The Structure of k-chromatic Graphs

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The graphs considered in this paper are such that a node is never joined to itself by an edge, and two nodes are never joined by more than one edge. A graph has *chromatic number* k, or is *k-chromatic* if there exists a smallest positive integer k such that the nodes of the graph can be divided into k mutually disjoint (colour) classes so that no two nodes in the same class are joined by an edge. There is a conjecture, the history of which I have not been able to trace, according to which a k-chromatic graph contains a subgraph which is homoeomorphic to the complete k-graph. (Two graphs are *homoeomorphic* if they are isomorphic or if they can be made isomorphic by (repeatedly) dividing edges into two through the insertion of nodes. A *complete k-graph* is a graph with k nodes, every pair of distinct nodes being joined by an edge). The conjecture is trivial for $k \leqslant 3$, and I have been able to prove it for $k=4$ [1]). For $k=5$ the truth of the conjecture would clearly imply the truth of the four-colour hypothesis, so that to prove it in full generality is at present quite hopeless. But it might be possible, with the help of Theorem 5 of this paper, to prove something implied by the conjecture; that a 6-chromatic graph always contains a subgraph which is homoeomorphic to the complete 5-graph. This would also constitute a new proof of the five-colour theorem which would be almost entirely combinatorial in character. The results established in this paper arose from an investigation of the conjecture in its general form.

If a graph is disconnected, its chromatic number is the greatest among the set of chromatic numbers of its connected components. Accordingly, a graph of chromatic number k must have at least one connected component with the same chromatic number, so that if we wish to find relations between the chromatic numbers of graphs and their structure, we may confine ourselves to connected graphs.

De Bruijn and Rado proved that *if the chromatic number of every finite subgraph of an infinite graph Γ is less than or equal to k, the chromatic number of Γ is less than or equal to k* [2]). This theorem shows that the chromatic number of an infinite graph is determined by the chromatic numbers of its finite subgraphs. It follows from the theorem that *an infinite k-chromatic graph necessarily contains a finite connected k-chromatic subgraph.* One consequence of this is that the conjecture mentioned above need only be proved for finite connected graphs.

[1]) G. A. Dirac, *A property of 4-chromatic graphs and some remarks on critical graphs*, Journal London Math. Soc. **27** (1952), p. 85.

A k-chromatic graph will be called *critical* if it has no proper subgraph with chromatic number k. It follows from the preceding remarks that *a critical graph is finite and connected.* Further, we saw above that any k-chromatic graph contains a finite connected k-chromatic subgraph. It is easy to see that any finite connected k-chromatic graph contains one or more critical k-chromatic subgraphs. Thus *a k-chromatic graph always contains a critical k-chromatic subgraph.*

This notion of critical graphs is useful because every general feature of k-chromatic graphs is possessed also by critical k-chromatic graphs, and a critical graph is more sharply defined and less arbitrary than a non-critical one.

Critical graphs are finite and connected, and in addition have the following two characteristics:

1. The degree of every node of a critical k-chromatic graph is at least $k-1$.

2. A critical graph has no isthmus.

(An *isthmus* is a node of degree $\geqslant 2$ such that it divides the graph into two or more separate parts).

The object of this paper is to generalize 1 and 2 in a significant way. In the first part we will investigate the connection between the chromatic number of a graph and the way in which the subgraphs are joined to the graph by edges. In the second part we will consider how the graph may be split into disconnected components by isthmoids.

Notation. The nodes will be denoted by l. c. letters a, b etc., possibly with suffixes, and the edges by (a,b) etc., where $a \neq b$ and $(a,b)=(b,a)$.

If Γ is a graph, $K(\Gamma)$ will denote its chromatic number.

If the nodes a and b are joined by an edge, we write $a \times b$, if they are not joined by an edge we write $a \circ b$.

If the node a belongs to the graph Γ (in symbols if $a \in \Gamma$), the graph obtained from Γ by deleting a and all edges incident in a, leaving all other nodes and edges unchanged, will be denoted by $\Gamma - a$. Similarly if $a, b, \ldots, c \in \Gamma$ the graph obtained from Γ by deleting $a, b, \ldots, c$ and all edges incident with at least one of them will be denoted by $\Gamma - a - b - \ldots - c$. Similarly if $(d,e) \in \Gamma$, the graph obtained from Γ by deleting (d,e) will

[2]) N. G. De Bruijn and P. Erdös, *A colour problem for infinite graphs and a problem in the theory of relations*, Proc. Konінkl. Nederl. Akademie van Wetenschappen. Series A, **54** no. 5, p. 371.

be denoted by $\Gamma-(d,e)$. The node a will be called a *critical node* of Γ if $K(\Gamma-a)=K(\Gamma)-1$.

A colouring of the nodes of Γ using at most k colours, or a k-colouring, is a function C defined over the nodes of Γ and taking one of the values $1,2,...,k$ at each node, with the condition that $C(x)\neq C(y)$ if x and y are joined by an edge.

1. The following definitions will provide a precise terminology:

Suppose Γ_1 and Γ_2 are two mutually disjoint graphs and Γ is a graph whose nodes are the nodes of Γ_1 and of Γ_2 and whose edges are the edges of Γ_1 and Γ_2 together with some edges joining nodes of Γ_1 to nodes of Γ_2 (*i. e.* Γ consists of Γ_1 and Γ_2 and edges joining nodes of Γ_1 to nodes of Γ_2). The number of edges joining nodes of Γ_1 to nodes of Γ_2 will be called the *adjoint number* of Γ_1 and Γ_2.

