha_0}$ depends only on the substitution $F_1^1 = \{X_i\}$ and is equal to

$$\mathbf{C}\big(-\sum_{i=1}^{\infty} X_i(\mathbf{C}-\mathbf{C}X_i)\big)$$

i. e. to

$$\overline{\mathfrak{X}-\sum_{j=1}^{\infty} X_i\cdot\overline{\mathfrak{X}-\overline{X}_i}}$$

using the usual topological notation. By Baire's theorem, we find that, in every complete metric space [32]) and in every locally compact regular space [20]),

(a) $$\overline{\mathfrak{X}-\sum_{i=1}^{\infty} X_i\cdot\overline{\mathfrak{X}-\overline{X}_i}} = \mathfrak{X},$$

i. e. that $\big(I_0, C(\mathfrak{X})\big)\Phi_{\alpha_0} = \mathfrak{X} =$ the unit of $C(\mathfrak{X})$. On the other hand the formula α_0 is not provable since the equation (a) does not hold in every topological space.

The question whether there is a metric space which is functionally free or functionally σ-free, is unsolved.

§ 11. The Heyting calculus. Consider now the case where $\mathcal{S}$ is the Heyting sentential calculus [33]) denoted here by $\mathcal{S}_\chi$. Besides the sign $+, \cdot, \rightarrow$ the system $\mathcal{S}_\chi$ contains also the negation sign [34]) $\neg$. The axioms of $\mathcal{S}_\chi$ are the formulas T_1-T_8 and the formulas T_9 and T_{10} where the sign $-$ is replaced by $\neg$. It is known that the formulas (N) (where $-$ is relaced by $\neg$) are theorems in $\mathcal{S}_\chi$.

The functional calculus obtained by the method of § 2 where $\mathcal{S} = \mathcal{S}_\chi$ and $I = I_0$ will be denoted by $\mathcal{S}_\chi^*$. Clearly $\mathcal{S}_\chi^*$ is the functional calculus of Heyting.

[32]) See Kuratowski [2], p. 320 and p. 323.

[33]) See Heyting [1], p. 53 and Łukasiewicz [1], p. 86.

[34]) Since we shall simultaneously examine the systems S_χ^* and S_λ^* and the S_χ- and S_λ-algebras, it is convenient to use different negation signs for these systems.

By S_χ and S_χ^* we shall denoted the sets of all formulas in $\mathcal{S}_\chi$ and $\mathcal{S}_\chi^*$ respectively. Analogously L_χ will denote the Lindenbaum algebra described in § 4 where $\mathcal{S}^* = \mathcal{S}_\lambda^*$ and $R =$ the empty set.

The $\mathcal{S}_\chi$-algebras will be called *Heyting algebras*. It is known that $\langle H; e; +, \cdot, \rightarrow, \neg\rangle$ is a Heyting algebra if and only if

(a) $\langle H; +, \cdot, \rightarrow\rangle$ is a pseudocomplemented lattice [11]) with the unit element e and the zero element $0 = \neg e$, and

(b) $\neg a = a \rightarrow 0$ for every $a \in H$ [35]).

An $\mathcal{S}_\chi^*$-algebra is a complete Heyting algebra and conversely.

The letter H will exclusively denote a Heyting algebra.

If C is a closure algebra, then the class $\boldsymbol{H}(C)$ of all open elements of C is a Heyting algebra with the following operations [36]): join $a+b$ and meet $a\cdot b$ are in $\boldsymbol{H}(C)$ the same as in C; the operation $a\rightarrow b$ in $\boldsymbol{H}(C)$ is defined by the equation $a\rightarrow b = \mathbf{I}\big((-a)+b\big)$; the "complementation" $\neg a$ is defined by (b). Note that the operation $\rightarrow$ in $\boldsymbol{H}(C)$ is other than in C. In the sequel we shall often consider lattices $\boldsymbol{H}(C)$ and their sublattices; $\rightarrow$ will always denote the operation in $\boldsymbol{H}(C)$ (not in C), and $-a$ will always denote the complement of a in C.

If $\mathfrak{X}$ is a topological space, we write $\boldsymbol{H}(\mathfrak{X})$ instead of $\boldsymbol{H}\big(C(\mathfrak{X})\big)$ $=$ the Heyting algebra of all open subsets of $\mathfrak{X}$.

The system $\mathcal{S}_\chi$ has the property (E), *i. e.* for every Heyting algebra H there is a complete Heyting algebra H^* and an $\mathcal{S}_\chi$-isomorphism h of H into H^* which preserves a given enumerable set of infinite sums and products. One can even construct (by means of MacNeille's minimal extensions [14]) of Boolean algebras) H^* and h so that h preserves all infinite sums and products in H [37]). However, for our purpose it is convenient to construct another complete extension of H. This construction will be the subject of Theorem 11.2, which clearly implies the property (E). Theorem 11.2 must be preceded by the following lemma.

