On a System of Axioms Which Has no Recursively Enumerable Arithmetic Model

By

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According to a well-known result of Löwenheim, Skolem, and Gödel every consistent axiomatic system S based on the functional calculus of the first order has an interpretation in the set of positive integers [1]). Hence if A is the conjunction of the axioms of S [2]) and $R_1, R_2, \ldots, R_p$ are the predicates [3]) which occur in A, then there are relation

$$\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_p \tag{1}$$

(with the same number of arguments as the predicates R_j) defined in the set of positive integers which satisfy formula A in the domain of positive integers [4]).

We shall denote by $c_1, c_2, \ldots, c_p$ the numbers of arguments in relations (1).

The ordered p-tuple (1) is called an *arithmetic model* of S. The model (1) is said to *belong to the class* $\mathbf{P}_n$ (or *to the class* $\mathbf{Q}_n$) if

$$\mathbf{R}_j \in \mathbf{P}_n^{(c_j)} \quad \text{or} \quad \mathbf{R}_j \in \mathbf{Q}_n^{(c_j)} \quad \text{for} \quad j = 1, 2, \ldots, p \text{ [5])}.$$

It has been proved by S. C. Kleene [6]) that every consistent and finitely axiomatizable system S possesses a model of class $\mathbf{P}_2 \cdot \mathbf{Q}_2$. The aim of this paper is to construct a finitely axiomatizable system S which possesses no model of class $\mathbf{P}_1$.

We shall obtain a required system suitably modifying the axiomatic system of set theory proposed by Bernays [7]). The modification consists in allowing a far larger number of primitive notions.

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[1]) Cf. for instance [2], p. 182-189. Numbers in square brackets refer to the bibliography at the end of this paper.

[2]) S is assumed to be finitely axiomatizable.

[3]) I assume that S does not contain symbols for mathematical functions but exclusively symbols for relations. Standard logical sings will be used in S and in the metamathematical discussion of this system.

[4]) The notion of satisfaction is meant in the sense of Tarski. Cf. [4].

[5]) Cf. [3] for the explanation of symbols used in this definition.

[6]) Kleene [6], p. 394.

[7]) Cf. [1].

The primitive notions of S are:

$S(a)$	[a is a set],
$C(a)$	[a is a class],
$I(a,b)$	[sets a and b are identical],
$J(a,b)$	[classes a and b are identical],
$E(a,b)$	[the set a belongs to the set b],
$H(a,b)$	[the set a belongs to the class b],
$A_0(a)$	[a is a void set],
$A_1(a,b,c)$	[a arises from the set b by adjunction of the element c],
$A_2(a,b,c)$	[a is the ordered pair of sets b and c],
$A_3(a,b,c,d)$	[a is the ordered triple of sets b, c, and d],
$B_0(a,b)$	[a is a class with the single element b],
$B_1(a)$	[a is the universal class],
$B_2(a)$	[a is the class of all one-element sets],
$B_3(a)$	[a is the class of all ordered pairs $\langle x,y \rangle$ such that $E(x,y)$],
$B_4(a,b)$	[a is the complement of the class b],
$B_5(a,b)$	[a is the class of ordered pairs $\langle x,y \rangle$ such that $E(x,b)$],
$B_6(a,b)$	[a is the domain of b],
$B_7(a,b)$	[a is the converse class to b],
$B_8(a,b)$	[a is the class of triples which arise by "coupling to the left" the triples which belong to b],
$B_9(a,b,c)$	[a is the union of b and c].

The axioms of S are those given by Bernays with obvious changes necessitated by our choice of primitive notions. For instance, instead of Bernays' single axiom IIIc(1) we have to assume the following two axioms:

$$C(b) \supset (\exists a)[C(a) \cdot B_6(a,b)],$$

$$B_6(a,b) \supset \{H(x,a) \equiv (\exists y,z)[A_2(y,x,z) \cdot H(y,b)]\}.$$

In a similar way we adapt the remaining axioms of Bernays to our choice of primitive notions.

We define now inductively a class $\mathcal{C}$ of formulae with one free variable. The formulae of class $\mathcal{C}$ will be said to *define classes*.

(I) Formulae $B_1(a)$, $B_2(a)$, $B_3(a)$ belong to $\mathcal{C}$;

(II) If $\Gamma(a)$ belongs to $\mathcal{C}$, then so do the formulae

$$(\exists x)[B_j(a,x) \cdot \Gamma(x)] \qquad (j = 4, 5, 6, 7, 8),$$

where x is any variable which does not occur in $\Gamma(a)$;

(III) If $\Gamma(a)$ and $\Delta(a)$ belong to $\mathcal{C}$, then so does the formula

$$(\exists x,y)[B_9(a,x,y) \cdot \Gamma(x) \cdot \Delta(y)],$$

where x and y are any variables not occurring in $\Gamma(a)$ and in $\Delta(a)$.