Nodes in Γ_1 joined to nodes in Γ_2 and nodes in Γ_2 joined to nodes in Γ_1 will be called *clasp-nodes*. The number of clasp-nodes in Γ_1 will be called the *clasp-number of* Γ_1 and the number of clasp-nodes in Γ_2 will be called the *clasp-number of* Γ_2.

These definitions are framed so that Γ can be regarded as the complete graph, either Γ_1 or Γ_2 as the subgraph and the other as the complementary subgraph.

In what follows we shall denote by k_1,k_2 and k the chromatic numbers of Γ_1,Γ_2 and Γ respectively, γ_1 will denote the clasp-number of Γ_1 and γ_2 the clasp-number of Γ_2.

The following simple inequality applies to all graphs:

(1) *If* $k_1\geqslant k_2$ *then* $k\leqslant k_1+\gamma_2$.

Proof. Γ_1 may be coloured by k_1 colours, and the γ_2 nodes of Γ_2 which are joined to nodes of Γ_1 require at most γ_2 additional colours to ensure that no two nodes joined by an edge are coloured the same. Since $k_2\leqslant k_1$, the remaining nodes of Γ_2 may now be coloured with the k_1 colours which were used to colour Γ_1. Thus Γ may be coloured using $k_1+\gamma_2$ colours and the result follows.

It is easy to see that the above inequality is best-possible. For, suppose Γ is the complete k-graph $[a_1,a_2,...,a_k]$, Γ_1 is the complete $k-1$-graph $[a_1,a_2,...,a_{k-1}]$ and Γ_2 is the node a_k. Then $k_1=k-1$, $k_2=1$ so that $k_1>k_2$, and $k=k_1+\gamma_2$.

A more precise result, expressed in the above notation, is the following:

(2) $k\leqslant\max(k_1,k_2,\gamma_1+\gamma_2)$.

Proof. If $k=k_1$, or if $k=k_2$ the theorem is trivial, so we need consider only the case in which $k_1<k$ and $k_2<k$. Suppose the theorem fails in this case, so that for the graph to be considered $\gamma_1+\gamma_2\leqslant k-1$.

If $\gamma_1+\gamma_2\leqslant k-1$, we shall show how Γ can be coloured using at most $k-1$ colours $1,2,...,k-1$. We are given that $k_1\leqslant k-1$ so that Γ_1 can be coloured using at most $k-1$ colours $1,2,3,...,k-1$. In such a colouring the γ_1 clasp-nodes of Γ_1 will receive at most γ_1 different colours. Similarly $k_2\leqslant k-1$ so that Γ_2 can be coloured using at most $k-1$ colours $1,2,3,...,k-1$ and in such a colouring the γ_2 clasp-nodes of Γ_2 will receive at most γ_2 different colours. If $\gamma_1+\gamma_2\leqslant k-1$ we can obviously colour Γ_1 and Γ_2 with these $k-1$ colours in such a way that no clasp-node of Γ_2 has the same colour as a clasp-node of Γ_1. But then this colouring of Γ_1 and Γ_2 is clearly also a permissible colouring of Γ using at most $k-1$ colours, which contradicts the assumption that $K(\Gamma)=k$. This proves the theorem.

An argument very similar to the one just given yields the following:

(3) *If* $k_1<k$ *and* $k_2<k$ *then* $k\leqslant k_1+\gamma_2$ *and* $k\leqslant k_2+\gamma_1$.

Proof. Suppose on the contrary that *e. g.* $k_1+\gamma_2\leqslant k-1$. We can colour Γ_1 using only the colours $1,2,...,k_1$; and we can colour Γ_2 using only the colours $1,2,...,k-1$ in such a way that the clasp-nodes of Γ_2 receive colours from among $k_1+1,k_1+2,...,k_1+\gamma_2$, since $k_1+\gamma_2\leqslant k-1$. But this colouring of Γ_1 and Γ_2 is clearly also a permissible colouring of Γ using at most $k-1$ colours, which contradicts the datum that $K(\Gamma)=k$. Hence $k\leqslant k_1+\gamma_2$, and by a change of notation $k\leqslant k_2+\gamma_1$.

The example given after the proof of (1) shows that both (2) and (3) state best-possible results.

The next theorem is concerned with the adjoint number of Γ_1 and Γ_2, and is a generalization of the fact that a critical node of a k-chromatic graph must be of degree $\geqslant k-1$.

Theorem 1. *If* $k_1<k$ *and* $k_2<k$ *the adjoint number of* Γ_1 *and* Γ_2 *is at least* $k-1$.

Proof. The number of nodes of Γ_2 to which a clasp-node c of Γ_1 is joined by an edge will be called the *multiplicity* of c. The number of clasp-nodes of Γ_1, each counted according to its multiplicity, is equal to the adjoint number of Γ_1 and Γ_2 which we denote by A. Now suppose that the theorem is false so that for some graph Γ and its subgraphs Γ_1 and Γ_2 we have $k_1\leqslant k-1, k_2\leqslant k-1$ and the adjoint number A of Γ_1 and Γ_2 is $\leqslant k-2$. Then the number of clasp-nodes of Γ_1 each counted according to its multiplicity is equal to A and so $\leqslant k-2$.

Let a be a clasp-node of Γ_1 joined to the nodes $a_1,a_2,...,a_s$ of Γ_2. Colour Γ_1 using the colours $1,2,...,k_1$ so that a is coloured 1. Colour Γ_2 using the colours $1,2,...,k_2$ so that a_1 is coloured 2. If these colourings of Γ_1 and Γ_2 are denoted by C and C_1 respectively, we require that $C(a)=1$ and $C_1(a_1)=2$.