11.1. *Let* C *be a closure algebra,* $a, b \in C$, $a_u, b_v \in \boldsymbol{H}(C)$. *The equation* $a = (C)\sum_{u\in U} a_u$ *holds if and only if* $a = \big(\boldsymbol{H}(C)\big)\sum_{v\in U} a_u$.

If C *is complete, then* $\boldsymbol{H}(C)$ *is also complete. Moreover, in this case, the equation* $b = \mathbf{I}\big((C)\prod_{v\in V} b_v\big)$ *is equivalent to* $b = \big(\boldsymbol{H}(C)\big)\prod_{v\in V} b_v$.

The simple proof [38]) is omitted.

[35]) See McKinsey-Tarski [3], p. 12. Clearly each pseudocomplemented lattice may be treated as a Heyting algebra. See § 12, p. 88. See also Birkhoff [1], p. 147.

[36]) See McKinsey-Tarski [2], p. 130.

[37]) Rasiowa [1], p. 112.

[38]) The proof is similar to Rasiowa [1], lemmas 3.15 and 3.16. The Brouwerian algebras examined by Rasiowa [1] are dual to Heyting algebras. Therefore Rasiowa [1] examines the class of all closed elements of a closure algebra C instead of $\boldsymbol{H}(C)$.

11.2. *Let H be a Heyting algebra, $a_n, a_{nu} \in H$ ($u \in U_n$, $n=1,2,\ldots$). Suppose that*

$$(+) \qquad a_n = (H) \sum_{u \in U_n} a_{nu} \qquad (n=1,2,\ldots).$$

Then there are a topological space $\mathfrak{X} \neq 0$ and an $\mathcal{S}_\chi$-isomorphism h of H into $\boldsymbol{H}(\mathfrak{X})$ which preserves all the sums (+) and all infinite products, i. e.

$$h(a_n) = \sum_{u \in U_n} h(a_{nu}) \qquad \text{for } n=1,2,\ldots$$

and

$$h(b) = \mathbf{I}\Big(\prod_{v \in V} h(b_v)\Big) \quad \text{whenever} \quad b = (H) \prod_{v \in V} b_v.$$

By a theorem of McKinsey and Tarski [39]) we may suppose that $H = \boldsymbol{H}(C)$ where C is a closure algebra. By 11.1 we have

$$(++) \qquad a_n = (C) \sum_{u \in U_n} a_{nu} \qquad \text{for } n=1,2,\ldots$$

By 9.1 there is an $\mathcal{S}_\chi$-isomorphism h of the Boolean algebra C into the Boolean algebra of all subsets of a set $\mathfrak{X} \neq 0$ such that h preserves all the sums (++). We define the topology in $\mathfrak{X}$ as follows: a set $X \subset \mathfrak{X}$ is said to be open if it is the set-theoretical union of all sets $h(a)$ where $a \in \boldsymbol{H}(C)$ and $h(a) \subset X$. We obtain from 11.1

$$h(a_n) = \sum_{u \in U_n} h(a_{nu}) = \big(\boldsymbol{H}(\mathfrak{X})\big) \sum_{u \in U_n} h(a_{nu}) \qquad \text{for } n=1,2,\ldots$$

Suppose $b = \big(\boldsymbol{H}(C)\big) \prod_{v \in V} b_v$, *i. e.* b is the greatest open element in $\boldsymbol{H}(C)$ such that $b \subset b_v$ for every $v \in V$. If $h(c) \subset h(b_v)$ for every $v \in V$, then $c \subset b$, and consequently $h(c) \subset h(b)$. Hence $h(b)$ is the greatest open subset of $\mathfrak{X}$ which is simultaneously contained in all $h(b_v)$. This means that $h(b) = \mathbf{I}\big(\prod_{v \in V} h(b_v)\big)$, *i. e.* (see 11.1) that $h(b) = \big(\boldsymbol{H}(\mathfrak{X})\big) \prod_{v \in V} h(b_v)$. The $\mathcal{S}_\chi$-isomorphism h, restricted to $\boldsymbol{H}(C)$, is the required $\mathcal{S}_\chi$-isomorphism.