To every formula of class $\mathfrak{C}$ we let correspond a sequence of integers. To formulae (I) correspond sequences consisting of single integers 1, 2, and 3. If a sequence

(2) $$n_1, \dots, n_s$$

corresponds to a formula $\Gamma(a)$, then the sequences

$$n_1, \dots, n_s, j \qquad (j = 4, 5, 6, 7, 8)$$

correspond to the formulae (II). If sequences

$$n_1, \dots, n_s; \qquad m_1, \dots, m_t$$

correspond to formulae $\Gamma(a)$ and $\Delta(a)$, then the sequence

$$n_1, \dots, n_s, m_1, \dots, m_t, 9+s$$

corresponds to the formula (III).

It is well known that it is possible to enumerate all finite sequences (2) in such a way that every integer g will be a number of exactly one sequence (2) and s and n_j $(j=1,\dots,s)$ be primitive recursive functions of g:

$$s = L(g), \quad n_j = \bar{g}_j \qquad (j = 1, 2, \dots, L(g)).$$

If the sequence (2) corresponds to a formula $\Gamma(a)$ of class $\mathfrak{C}$, then the number of the sequence (2) is said to *represent the formula* $\Gamma(a)$

Let

(3) $$\boldsymbol{S}, \boldsymbol{C}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{A}_0, \dots, \boldsymbol{A}_3, \boldsymbol{B}_0, \dots, \boldsymbol{B}_9$$

be an arithmetic model of the system S. We write $\vdash \Gamma(n)$ (resp. $\vdash \Gamma$) instead of: *n satisfies $\Gamma(a)$ in the model* (resp. *Γ is true in the model*).

We put for arbitrary integers g and k [8])

(4) $$\Pi(g,k) \equiv \{\sum_{j=1}^{3}[(\bar{g}_1 = j)\cdot \boldsymbol{B}_j(\bar{k}_1)]\} \cdot (h)_{L(g)}\{\sum_{j=1}^{3}[(\bar{g}_{h+1}=j)\cdot \boldsymbol{B}_j(\bar{k}_{h+1})] \vee \vee \sum_{j=4}^{8}[(\bar{g}_{h+1}=j)\cdot \boldsymbol{B}_j(\bar{k}_{h+1}, \bar{k}_h)] \vee [(\bar{g}_{h+1} > 9)\cdot \boldsymbol{B}_9(\bar{k}_{h+1}, \bar{k}_{\bar{g}_{h+1}-9}, \bar{k}_h)]\}.$$

Lemma 1. *If g represents a formula $\Gamma(a)$ of class $\mathfrak{C}$, then*

$$\vdash \Gamma(n) \equiv (\exists k)\{[L(k) = L(g)]\cdot \Pi(g,k)\cdot [\bar{k}_{L(k)} = n]\}.$$

Proof. If $\Gamma(a)$ is the expression $B_1(a)$, then $L(g)=1$, $\bar{g}_1 = 1$. Assume that $\boldsymbol{B}_1(n)$ and let k be the number of a sequence containing the single term n. Hence $L(k)=1$, $\bar{k}_1 = n$. Since $\bar{g}_1 = 1$ and $\boldsymbol{B}_1(\bar{k}_1)$ and since the

[8]) I use the sign Σ for alternations with finitely many terms. $(h)_n$ is to be read: for every h less than n.

second part of formula (4) is vacuously satisfied, we obtain $\Pi(g,k)$ whence

(5) $$[L(k) = L(g)]\cdot \Pi(g,k)\cdot [\bar{k}_{L(k)} = n].$$

Assume conversely that there exists an integer k satisfying (5). It follows from $\Pi(g,k)$ and $\bar{g}_1 = 1$ that $\boldsymbol{B}_1(\bar{k}_1)$ and hence $\boldsymbol{B}_1(n)$, *i. e.*, $\vdash \Gamma(n)$. Thus the lemma is proved in the case for which $\Gamma(a)$ is the formula $B_1(a)$.

The proof in the cases $\Gamma(a) = B_2(a)$ and $\Gamma(a) = B_3(a)$ is similar.

Assume now that the lemma holds for a formula $\Gamma(a)$ (represented by the integer g) and let $\Delta(a)$ be the formula (II) with $j=4$. Let f be the integer representing $\Delta(a)$. Hence

(6) $$L(f) = L(g)+1, \ \bar{f}_1 = \bar{g}_1, \dots, \bar{f}_{L(g)} = \bar{g}_{L(g)}, \ \bar{f}_{L(f)} = 4.$$