If now $C_1(a_2)=2$ leave the colouring, if $C_1(a_2)\neq 2$ define a new $(k-1)$-colouring C_1' *of* Γ_2 as follows: If $C_1(x)\neq C_1(a_2)$ and $C_1(x)\neq 3$ then $C_1(x)=C_1'(x)$, if $C_1(x)=C_1(a_2)$ then $C_1'(x)=3$ and if $C_1(x)=3$ then $C_1'(x)=C_1(a_2)$. Now $C_1'(a_1)=2$ and $C_1'(a_2)=3$. This operation consists simply of interchanging the colours $C_1(a_2)$ and 3 in Γ_2 and leaving all other colours undisturbed. It is obviously a permissible re-colouring, and in future when we make such an interchange we shall describe it in this way and not use the functional symbol each time. Of course if $C_1(a_2)=3$, no change is made.

If in the new colouring a_3 is coloured 2 or 3, leave it, otherwise interchange the colours $C_1'(a_3)$ and 4 in the same way as above, to obtain a new $(k-1)$-colouring C_1'' *of* Γ_2 in which $C_1''(a_3)=4$. We proceed in this way until we reach a_s. In any re-colouring associated with a_i $(i\leqslant s)$ the colours given to the nodes $a_1,a_2,\dots,a_{i-1}$ are left undisturbed and different from 1, and finally none of the nodes $a_1,a_2,\dots,a_s$ will have the colour 1. The number of colours needed to carry out these operations is at most $s+1$ (the extreme case is when a_i is coloured $i+1$ for $1\leqslant i\leqslant s$) and since $s\leqslant k-2$ by hypothesis, $s+1\leqslant k-1$, so that we need at most $k-1$ colours.

If a is the only clasp-node of Γ_1 the above colouring is clearly also a permissible colouring of Γ using at most $k-1$ colours, which contradicts the datum that $K(\Gamma)=k$. If a is not the only clasp-node of Γ_1, let b be another, having multiplicity t, joined to the nodes $b_1,b_2,\dots,b_t$ of Γ_2. The sets of nodes $\{a_1,a_2,\dots,a_s\}$ and $\{b_1,b_2,\dots,b_t\}$ need not be disjoint but may overlap. Now let Γ_1 be coloured so that a has the colour 1 and b has the colour 1 or 2. We can now proceed in exactly the same way with $b_1,b_2,\dots,b_t$ as we did with $a_1,a_2,\dots,a_s$: If b_1 has the same colour as a or b, or if it has a colour $\geqslant s+3$, interchange this colour and the colour $s+2$, otherwise leave the colouring unchanged, and so on for $b_2,\dots,b_t$. After an exchange connected with b_j all the preceding nodes $a_1,a_2,\dots,a_s$ and $b_1,b_2,\dots,b_{j-1}$ either retain their colour (if the node b_j is not also one of $\{a_1,\dots,a_s\}$) or receive the colour to which we change b_j (if the node b_j is also one of $\{a_1,\dots,a_s\}$), and this colour is certainly different from the colours assigned to those nodes among $a_1,\dots,a_s$ and $b_1,\dots,b_{j-1}$ which are different from b_j. The number of colours needed to carry out these operations is at most $s+t+1$ and since $s+t\leqslant k-2$, $s+t+1\leqslant k-1$ so that we need at most $k-1$ colours. At the end clearly neither a nor b is joined to a clasp-node of Γ_2 having the same colour.

We can proceed in this way with all the clasp-nodes of Γ_1, and the total number of colours needed is at most one greater than the number of clasp-nodes of Γ_1 each counted according to its multiplicity, *i. e.* it is at most $A+1$. Now by hypothesis $A\leqslant k-2$ so that $A+1\leqslant k-1$. But when we have completed these processes for all the clasp-nodes of Γ_1 we clearly obtain a colouring of Γ using at most $k-1$ colours, in contradiction to the datum that $K(\Gamma)=k$. The assumption that the theorem is false therefore leads to a contradiction, and so the theorem is proved.

The result obtained is again best possible. A simple example showing this is the case of the complete k-graph given just after the proof of (1). A less trivial example in which Γ_2 is not just a node by itself is given in the Fig. 1. Here $k_1=k_2=2$, $k=3$ and the adjoint number of Γ_1 and Γ_2 is 2.

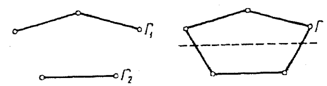

Fig. 1.

2. We now go on to the second part of this paper, in which we will consider the isthmoids of a graph. An isthmoid is a simple extension of the idea of an isthmus defined earlier, and its precise meaning is given in the following

Definition. Let Γ be a connected graph. The set of nodes $S=\{a,b,c,\dots,d\}$ of Γ is said to be an *isthmoid* of Γ if

(i) $\Gamma-S$ is a graph having more than one connected component,

(ii) If S' is any proper subset of S, the number of connected components of $\Gamma-S'$ is less than the number of connected components of $\Gamma-S$.

($\Gamma-S$ denotes $\Gamma-a-b-c-\dots-d$, $\Gamma-S'$ is defined similarly).

The *order* of an isthmoid is the number of nodes which it contains.

An isthmus is an isthmoid consisting of only one node, *i. e.* an isthmoid of order 1.

We shall prove a theorem which connects the structure of the isthmoid with the chromatic numbers of the whole graph and the connected components into which it is split by removing the isthmoid.