Since $\mathcal{S}_\chi$ has the property (E) and since (N) is a theorem of $\mathcal{S}_\chi$, all the theorems 6.1, 6.2, 7.2, 7.3 hold in the case of $\mathcal{S}^* = \mathcal{S}^*_\chi$. Those theorems may be treated as analogous to the Gödel and Skolem-Löwenheim theorems, proved for the classical functional calculus. However, as we shall see later (11.5 and 11.6), in all the definitions of satisfiability and validity (§ 6), we may restrict the domain of all arbitrary complete Heyting algebras to the domain of all Heyting algebras $\boldsymbol{H}(\mathfrak{X})$ where $\mathfrak{X} \neq 0$ is a topological space. Moreover, we can construct a universal topological space $\mathfrak{X}_\chi$ such that $\boldsymbol{H}(\mathfrak{X}_\chi)$ plays the same part with regard to the system $\mathcal{S}^*_\chi$ as the two-element Boolean algebra B_0 with regard to the classical functional calculus.

[39]) McKinsey and Tarski [2], p. 130. The Brouwerian algebras examined in that paper are dual to Heyting algebras.

For this purpose, besides the system $\mathcal{S}^*_\chi$ we shall also examine the system $\bar{\mathcal{S}}^*_\chi$, constructed by the method of § 2 where we set $\mathcal{S} = \mathcal{S}_\chi$ and $I =$ the set $\bar{I}_0$ of all integers (see § 8). $\bar{S}^*_\chi$ is the set of all formulas in $\bar{\mathcal{S}}^*_\chi$.

11.3. (*Deduction Theorem*). *Every set $R \subset S^*_\chi$ (or: $\subset \bar{S}^*_\chi$) of closed formulas has the property* (D) *in $\mathcal{S}^*_\chi$ (in $\bar{\mathcal{S}}^*_\chi$).*

The proof is the same as for the classical functional calculus.

Let $Q \subset \bar{S}^*_\chi$ be a consistent (in $\bar{\mathcal{S}}^*_\chi$) set of closed formulas. Let $\bar{L}_\chi(Q)$ be the Lindenbaum algebra constructed by the method of § 4 where we set $\mathcal{S}^* = \bar{\mathcal{S}}^*_\chi$ and $R = Q$. By 11.2 there is a topological space $\mathfrak{X}_{\chi Q}$ such that $\boldsymbol{H}(\mathfrak{X}_{\chi Q})$ is an $\bar{\mathcal{S}}^*_\chi$-extension of $\bar{L}_\chi(Q)$. Analogously there is a topological space $\mathfrak{X}^0_\chi$ such that $\boldsymbol{H}(\mathfrak{X}^0_\chi)$ is an $\mathcal{S}^*_\chi$-extension of L_χ. Let $\mathfrak{X}_\chi$ be the Cartesian product of all spaces $\mathfrak{X}^0_\chi$ and $\mathfrak{X}_{\chi Q}$ where Q fulfils the conditions mentioned above.

11.4. *Each Heyting algebra $\boldsymbol{H}(\mathfrak{X}^0_\chi)$ and $\boldsymbol{H}(\mathfrak{X}_{\chi Q})$ is $\mathcal{S}_\chi$-isomorphic to a complete subalgebra of $\boldsymbol{H}(\mathfrak{X}_\chi)$. The isomorphism mentioned above preserves all infinite sums and products.*

Consequently, $\boldsymbol{H}(\mathfrak{X}_\chi)$ is an $\mathcal{S}^*_\chi$-extension of L_χ and an $\bar{\mathcal{S}}^*_\chi$-extension of $\bar{L}_\chi(Q)$.

This results from the following general theorem: Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two non-empty topological spaces. The transformation $h(X) = X \times \mathfrak{Y}$ is an $\mathcal{S}_\chi$-isomorphism of $\boldsymbol{H}(\mathfrak{X})$ into $\boldsymbol{H}(\mathfrak{X} \times \mathfrak{Y})$ which preserves all infinite sums and products.

Theorem 7.3 can be formulated in the following stronger form:

11.5. *A set $R \subset S^*_\chi$ is satisfiable if and only if it is satisfiable in I_0 and $\boldsymbol{C}(\mathfrak{X}_\chi)$.*

The proof is the same as that of 10.5.

Suppose $R \subset S^*_\chi$ is satisfiable. Apply the notations of § 8. By 11.3, the set $\bar{R} \subset \bar{S}^*_\chi$ has the property (D). By 8.2, $\bar{R}$ is satisfiable. By 7.2 $\bar{R}$ is consistent. By 6.2 and 11.4 $\bar{R}$ is satisfiable in $\bar{I}_0$ and in $\boldsymbol{H}(\mathfrak{X}_\chi)$. By 8.2 and 6.5 (B) R is satisfiable in I_0 and in $\boldsymbol{H}(\mathfrak{X}_\chi)$.

The following theorem is another formulation of the completeness theorem 6.1 for the system $\mathcal{S}^*_\chi$.