Assume that $\vdash \Delta(m)$, *i. e.*, that there is an integer n such that

(7) $$\boldsymbol{B}_4(m,n) \quad \text{and} \quad \vdash \Gamma(n).$$

It follows by the inductive assumption that there is an integer k satisfying (5). Let t be the number of the sequence

$$\bar{k}_1, \bar{k}_2, \dots, \bar{k}_{L(g)}, m$$

It is easy to infer from (5), (6), and (7) that

(8) $$[L(t) = L(f)]\cdot \Pi(f,t)\cdot [\bar{t}_{L(t)} = m].$$

Conversely, let us assume that there is an integer t satisfying the formula (8). Define k as the number of the sequence

$$\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{L(t)-1}.$$

It follows from (8) that

$$L(k) = L(g) \quad \text{and} \quad \Pi(g,k),$$

whence by the inductive assumption $\vdash \Gamma(\bar{k}_{L(k)})$. By (6) and (8) we obtain $\boldsymbol{B}_4(\bar{t}_{L(t)}, \bar{t}_{L(t)-1})$ and hence the equation $\bar{t}_{L(t)-1} = \bar{k}_{L(k)}$ gives $\boldsymbol{B}_4(\bar{t}_{L(t)}, \bar{k}_{L(k)})$, whence we obtain finally $\vdash \Delta(\bar{t}_{L(t)})$, *i. e.*, $\vdash \Delta(m)$. This proves the lemma for formulae (II) with $j=4$.

The proof for the remaining formulae (II) and for formulae (III) is similar.

Let

$$M_1, M_2, \dots$$

be a sequence of all formulae of S without free variables in which exclusively quantifiers of the form

$$(x)[S(x) \supset \dots] \quad \text{and} \quad (\exists x)[S(x) \dots]$$

occur. Formulae M_j are essentially what Bernays[9]) calls *constitutive expressions without free variables.* We denote by $\xi(j)$ the Gödel number of M_j; it is known that $\xi(j)$ is a primitive recursive function.

Analyzing the proof of the class-theorem of Bernays[10]) we arrive at the following

Lemma 2. *For every j there exists in $\mathfrak{C}$ a formula $\Gamma_j(a)$ such that the equivalence*

$$(\exists a)[\Gamma_j(a)\cdot B_1(a)]\equiv M_j$$

is provable in S. The integer $\gamma(j)$ representing the formula $\Gamma_j(a)$ is a primitive recursive function of j.

From the general theory of models we obtain the following

Lemma 3. *If (3) is an arithmetic model of S, then formulae M_j for which*

$$\vdash M_j$$

form a complete and consistent extension of S.

In order to prove our theorem we assume that (3) is a model of S of class $\boldsymbol{P}_1$. The set

$$Z=E_j\{\vdash(\exists a)[\Gamma_j(a)\cdot B_1(a)]\}$$

belongs to the class $\boldsymbol{P}_1^{(1)}$. Indeed, by lemma 1

$$j\in Z\equiv(\exists n)[\vdash\Gamma_j(n)\cdot B_1(n)]$$

$$\equiv(\exists n,k)\{[L(k)=L(\gamma(j))]\cdot\Pi(\gamma(j),k)\cdot[\bar{k}_{L(k)}=n]\cdot\boldsymbol{B}_1(n)\}.$$

The formula enclosed in braces { } defines a relation of class $\boldsymbol{P}_1^{(3)}$ because $B_1\in\boldsymbol{P}_1^{(1)}$, relations $L(k)=L(\gamma(j))$ and $\bar{k}_{L(k)}=n$ are primitive recursive, and $\Pi(\gamma(j),k)$ as is evident from (4) defines a relation of class $\boldsymbol{P}_1^{(2)}$.

It follows that the set

$$T=\xi(Z)=E_k[(\exists j)(\xi(j)=k)\cdot(j\in Z)]$$

belongs to $\boldsymbol{P}_1^{(1)}$ and from lemmas 2 and 3 that T is the set of Gödel numbers of a complete and consistent extension of S.

Since S contains the arithmetic of non-negative integers, the existence of the set $T\in\boldsymbol{P}_1^{(1)}$ with this property contradicts the well-known result of Rosser[11]). Thus S has no model of class $\boldsymbol{P}_1$, q. e. d.

I have not succeeded in finding an example of a finitely axiomatizable system which has no model of class $\boldsymbol{Q}_1$. On the other hand it does not seem probable that the evaluation of the class of models found by **Kleene** can be amended. (See note on p. 61).

[9]) See l. c., p. 71.
[10]) See l. c., p. 72-76.
[11]) Cf. [5], theorem II, p. 89.

References

[1] P. Bernays, *A system of axiomatic set-theory. Part I*, The Journal of Symbolic Logic **2** (1937), p. 65-77.

[2] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, vol. 2, Berlin 1939.

[3] A. Mostowski, *On Definable Sets of Positive Integers*, Fundamenta Mathematicae **34** (1947), p. 81-112.

[4] A. Tarski, *Der Wahrheitsbegriff in formalisierten Sprachen*, Studia Philosophica **1** (1935), p. 261-405.

[5] J. B. Rosser, *Extension of Some Theorems of Gödel and Church*, The Journal of Symbolic Logic **1** (1936), p. 87-91.

[6] S. C. Kleene, *Introduction to Metamathematics*, Groningen 1953.

Note added reading the proofs. The problem mentioned at bottom of p. 60 has been meanwhile solved: There exist finitely axiomatizable systems which have no models of class $\boldsymbol{P}$ and no models of class $\boldsymbol{Q}$. One such system will be exhibited in my next paper forthcoming in Fundamenta Mathematicae.