Suppose $I=\{c_1,c_2,\dots,c_n,\dots,c_i\}$ is an isthmoid of order i of the graph Γ in which the nodes $c_1,c_2,\dots,c_n$ $(n\geqslant 1)$ together with the edges connecting them in Γ form a complete n-graph and no nodes of I form a complete m-graph in Γ with $m>n$. By removing the nodes of I and all edges incident with them from Γ we obtain two or more mutually disjoint connected subgraphs of Γ. In case $\Gamma-I$ consists of more than two connected components, we subdivide them in any way into two non-empty sets, and regard those in one set as forming together the graph Γ_a, those in the other the graph Γ_b; in case $\Gamma-I$ consists of two connected components we call one Γ_a, the other Γ_b. Such a division will be supposed to have been carried out in what follows. We shall denote Γ_a also by $[a_1,a_2,\dots,a_s]$ and Γ_b by $[b_1,b_2,\dots,b_t]$. It is to be noted that either Γ_a or Γ_b may consist entirely of one of the connected components of $\Gamma-I$.

The graph $\Gamma-b_1-b_2-\dots-b_t$ (*i. e.* the graph obtained from Γ by deleting the nodes of Γ_b and all edges incident with them) will be denoted by Γ' and also by $[a_1,a_2,\dots,a_s,c_1,c_2,\dots,c_i]$, the graph $\Gamma-a_1-a_2-\dots-a_s$ will be denoted by Γ'' and also by $[b_1,b_2,\dots,b_t,c_1,c_2,\dots,c_i]$. In each case the nodes enumerated in the square brackets are precisely all the nodes of the graph concerned. In what follows we shall denote by k, k' and k'' the chromatic numbers of Γ, Γ' and Γ'' respectively. With this notation we now prove the following

Theorem 2. *If $k'<k$ and $k''<k$ then $k \leqslant k'+i-n$ and $k \leqslant k''+i-n$.*

Proof. If C' is a colouring of Γ' using at most the colours $1,2,\ldots,k$ and if C'' is a colouring of Γ'' using at most the colours $1,2,\ldots,k$ and if in addition $C'(c_j)=C''(c_j)$ for $1 \leqslant j \leqslant i$, then the colouring C defined by

$$C(a_\alpha) = C'(a_\alpha), \quad 1 \leqslant \alpha \leqslant s;$$
$$C(b_\beta) = C''(b_\beta), \quad 1 \leqslant \beta \leqslant t;$$
$$C(c_\gamma) = C'(c_\gamma) = C''(c_\gamma), \quad 1 \leqslant \gamma \leqslant i$$

is clearly a permissible colouring of Γ using only the colours $1,2,\ldots,k$. To prove the theorem it is obviously sufficient to prove that $k \leqslant k'+i-n$ assuming that $k' \leqslant k''$. We shall do this by showing how two colourings C' and C'' of Γ' and Γ'' respectively may be obtained with $C'(c_j)=C''(c_j)$ for $1 \leqslant j \leqslant i$ using at most the colours $1,2,\ldots,k'+i-n$. It will then follow that Γ itself can be coloured using at most $k'+i-n$ colours, and therefore that $k \leqslant k'+i-n$.

We consider first the case $k'=k''$. Let C_1' and C_1'' denote colourings of Γ' and Γ'' respectively using only the colours $1,2,\ldots,k'$ and such that $C_1'(c_j)=C_1''(c_j)$ for $1 \leqslant j \leqslant n$. This is obviously always possible because $c_1, c_2, \ldots, c_n$ form a complete n-graph in Γ' and in Γ''. It now remains to adjust the colourings so that they agree over the remaining nodes of the isthmoid as required. We now introduce $i-n$ new colours, $k'+1, k'+2, \ldots, k'+i-n$ and define new colourings C' and C'' of Γ' and Γ'' respectively as follows:

$$C'(c_{n+h}) = k'+h \quad \text{for} \quad 1 \leqslant h \leqslant i-n,$$
$$C'(a_\alpha) = C_1'(a_\alpha) \quad \text{for} \quad 1 \leqslant \alpha \leqslant s$$

and

$$C'(c_j) = C_1'(c_j) \quad \text{for} \quad 1 \leqslant j \leqslant n;$$
$$C''(c_{n+h}) = k'+h \quad \text{for} \quad 1 \leqslant h \leqslant i-n,$$
$$C''(b_\beta) = C_1''(b_\beta) \quad \text{for} \quad 1 \leqslant \beta \leqslant t$$

and

$$C''(c_j) = C_1''(c_j) \quad \text{for} \quad 1 \leqslant j \leqslant n$$

(*i. e.* we give a new colour to each of the nodes $c_{n+1}, c_{n+2}, \ldots, c_i$ of the isthmoid, and leave the colours of all other nodes unchanged).

The colourings C' and C'' now agree over all the nodes of the isthmoid, so that we can certainly colour Γ using at most $k'+i-n$ colours. Hence if $k'=k''$, then $k \leqslant k'+i-n$.

It remains to show that if $k'<k''$ then $k \leqslant k'+i-n$. We shall first show that we can find colourings of Γ' and Γ'' using only the colours $1,2,\ldots,k''$ which agree over the nodes $c_1, c_2, \ldots, c_n, \ldots, c_{n+k''-k'}$ of the isthmoid. Then as before we can if necessary give a new colour to each of the remaining $i-(n+k''-k')$ nodes of the isthmoid and after this the two colourings will agree over all the nodes of the isthmoid. The total number of colours used is $k''+i-(n+k''-k')=k'+i-n$; so that $k \leqslant k'+i-n$.