11.6. *The following conditions are equivalent for every formula $a \in S^*_\chi$:*

(i) *a is provable in $\mathcal{S}^*_\chi$;*

(ii) *a is valid;*

(iii) $\big(J, \boldsymbol{H}(\mathfrak{X})\big) \Phi_a = \mathfrak{X}$ [40]) *for every set $J \neq 0$ and for every topological space $\mathfrak{X} \neq 0$;*

(iv) $\big(I_0, \boldsymbol{H}(\mathfrak{X}_\chi)\big) \Phi_a = \mathfrak{X}_\chi$.

[40]) Clearly $\mathfrak{X}$ is the unit element of $\boldsymbol{H}(\mathfrak{X})$.

The proof is the same as that of 10.6. The equivalence (i)$\equiv$(iv) in 11.6 can be otherwise formulated as follows:

11.7. *The Heyting algebra $\boldsymbol{H}(\mathcal{X}_\chi)$ is functionally σ-free.*

The space $\mathcal{X}_\chi$ may be replaced in 11.6 (iv) and in 11.7 by the space $\mathcal{X}_\chi^0$. The space $\mathcal{X}_\chi^0$ has an enumerable basis.

The problem whether there exists a metric space $\mathcal{X}$ such that $\boldsymbol{H}(\mathcal{X})$ is functionally free or functionally σ-free, is unsolved.

§ 12. The positive calculus. Consider now the case where $\mathcal{S}$ is the positive sentential calculus denoted here by $\mathcal{S}_\pi$. The system $\mathcal{S}_\pi$ contains only the three logical signs $+, \cdot, \rightarrow$. The axioms of $\mathcal{S}_\pi$ are the formulas T_1-T_8.

The functional calculus obtained by the method of § 2, where $\mathcal{S}=\mathcal{S}_\pi$ and $I=I_0$, will be denoted by $\mathcal{S}_\pi^*$ and referred to as the positive functional calculus.

By S_π and S_π^* we shall denote the sets of all formulas in $\mathcal{S}_\pi$ and $\mathcal{S}_\pi^*$ respectively. Analogously L_π will denote the Lindenbaum algebra described in § 4 where $\mathcal{S}^*=\mathcal{S}_\pi^*$ and $R=$ the empty set.

It is easy to verify that $\mathcal{S}_\pi$-algebras are relatively pseudocomplemented algebras and conversely.

If $\langle H; e; +, \cdot, \rightarrow, \neg\rangle$ is a Heyting algebra, then $\langle H; e; +, \cdot, \rightarrow\rangle$ is a pseudocomplemented lattice having the zero element. Conversely, if $\langle P; e; +, \cdot, \rightarrow\rangle$ is a pseudocomplemented lattice with the zero element 0, then, if we set $\neg a = a\rightarrow 0$, the algebra $\langle P; e; +, \cdot, \rightarrow, \neg\rangle$ is a Heyting algebra.

Consequently $\mathcal{S}_\pi^*$-algebras are complete Heyting algebras and conversely.

If C is a closure algebra, let $\boldsymbol{H}'(C)$ denote the class of all dense open elements of C (an element $a \in C$ is dense if $\mathbf{C}a =$ the unit of C). If $C=\boldsymbol{C}(\mathcal{X})$, where $\mathcal{X}$ is a topological space, we write $\boldsymbol{H}'(\mathcal{X})$ instead of $\boldsymbol{H}'(\boldsymbol{C}(\mathcal{X}))$.

If C is a closure algebra, then $\boldsymbol{H}(C)$ and $\boldsymbol{H}'(C)$ are examples of relatively pseudocomplemented lattices. The operations $+$ and $\cdot$ are the same as in C. The operation $a\rightarrow b$ is defined as $\mathbf{I}(-a+b)$. Conversely

12.1. *For every relatively pseudocomplemented lattice P there is a closure algebra C such that $P=\boldsymbol{H}'(C)$. Moreover, $\boldsymbol{H}'(C)$ is the class of all non-zero open elements in C.*

Let P_0 denote the relatively pseudocomplemented lattice defined as follows. Elements of P_0 are elements of P and a new added element 0. The operations in P are extended over P_0 in the following way:

$a+0=a$ and $a\cdot 0=0$ for every $a \in P_0$,
$0\rightarrow a=e$ for every $a \in P_0$,
$a\rightarrow 0=0$ for every $a \in P$.

It is easy to verify that P_0 is a Heyting algebra with the unit e and the zero element 0. By a theorem of McKinsey and Tarski [39]), there is a closure algebra C such that $P_0=\boldsymbol{H}(C)$. Consequently P is the class of all non-zero open elements of C. If $a, b \in P$, then $ab\neq 0$ since $ab \in P$. Therefore, if $a \in P$, there is no open $b\subset -a$, $b\neq 0$. Consequently each $a \in P$ is dense in C.