Using the colours $1,2,\ldots,k$ for Γ' and $1,2,\ldots,k''$ for Γ'' we can certainly colour them so that the colours given to $c_1, c_2, \ldots, c_n$ agree in the two colourings. Now suppose that C_1' denotes a colouring of Γ' using at most the colours $1,2,\ldots,k'+r$, where $0 \leqslant r \leqslant k''-k'-1$, and C_1'' denotes a colouring of Γ'' using only the colours $1,2,\ldots,k''$ and suppose that $C_1'(c_j)=C_1''(c_j)$ for $1 \leqslant j \leqslant n+r$ (we can, by the preceding remark, certainly realise this situation with $r=0$). We shall show how to construct a colouring C_2' of Γ' using at most the colours $1,2,\ldots,k'+r+1$ and a k''-colouring C_2'' of Γ'' such that $C_2'(c_j)=C_2''(c_j)$ for $1 \leqslant j \leqslant n+r+1$.

We may obviously assume that $C_1'(c_j)=C_1''(c_j)=j$ for $1 \leqslant j \leqslant n$ and that $C_1'(c_j)=C_1''(c_j) \leqslant j$ for $n \leqslant j \leqslant n+r$. If possible let C_1'' be chosen so that in addition to the conditions just preceding, $C_1''(c_{n+r+1}) \neq C_1''(c_j)$ for $1 \leqslant j \leqslant n+r$, then we construct C_2'' from C_1'' by interchanging the colours $C_1''(c_{n+r+1})$ and $k'+r+1$. This will not affect the colours given to $c_1, c_2, \ldots, c_{n+r}$ because obviously $k' \geqslant n$. We now construct C_2' from C_1' by making $C_2'(c_{n+r+1})=k'+r+1$ and leaving the colours of all the other nodes unchanged. Then C_2' is a colouring of Γ' using at most the colours $1,2,3,\ldots,k'+r+1$, C_2'' is a colouring of Γ'' using the colours $1,2,\ldots,k''$ and $C_2'(c_j)=C_2''(c_j)$ for $1 \leqslant j \leqslant n+r+1$. Further, by interchanging the colours $k'+r+1$ and *e. g.* $n+r+1$ if necessary, we can arrange that for $1 \leqslant j \leqslant n+r+1$ the colour assigned to c_j should have a number not exceeding j.

If on the other hand C_1'' cannot be chosen so that $C_1''(c_{n+r+1}) \neq C_1''(c_j)$ for $1 \leqslant j \leqslant n+r$ then we know that $C_1''(c_{n+r+1}) \leqslant k'+r$. (We have assumed that $C_1'(c_j)=C_1''(c_j) \leqslant j$ for $1 \leqslant j \leqslant n+r$ and obviously $n \leqslant k'$). In this case we construct C_2' and C_2'' as follows: C_2'' is obtained from C_1'' by interchanging the colours $k'+r+1$ and $C_1''(c_{n+r+1})$ and leaving everything else unchanged, so that $C_2''(c_{n+r+1})=k'+r+1$. C_2' is obtained from C_1' by giving to c_j the colour $k'+r+1$ if $C_2''(c_j)=k'+r+1$, and leaving the colours of all the other nodes unchanged. This is a permissible colouring of Γ' because the colour $k'+r+1$ does not occur in C_1', so that the only obstacle would be if two nodes joined by an edge in Γ' both received the same colour $k'+r+1$. This cannot happen because the two nodes receive the colour $k'+r+1$ only if they receive this colour in C_2'', so that they cannot be joined by an edge in Γ'' and therefore cannot be joined by an edge in Γ' either (see the definitions of Γ' and Γ''). So C_2' is a colouring of Γ' using at most the colours $1,2,\ldots,k'+r+1$, C_2'' is a colouring of Γ'' using the colours $1,2,\ldots,k''$ and $C_2'(c_j)=C_2''(c_j)$

for $1 \leqslant j \leqslant n+r+1$. Further, by interchanging colours if necessary, we can clearly arrange that for $1 \leqslant j \leqslant n+r+1$ the colour assigned to c_j should have a number not exceeding j.

If $0 \leqslant r \leqslant k''-k'-1$ then $k'+r+1 \leqslant k''$, and so this process can be carried through for $r=0,1,\ldots,k''-k'-1$. Hence Γ' and Γ'' can be coloured using the colours $1,2,\ldots,k''$ in such a way that the colours of $c_1,c_2,\ldots,c_{n+k''-k'}$ agree in the two colourings. The two colourings cannot agree over all the nodes of the isthmoid though, because if they did agree then we should have a colouring of Γ using k'' colours, and this would contradict the assumption that $k''<k$. Accordingly we can say that using k'' colours, $1,2,\ldots,k''$, we can colour Γ' and Γ'' and secure agreement in the colourings over the nodes $c_1,c_2,\ldots,c_\beta$ where $n+k''-k' \leqslant \beta \leqslant i-1$. There remain $i-\beta\,(\geqslant 1)$ nodes over which agreement has not been secured. If we give to each of these nodes a new colour, then we shall obtain a colouring of Γ itself. The total number of colours used in this is $k''+i-\beta$, which is $\leqslant k''+i-(n+k''-k')$ since $\beta \geqslant n+k''-k'$, and $k''+i-(n+k''-k')=k'+i-n$. Hence $k \leqslant k'+i-n$.

We have shown that if $k' \leqslant k'' < k$, then $k \leqslant k'+i-n$, this proves the theorem.

As we have explained just before Theorem 2, if the isthmoid splits the graph into more than two connected components, the graphs Γ_a and Γ_b and consequently also Γ' and Γ'' can be constructed in various ways. If we label the possible choices for Γ' as $\Gamma_1',\Gamma_2',\ldots,\Gamma_g'$ and denote their chromatic numbers by $k_1',k_2',\ldots,k_g'$ respectively then we can deduce from the theorem just proved that

(4) $$k \leqslant i-n+k_\varrho' \quad \text{for} \quad 1 \leqslant \varrho \leqslant g.$$

From Theorem 2 we conclude at once that (in the above notation)

(5) $$\textit{if} \quad k' \leqslant k'' < k \quad \textit{then} \quad i-n \geqslant k-k' \geqslant 1.$$

The condition that $k'<k$ and $k''<k$ is always satisfied for a critical graph so that:

(6) *An isthmoid of a critical graph must contain two nodes which are not joined by an edge.*

That a critical graph cannot have an isthmus, is a special case of this more general result.

We can deduce a result concerning isthmoids of order 2; we use the notation introduced previously:

Theorem 3. *Let Γ be a critical graph and let the nodes c_1 and c_2 form an isthmoid. Then*

(i) *c_1 and c_2 are not connected by an edge,*

(ii) *by removing c_1 and c_2 Γ is split into exactly two connected components Γ_a and Γ_b,*

(iii) *Γ' and Γ'' have chromatic number $k-1$.*

The proof is straightforward. Suppose on the contrary that the graph is split into more than two connected components by the isthmoid, which we label by $\Gamma_1,\Gamma_2,\ldots,\Gamma_f$ with $f \geqslant 3$. We denote by Γ_1' the graph obtained by deleting from Γ the nodes of $\Gamma_2,\ldots,\Gamma_f$, in symbols $\Gamma_1'=\Gamma-\Sigma_i\Gamma_i$; we define $\Gamma_2',\ldots,\Gamma_f'$ similarly. Then c_1 and c_2 belong to each of $\Gamma_1',\Gamma_2',\ldots,\Gamma_f'$. The chromatic numbers of $\Gamma_1',\Gamma_2',\ldots,\Gamma_f'$ are equal to $k-1$. For *e. g.* $k_1' \leqslant k-1$ since Γ is critical, and $k \leqslant k_1'+i-n$ by Theorem 2. But $i-n \leqslant 1$ so that we must have $i-n=1$ and $k=k_1'+1$.

If we could colour $\Gamma_1',\Gamma_2',\ldots,\Gamma_f'$ using the colours $1,2,\ldots,k-1$ so that these colourings all agree over the nodes c_1 and c_2, we should be able to colour Γ itself with $k-1$ colours, and this we know is not possible. It follows that there must be two of $\Gamma_1',\Gamma_2',\ldots,\Gamma_f',\Gamma_i'$ and $\Gamma_j'(i \neq j)$, such that we cannot colour them both using the colours $1,2,\ldots,k-1$ only in such a way that the colourings agree over c_1 and c_2. To colour these two in such a way as to obtain agreement over c_1 and c_2 needs k colours. But by hypothesis $f \geqslant 3$, so that Γ must have a k-chromatic proper subgraph, obtained by deleting from Γ all nodes (and edges incident with them) except those in Γ_i' or Γ_j'. This contradicts the datum that Γ is critical.

We have therefore proved (ii). Part (i) is an immediate consequence of (6) and part (iii) follows from the inequality $k \leqslant k'+i-n$ since $i-n=1$.

We may observe that Theorem 3 has been proved from Theorem 2 for the special case $i=2$. The proof of Theorem 2 itself in this special case is of course very simple, the argument is almost obvious. It is very easy to see also that after colouring Γ' and Γ'' each with the colours $1,2,\ldots,k-1$, and giving c_1 the colour 1 in each colouring, we may obtain a colouring of Γ itself either by giving c_1 and c_2 both a new colour k $(c_1 \circ c_2)$ or by giving c_2 alone the new colour k. Hence *Γ can be coloured using k colours, and the two nodes of the isthmoid may both be given the same colour, or they may be given different colours.*

We shall now obtain another result for critical graphs having an isthmoid of order 2. This result will concern the degrees of the nodes forming the isthmoid. First we observe that if the critical graph Γ is split as before by an isthmoid of order 2 into two connected components, and Γ' and Γ'' are the same two $(k-1)$-chromatic subgraphs as before, then each of Γ' and Γ'' can be coloured using the colours $1,2,\ldots,k-1$, but these colourings cannot be so chosen that they agree over the nodes c_1 and c_2. In symbols, if C' and C'' denote colourings of Γ' and Γ'' respectively using the colours $1,2,\ldots,k-1$ and if $C'(c_1)=C''(c_1)$ then $C'(c_2) \neq C''(c_2)$. It follows that *either* $C'(c_1)=C'(c_2)$ and $C''(c_1) \neq C''(c_2)$ for all possible colourings C' and C'', *or* $C'(c_1) \neq C'(c_2)$ and $C''(c_1)=C''(c_2)$ for all possible colourings C' and C''. Expressed in words this means that *either* in any colouring of Γ' using $k-1$ colours c_1 and c_2 are bound to have the same colour and in any colouring of Γ'' using $k-1$ colours c_1 and c_2 are bound to have different colours *or* in any colouring of Γ' using $k-1$ colours c_1 and c_2 are bound to have different colours and in any colouring of Γ'' using $k-1$ colours c_1 and c_2 are bound to have the same colour. In what follows, we may assume without loss of generality that the first of these two alternatives is the case.

Let $d(c_1,\Gamma)$, $d(c_1,\Gamma')$, $d(c_1,\Gamma'')$ denote the degrees of c_1 in Γ, Γ' and Γ'' respectively (*i. e.* the number of nodes in Γ, Γ' and Γ'' to which c_1 is joined by an edge) with a similar notation for c_2. Clearly $d(c_1,\Gamma)=d(c_1,\Gamma')+d(c_1,\Gamma'')$ and analogously for c_2.