The system $\mathcal{S}_\pi$ has the property (E). This results from the following theorem:

12.2. *Let P be a relatively pseudocomplemented lattice, $a_n, a_{nu} \in P$ ($u \in U_n$, $n=1,2,\ldots$). Suppose that*

$$(+)\qquad a_n=(P)\sum_{u\in U_n} a_{nu} \qquad (n=1,2,\ldots).$$

Then there are a topological space $\mathcal{X}\neq 0$ and an $\mathcal{S}_\pi$-isomorphism h of P into $\boldsymbol{H}(\mathcal{X})$ which preserves all the sums (+) and all infinite products.

Let P_0 have the meaning as in the proof of 12.1. Clearly (+) implies

$$(*)\qquad a_n=(P_0)\sum_{u\in U_n} a_{nu} \qquad (n=1,2,\ldots).$$

Analogously, if $b=(P)\prod_{v\in V} b_v$ then also

$$(**)\qquad b=(P_0)\prod_{v\in V} b_v.$$

By 11.2 there is a topological space $\mathcal{X}\neq 0$ and an $\mathcal{S}_\chi$-isomorphism h of P_0 into $\boldsymbol{H}(\mathcal{X})$ which preserves all the infinite sums (*) and all the infinite products (**). The isomorphism h reduced to the set P is the required $\mathcal{S}_\pi$-isomorphism.

Since the system $\mathcal{S}_\pi$ has the property (E), Theorem 6.1 may be applied to the positive functional calculus $\mathcal{S}_\pi^*$. Analogously to §§ 9-11, we can formulate this theorem in a more precise form (12.3). The domain of all $\mathcal{S}_\pi^*$-algebras can be reduced to S_π^*-algebras $\boldsymbol{H}(\mathcal{X})$ where $\mathcal{X}$ is a topological space.

By 12.2 there is a topological space $\mathcal{X}_\pi$ such that $\boldsymbol{H}(\mathcal{X}_\pi)$ is an $\mathcal{S}_\pi^*$-extension (see § 4) of the Lindenbaum algebra L_π.

12.3. *The following conditions are equivalent for every formula $a \in S_\pi^*$:*

(i) *a is provable in $\mathcal{S}_\pi^*$;*

(ii) *a is valid;*

(iii) *$(J, \boldsymbol{H}(\mathcal{X}))\Phi_a=\mathcal{X}$ [40]) for every set $J\neq 0$ and for every topological space $\mathcal{X}$;*

(iv) $(I_0, \boldsymbol{H}(\mathcal{X}_\pi))\Phi_a=\mathcal{X}_\pi$.

The proof is the same as that of 10.6. The equivalence (i)$\equiv$(iv) may be expressed as follows

12.4. *The pseudocomplemented algebra $\boldsymbol{H}(\mathcal{X}_\pi)$ is functionally σ-free.*

Theorem 6.2 is not interesting in the case of the system $\mathcal{S}_\pi^*$ since each set of formulas in $\mathcal{S}_\pi^*$ is satisfied in each set $J\neq 0$. In fact, if H is a Heyting algebra and $\varphi_m^k(i_1,\ldots,i_m)=e\in H$, then $(J,H)\,\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big)=e$.

Theorems 7.2 and 7.3 cannot be applied to $\mathcal{S}_\pi^*$ since $\mathcal{S}_\pi^*$ contains no negation sign.

§ 13. The minimal functional calculus. Consider at least the case where $\mathcal{S}$ is the minimal[41]) sentential calculus denoted here by $\mathcal{S}_\mu$. The system $\mathcal{S}_\mu$ contains the binary operators $+,\cdot,\rightarrow$, and a unary operator $\sim$ (the negation). The axioms of $\mathcal{S}_\mu$ are the formulas T_1-T_9 where the sign $-$ should be replaced by $\sim$.

The functional calculus obtained by the method of § 2 where $\mathcal{S}=\mathcal{S}_\mu$ and $I=I_0$, will be denoted by $\mathcal{S}_\mu^*$ and referred to as the minimal functional calculus.

By S_μ and S_μ^* we shall denoted the sets of all formulas in $\mathcal{S}_\mu$ and $\mathcal{S}_\mu^*$ respectively. Analogously L_μ will denote the Lindenbaum algebra described in § 4 where $\mathcal{S}^*=\mathcal{S}_\mu^*$ and R = the empty set.

$\mathcal{S}_\mu$-algebras are algebras $\langle M; e; +,\cdot,\rightarrow,\sim\rangle$ such that

(a) $\langle M; e; +,\cdot,\rightarrow\rangle$ is a relatively pseudocomplemented lattice with the unit e;

(b) $\sim a = a\rightarrow\sim e$.