Since c_1 and c_2 must have the same colour in any colouring of Γ' using $k-1$ colours it follows that:

$$(7) \qquad d(c_1,\Gamma')\geqslant k-2; \qquad d(c_2,\Gamma')\geqslant k-2.$$

For otherwise, if *e. g.* $d(c_1,\Gamma')<k-2$ then in any colouring of Γ' using $k-1$ colours we could change the colour of c_1 leaving the colours of all other nodes unchanged and so c_1 could receive a colour different from the colour of c_2, which contradicts our hypothesis.

We can also prove that:

$$(8) \qquad d(c_1,\Gamma'')+d(c_2,\Gamma'')\geqslant k-1.$$

We assumed that c_1 and c_2 must receive different colours in any colouring of Γ'' using $k-1$ colours.

If $d(c_1,\Gamma'')$ or $d(c_2,\Gamma'')$ is greater than $k-2$, there is nothing to prove.

We may therefore suppose that $d(c_1,\Gamma'')\leqslant k-2$ and $d(c_2,\Gamma'')\leqslant k-2$. Consider any colouring of Γ'' using $k-1$ colours. Taking the colours assigned to all nodes of Γ'' except c_1 and c_2 as fixed, we clearly have a choice of at least $k-1-d(c_1,\Gamma'')$ colours for the colour of c_1 and a choice of at least $k-1-d(c_2,\Gamma'')$ colours for the colour of c_2. *If the sum of the number of possible choices for c_1 and c_2 exceeded $k-1$, there would be at least one possible choice in which c_1 and c_2 received the same colour.* By hypothesis this is impossible, so that:

$$k-1-d(c_1,\Gamma'')+k-1-d(c_2,\Gamma'')\leqslant k-1$$

hence

$$k-1\leqslant d(c_1,\Gamma'')+d(c_2,\Gamma'');$$

this proves (8).

Remembering that

$$d(c_1,\Gamma')+d(c_1,\Gamma'')=d(c_1,\Gamma)$$

and similarly for c_2, we obtain from (7) that

$$d(c_1,\Gamma')+d(c_2,\Gamma')\geqslant 2k-4$$

and adding this equation to (8) we have that

$$d(c_1,\Gamma')+d(c_1,\Gamma'')+d(c_2,\Gamma')+d(c_2,\Gamma'')\geqslant 3k-5,$$

i. e.

$$(9) \qquad d(c_1,\Gamma)+d(c_2,\Gamma)\geqslant 3k-5.$$

We express this result in the following

Theorem 4. *If two nodes form an isthmoid of a k-chromatic critical graph, the sum of their degrees in the graph is at least $3k-5$.*

Both the inequalities (7) and (8) and Theorem 4 are best possible, the following simple example illustrates this:

Let Γ' consist of the complete $(k-2)$-graph $[a_1,a_2,\ldots,a_{k-2}]$ and the two nodes c_1 and c_2, each of them being joined by an edge to each of $a_1,a_2,\ldots,a_{k-2}$. Then Γ' is $k-1$ chromatic, in any colouring of Γ' using $k-1$ colours c_1 and c_2 must be given the same colour and $d(c_1,\Gamma')=d(c_2,\Gamma')=k-2$. This shows that (7) is best-possible.

Let Γ'' consist of the complete $(k-1)$-graph $[b_1,b_2,\ldots,b_{k-1}]$ and the two nodes c_1 and c_2, where $c_1\times b_1$, and $c_1\circ b_2, c_1\circ b_3,\ldots,c_1\circ b_{k-1}$; $c_2\circ b_1$ and $c_2\times b_2, c_2\times b_3,\ldots,c_2\times b_{k-1}$. Then Γ'' is $(k-1)$-chromatic, in any colouring of Γ'' using $k-1$ colours c_1 and c_2 must be given different colours, and $d(c_1,\Gamma'')+d(c_2,\Gamma'')=k-1$. This shows that (8) is best-possible.

The graph Γ whose isthmoids are c_1 and c_2, and whose components are Γ' and Γ'' is k-chromatic and critical (as may easily be seen) and the degrees of c_1 and c_2 are $k-1$ and $2k-4$ respectively, the sum of their degrees is $3k-5$; this shows that Theorem 4 is best-possible.

In the Fig. 2 the preceding graph is illustrated with $k=5$.

We also give in Fig. 3 an illustration for $k=3$.

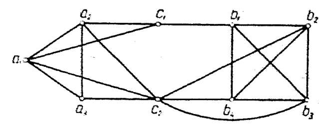

Fig. 2.

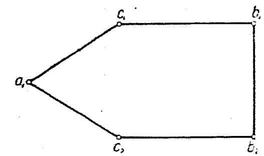

Fig. 3.

(N. B. We are not concerned with 2-chromatic graphs, because the only 2-chromatic critical graph is the complete 2-graph which has no isthmoid. Thus the case $k=2$ is excluded from our theorems).

We shall now deduce a specialised result concerning 6-chromatic graphs, using the inequalities which have just been established.

Suppose Γ is a critical 6-chromatic graph and $\mathcal{C}$ is any one of its longest circuits whose nodes are $a_1,a_2,\ldots,a_n,a_1$ in cyclic order round $\mathcal{C}$. (By considering one of the longest ways in the graph it is easy to see that $n\geqslant 6$).

A way between two non-neighbouring nodes of $\mathcal{C}$ having only its two end-nodes in common with $\mathcal{C}$ will be called a *chord* of $\mathcal{C}$. We shall call a chord *simple* if it is an edge by itself, otherwise we shall call it *composite*. Using this terminology we prove

Theorem 5. (i) *Any node a_i of $\mathcal{C}$ is connected by a chord to at leas one non-neighbouring node of $\mathcal{C}$.* (ii) *If it is connected by a chord to onet node a_j only, then it is not connected to a_j by an edge, and a_j is connected by a chord to at least one other node of $\mathcal{C}$ besides a_i. Further the sum of the degrees of a_i and a_j is at least* 13.

Because Γ is critical and 6-chromatic the degree of every node is at least five, so that each of the nodes in $\mathcal{C}$ is joined to at least three other nodes of Γ besides its two neighbours in $\mathcal{C}$.

Part (i) follows easily from the fact that Γ, being critical, contains no isthmus. Suppose on the contrary that one of the nodes of $\mathcal{C}, a_i$ say, is not connected by a chord to any node of $\mathcal{C}$. a_i has two neighbours on $\mathcal{C}$ which we denote by a_{i-1} and a_{i+1}, with the convention that if $i=1$ we understand a_{i-1} to mean a_n and if $i=n$ we understand a_{i+1} to mean a_1. By hypothesis a_i is not joined by an edge to any nodes of $\mathcal{C}$ except a_{i-1} and a_{i+1} so it is joined to a node b_i which is not a node of $\mathcal{C}$. Every way from b_i to a_{i-1} passes through a_i, hence a_i is an isthmus of Γ. For suppose a way from b_i does not pass through a_i. Let w denote this way. Starting from b_i and going towards a_{i-1} along w let a_x be the first node which w has in common with $\mathcal{C}$. Then $x \neq i$ by hypothesis and $x \neq i-1$ and $x \neq i+1$ because $\mathcal{C}$ is one of the longest circuits in Γ. Hence the portion of w connecting b_i and a_x together with the edge (a_i, b_i) forms a chord of $\mathcal{C}$ connecting a_i and a_x, which contradicts the hypothesis that a_i is not connected by a chord to any node of b. But Γ is critical and therefore has no isthmus. This contradiction proves (i).

Suppose that only one node of $\mathcal{C}$, a_j say, is connected to a_i by a chord. ($\mathcal{C}$ is one of the longest circuits of the graph so that $a_j \neq a_{i-1}$, $a_j \neq a_{i+1}$). Then the two nodes a_i and a_j form an isthmoid of the graph Γ, which splits Γ into two components Γ_a and Γ_b. Γ_a contains all those nodes which can be reached from a_i by ways which have no nodes except the starting node a_i in common with $\mathcal{C}$, Γ_b contains all other nodes of the graph, except of course a_i and a_j, in particular it contains all the nodes of $\mathcal{C}$ except a_i and a_j. In the same way we denote by Γ' and Γ'' the graphs obtained from Γ by deleting the nodes (and edges incident with them) of Γ_b and Γ_a respectively.

By Theorem 3 a_i and a_j cannot be joined by an edge in Γ. By (7) and (8) (with $k=6$) we have that: either

$$d(a_i, \Gamma'') \geqslant 4 \quad \text{and} \quad d(a_j, \Gamma'') \geqslant 4$$

or

$$d(a_i, \Gamma'') + d(a_j, \Gamma'') \geqslant 5.$$

The first alternative is the case if a_i and a_j are bound to have the same colour in any colouring of Γ'' with 5 colours, the second is the case if a_i and a_j are bound to have different colours in any colouring of Γ'' with 5 colours. It follows that which ever of these alternatives is the case, either $d(a_i, \Gamma'') \geqslant 3$ or $d(a_j, \Gamma'') \geqslant 3$.

By the definition of Γ' and Γ'' any node of Γ'' joined by an edge to a_i is a node of $\mathcal{C}$; and if a_j is joined by an edge to a third node of Γ'' besides its two neighbours a_{j-1} and a_{j-1} on $\mathcal{C}$ then a_j is joined to a node a_k of $\mathcal{C}$ ($a_k \neq a_i$ or a_{j-1} or a_{j+1}) by a simple or a composite chord.

(Proof. The first part of this statement follows immediately from the definition of Γ' and Γ''. If a_j is joined by an edge to a third node of Γ'' besides its two neighbours a_{j-1} and a_{j+1} on $\mathcal{C}$, this node cannot be a_i since by Theorem 3 $a_i \circ a_j$ in Γ. If $a_j \times a_k$ there is nothing to prove, so suppose that $a_j \times b_j, b_j$ not being among the nodes of $\mathcal{C}$. Now b_j is by hypothesis a node of Γ'' so that it cannot be reached from a_i by a way having no node except a_i in common with $\mathcal{C}$. If every way from b_j to a_i had to pass through a_j, a_j would be an isthmus of Γ, which is impossible by Theorem 2 since Γ is critical. There is therefore a way from b_j to a_i which does not pass through a_j, this way must have an intermediate node in common with $\mathcal{C}$, let the first of these nodes be a_k. The node a_k may depend on the way we have chosen, but $a_k \neq a_{j-1}$ and $a_k \neq a_{j+1}$ since otherwise $\mathcal{C}$ would not be one of the longest circuits of Γ. Then the way from b_j to a_k together with the edge (a_j, b_j) forms a composite chord linking a_j and a_k as required).

Since $d(a_i, \Gamma'') \geqslant 3$ or $d(a_j, \Gamma'') \geqslant 3$ either a_i or a_j must be joined by an edge to a third node of Γ'' besides its two neighbouring nodes on $\mathcal{C}$. Utilizing the above statement, we deduce that a_j is connected by a chord to another node of $\mathcal{C}$ besides a_i.

The last part follows at once from Theorem 4 on substituting $k=6$ since a_i and a_j form an isthmoid of Γ in the case contemplated.