In fact, an $\mathcal{S}_\mu$-algebra M is a pseudocomplemented lattice in which, identically, (see T_9)

$$(a\rightarrow\sim b)\rightarrow(b\rightarrow\sim a)=e,$$

i. e.

$$a\rightarrow\sim b\subset b\rightarrow\sim a.$$

Replacing a by b and b by a we obtain

$$a\rightarrow\sim b=b\rightarrow\sim a.$$

Consequently (see footnote [11])),

$$\sim a=e\rightarrow\sim a=a\rightarrow\sim e.$$

Conversely, if $\sim a=a\rightarrow\sim e$, then obviously (see T_3')

$$a\rightarrow\sim b=a\rightarrow(b\rightarrow\sim e)=b\rightarrow(a\rightarrow\sim e)=b\rightarrow\sim a,$$

hence identically

$$(a\rightarrow\sim b)\rightarrow(b\rightarrow\sim a)=e.$$

We see that the operation $\sim a$ is completely determined by the operations $+,\cdot,\rightarrow$ and by the element $\sim e$. Clearly the element $a_0=\sim e$ may be completely arbitrary but fixed. In particular, it is possible that $\sim e=e$. Notice that, if $\sim e$ is the zero element of M, then (if we replace $\sim$ by $\neg$) M is a Heyting algebra.

[41]) See Johansson [1] and Łukasiewicz [1], p. 86.

Let C be a closure algebra and let $P=\boldsymbol{H}(C)$ (or: $P=\boldsymbol{H}'(C)$) be the pseudocomplemented lattice of all open (or: open and dense) elements in C. Let a_0 be an arbitrary but fixed element in P. Setting $\sim a=a\rightarrow a_0$ we find that $\langle P; e; +,\cdot,\rightarrow,\sim\rangle$ is an $\mathcal{S}_\mu$-algebra denoted by $\boldsymbol{H}(C,a_0)$ (or: by $\boldsymbol{H}'(C,a_0)$). Clearly $a_0=\sim e$. If $C=\boldsymbol{C}(\mathfrak{X})$ where $\mathfrak{X}$ is a topological space, we write $\boldsymbol{H}(\mathfrak{X},a_0)$ instead of $\boldsymbol{H}(\boldsymbol{C}(\mathfrak{X}),a_0)$.

13.1. *For every $\mathcal{S}_\mu$-algebra M there is a closure algebra C and an open element $a_0\in C$ such that $M=\boldsymbol{H}'(C,a_0)$.*

This follows immediately from 12.1.

The system $\mathcal{S}_\mu$ has the property (E). In fact,

13.2. *Let M be an $\mathcal{S}_\mu$-algebra, $a_n,a_{nu}\in M$ $(u\in U_n,\ n=1,2,\ldots)$. Suppose that*

$$(+)\qquad a_n=(M)\sum_{u\in U_n}a_{nu}\qquad (n=1,2,\ldots).$$

Then there are a topological space $\mathfrak{X}$, an open set $G\subset\mathfrak{X}$ and an $\mathcal{S}_\mu$-isomorphism of M into $\boldsymbol{H}(\mathfrak{X},G)$ which preserves all the sums (+) and all infinite products.

This follows immediately from 12.2.

We infer that Theorem 6.1 may be applied to the minimal functional calculus $\mathcal{S}_\mu^*$. Analogously to §§ 9-11 we can formulate this theorem in a more precise form (13.3). The domain of all $\mathcal{S}_\mu^*$-algebras can be restricted to $\mathcal{S}_\mu^*$-algebras $\boldsymbol{H}(\mathfrak{X},G)$.

By 13.2 there is a topological space $\mathfrak{X}_\mu$ and an open set $G_\mu\subset\mathfrak{X}_\mu$ such that $\boldsymbol{H}(\mathfrak{X}_\mu,G_\mu)$ is an $\mathcal{S}_\mu^*$-extension (see § 4) of the Lindenbaum algebra L_μ.

13.3. *The following conditions are equivalent for every formula $\alpha\in S_\mu^*$:*

(i) *α is provable in $\mathcal{S}_\mu^*$;*

(ii) *α is valid;*

(iii) *$(J,\boldsymbol{H}(\mathfrak{X},G))\,\Phi_\alpha=\mathfrak{X}$ for every set $J\neq 0$, for every topological space $\mathfrak{X}\neq 0$ and for every open set $G\subset\mathfrak{X}$;*

(iv) *$(I_0,\boldsymbol{H}(\mathfrak{X}_\mu,G_\mu))\,\Phi_\alpha=\mathfrak{X}_\mu$.*

The proof is the same as that of 10.6. The equivalence (i)≡(iv) may be expressed as follows.

13.4. *The $\mathcal{S}_\mu^*$-algebra $\boldsymbol{H}(\mathfrak{X}_\mu,G_\mu)$ is functionally σ-free.*

The actual formulation of 6.2 is not interesting in the case of the system $\mathcal{S}_\mu^*$ since each set $R\subset S_\mu^*$ is satisfiable in each set $J\neq 0$. In fact, if M is an $\mathcal{S}_\mu^*$-algebra with $\sim e=e$, and if $\varphi_m^k(i_1,\ldots,i_k)=e\in M$, then $(J,M)\,\Phi_\alpha\big(\{j_i\},\{\varphi_m^k\}\big)=e$ for every α. The correct formulation of 6.2 should be as follows.

13.5. *Each consistent set $R\subset S_\mu^*$ is satisfiable in the set I_0 and in an $\mathcal{S}_\mu^*$-algebra M such that $\sim e\neq e$.*

The formulas (N) (with $\sim$ instead of $-$) are not theorems in $\mathcal{S}_\mu$. However, the following theorem, analogous to 7.2, is true.

13.6. *If a set $R \subset S_\mu^*$ of closed formulas is satisfiable in a set $J \neq 0$ and in an $\mathcal{S}_\mu^*$-algebra $\mathcal{M}$ with $\sim e \neq e$, then R is consistent.*

The proof is the same as that of 7.2. In fact, the proof of 7.2 is based only on the inequality $-e \neq e$. Each set $R \subset S_\mu^*$ of closed formulas has the property (D).

The necessary correction in the formulation of 7.3 is evident.

§ 14. The system $\mathcal{S}_\nu^*$. The content of the first part of this paper suggests the consideration of the following new system $\mathcal{S}_\nu$. The logical signs of $\mathcal{S}_\nu$ are $+, \cdot, \rightarrow$ and the negation sign $\dot{-}$. The axioms of $\mathcal{S}_\nu$ are the formulas T_1-T_8 and (N) where the sign $-$ is replaced by $\dot{-}$.

$\mathcal{S}_\nu^*$ is the corresponding functional calculus determined by $\mathcal{S}_\nu$ by the method described in § 2, where $\mathcal{S} = \mathcal{S}_\nu$ and $I = I_0$.

It is easy to see that $\langle \mathcal{N}; e; +, \cdot, \rightarrow, \dot{-} \rangle$ is an $\mathcal{S}_\nu$-algebra if and only if

(a) $\langle \mathcal{N}; e; +, \cdot, \rightarrow \rangle$ is a pseudocomplemented lattice;

(b) $\dot{-} e$ is the zero element of $\mathcal{N}$.

The condition (b) is the only restriction for the operation $\dot{-}$. For elements $a \neq e$ the operation $\dot{-} a$ may be defined in a completely arbitrary way.

Clearly, theorems analogous to 11.5 and 11.6 can be proved for the system $\mathcal{S}_\nu^*$. The formulation and the proof of those theorems is left to the reader.

Notice that the system $\mathcal{S}_\nu^*$ contains a subsystem which may be identified with the system $\mathcal{S}_\chi^*$. In fact, we can define the negation $\neg$ in $\mathcal{S}_\nu^*$ as follows

$$\neg a \equiv a \rightarrow \dot{-} (a \rightarrow a)$$

(and analogously, in $\mathcal{S}_\nu$-algebras, $\neg a = a \rightarrow \dot{-} e$). This subsystem is composed of all formulas $a \in S_\nu^*$ which can be written with the help of quantifiers, the signs $+, \cdot, \rightarrow, \neg$, but without $\dot{-}$. Clearly each formula from this subsystem may be treated as a formula in $\mathcal{S}_\chi^*$, and it is provable in $\mathcal{S}_\nu^*$ if and only if it is provable in $\mathcal{S}_\chi^*$. This follows from the Gödel theorems for the systems $\mathcal{S}_\nu^*$ and $\mathcal{S}_\chi^*$.

On the other hand, $\mathcal{S}_\nu^*$ may be interpreted as a subsystem of $\mathcal{S}_\chi^*$. It is sufficient for this purpose to replace the sign $\dot{-}$ by the sign $\neg$ in each formula $a \in S_\nu^*$.

A. Mostowski has remarked that formulas T_1-T_9 and (N) form a new set of axioms for the Heyting propositional calculus S_χ (the sign $-$ should be replaced by $\neg$).

§ 15. Applications. Let ψ be the transformation of S_χ^* into S_λ^* defined by induction as follows [42] (a, β belong to S_χ^*):

(i) $\psi(F_m^k(x_{i_1}, \ldots, x_{i_k})) = \mathbf{I} F_m^k(x_{i_1}, \ldots, x_{i_k})$

(ii) $\psi(a + \beta) = \psi(a) + \psi(\beta)$

(iii) $\psi(a \cdot \beta) = \psi(a) \cdot \psi(\beta)$

(iv) $\psi(a \rightarrow \beta) = \mathbf{I}((-\psi(a)) + \psi(\beta))$

(v) $\psi(\neg a) = \mathbf{I} - \psi(a)$

(vi) $\psi(\sum_{x_k} a) = \sum_{x_k} \psi(a)$

(vii) $\psi(\prod_{x_k} a) = \mathbf{I} \prod_{x_k} \psi(a)$.

15.1. *Let C be a complete closure algebra and $J \neq 0$ any set. For every $a \in S_\chi^*$ and for every substitution*

$$x_i = j_i \in J \qquad F_m^k = \varphi_m^k \in F^k(J, C),$$

we have [43]

$$(J, C)\, \Phi_{\psi(a)}(\{j_i\}, \{\varphi_m^k\}) = (J, \boldsymbol{H}(C))\, \Phi_a(\{j_i\}, \{\mathbf{I}\varphi_m^k\}).$$

Consequently, $(J, C)\, \Phi_{\psi(a)} = e$ if and only if $(J, \boldsymbol{H}(C))\, \Phi_a = e$.

The easy proof of the first part (by induction on the length of a) is omitted. The second part follows immediately from the first and from the fact that each mapping $\in F^k(J, \boldsymbol{H}(C))$ is of the form $\mathbf{I}\varphi$ where $\varphi \in F^k(J, C)$.

15.2. *A formula $a \in S_\chi^*$ is provable in $\mathcal{S}_\chi^*$ if and only if $\psi(a)$ is provable in $\mathcal{S}_\lambda^*$.*

This follows immediately from 15.1 (where $C = C(\mathcal{X})$, $\mathcal{X}$ — an arbitrary non-empty topological space) and from the equivalences (i) $\equiv$ (iii) in Theorems 10.6 and 11.6.

15.3. *There is an $\mathcal{S}_\chi^*$-isomorphism h of L_χ into $\boldsymbol{H}(L_\lambda)$.*

The isomorphism h is defined by the equation

$$h(\|a\|) = \|\psi(a)\| \quad \text{for} \quad a \in S_\chi^*.$$

Consequently,

15.4. *If C in an $\mathcal{S}_\lambda^*$-extension of L_λ, then $\boldsymbol{H}(C)$ is an $\mathcal{S}_\chi^*$-extension of L_χ.*

It follows directly from the definition that $S_\pi^* \subset S_\chi^*$. More exactly, S_π^* is the set of all $a \in S_\chi^*$ which do not contain the negation sign $\neg$.

15.5. *A formula $a \in S_\pi^*$ is provable in $\mathcal{S}_\pi^*$ if and only if it is provable in $\mathcal{S}_\chi^*$.*

This follows immediately from the equivalences (i) $\equiv$ (iii) in Theorems 11.6 and 12.3.

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[42]) The analogous transformation for the sentential calculi of Heyting and Lewis has been examined by McKinsey-Tarski [3], p. 13.

[43]) Clearly $\mathbf{I}\varphi_m^k$ denotes the mapping $\mathbf{I}(\varphi_m^k(j_1, \ldots, j_k))$.

Theorem 15.5 implies that

15.6. *L_π is S^*_π-isomorphic to the $\mathcal{S}_\pi$-subalgebra of L_χ formed of all $|\alpha|$ where α does not contain the negation sign.*

Hence

15.7. *Each $\mathcal{S}^*_\chi$-extension of L_χ is also an $\mathcal{S}^*_\pi$-extension of L_π.*

Therefore the space $\mathfrak{X}_\pi$ may be replaced in Theorems 12.3 and 12.4 by $\mathfrak{X}_\chi$ or $\mathfrak{X}^0_\chi$.

If we replace the sign $\sim$ (or: $\dot{-}$) by $\neg$, we may treat each formula $\alpha \in S^*_\mu$ (or: $\alpha \in S^*_\nu$) as a formula $\in S^*_\chi$. If α is provable in $\mathcal{S}^*_\mu$ (in $\mathcal{S}^*_\nu$) it is also provable in $\mathcal{S}^*_\chi$. If $\alpha \in S^*_\pi$ is provable in $\mathcal{S}^*_\pi$, it is also provable in $\mathcal{S}^*_\mu$ (in $\mathcal{S}^*_\nu$). Therefore the subscript χ in Theorems 15.5-7 may be replaced by μ (by ν).

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