Les remarques qui précèdent ne concernent, bien entendu, que la démonstration de l'existence de la courbe C *plane* ayant pour l'ensemble de ses bouts l'ensemble $\Gamma\subset\mathcal{C}$ *quelconque*, donné d'avance. *Si l'on renonce à la première de ces conditions*, on peut remplacer, pour tout $q\in\Phi$, l'ensemble $W(q)+\Omega(q)$ par la circonférence de rayon égal à celui de $\Omega(q)$, mais située dans la moitié $z\geqslant 0$ du plan $x=q$ et passant par le point $(q,0,0)$. La démonstration devient alors tout à fait simple.

La construction continue cependant de fournir les courbes C non localement connexes aux points de $W(q)-\Omega(q)$ toutes les fois que q est un point-limite d'un sommande F_i de la série (6), c'est-à-dire lorsqu'il y existe un F_i infini, voire parfait — cas inévitable avec des G_δ $\Gamma\subset\mathcal{C}$ arbitraires. Mr Kuratowski nous a posé donc la question si tout ensemble B qui est un G_δ de dimension nulle est homéomorphe à l'ensemble des bouts d'une courbe C localement connexe. Or, *si l'on renonce à la seconde condition du théorème*, on peut (sans modifier essentiellement la construction) ramener cette question à celle du choix convenable d'un $\Gamma\subset\mathcal{C}$ homéomorphe à B. La solution sera publiée dans un autre travail [16]) sous la forme de celle du problème suivant [17]), qui nous semble présenter l'intérêt pour la topologie des courbes: le théorème (p. 17) subsiste-t-il en demandant que la courbe C dont il y est question soit une dendrite? En d'autres termes, *un G_δ de dimension 0 est-il toujours homéomorphe à l'ensemble de tous les bouts d'une dendrite?*

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[16]) B. Knaster et K. Urbanik, *Sur les espaces complets séparables de dimension* 0, ce volume, p. 194-202.

[17]) Nous devons ce problème à E. Čech.

An Example of an Absolute Neighbourhood Retract, Which Is the Common Boundary of Three Regions in the 3-dimensional Euclidean Space

By

M. Lubański (Warszawa)

1. Introduction. Continuum K lying in an n-dimensional Euclidean space E_n is said to show the *phenomenon of Brouwer* [1] if it disconnects the space E_n into m $(m\geqslant 3)$ regions and is their common boundary.

Examples of plane continua showing the above phenomenon have been given by L. E. J. Brouwer [2], A. Denjoy [3], Yoneyama [4] and B. Knaster [5]. All those continua have a highly complicated topological structure. According to C. Kuratowski [6], the complication of their structure is not accidental, because every such plane continuum is either an indecomposable continuum or a sum of two indecomposable continua.

An absolute neighbourhood retract [1]) R is said to show the *phenomenon of Mazurkiewicz* [1] if it cannot be decomposed into a finite sum of absolute retracts whose diameters are smaller than the diameter of R.

Examples of absolute retracts showing the latter phenomenon have been given by K. Borsuk and S. Mazurkiewicz [7].

The aim of this paper is to establish the following

Theorem. There exists an absolute neighbourhood retract $0\neq W\subset E_3$ which shows the phenomenon of Brouwer and which can be decomposed into a finite sum of absolute retracts, whose diameters are arbitrarily small [2]).

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[1]) A subset A of a space E is called a *retract* of E, if there exists a continuous mapping f (called a *retraction*) of E onto A, so that $f(x)=x$ for every $x\in A$. A compactum A is said to be an *absolute retract* resp. *absolute neighbourhood retract*, provided it is a retract of every space $E\supset A$ resp. of some neighbourhood in E. See K. Borsuk, *Sur les rétractes*, Fund. Math. 17 (1931), p. 152-170 and K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fund. Math. 19 (1932), p. 220-242.

[2]) The first example of an absolute neighbourhood retract which is the common boundary of three regions of the 3-dimensional Euclidean space was given by Mr. Gruba in 1937. That paper was never published and its manuscript was lost during the last war. It is not certain whether the absolute neighbourhood retract constructed by Mr. Gruba was decomposable into a finite sum of arbitrarily small absolute retracts.

The proof of this theorem will be given by means of constructing a set W having the properties mentioned. Let m denote the number of regions of the space E_3, the common boundary of which is W. The following reasoning concerns the case of $m=3$. The case of any finite $m>3$ should be treated in the same manner and hardly presents any difficulty.

The above theorem shows that, contrary to the plane E_2, in the space E_3 the phenomenon of Brouwer appears already among continua having a highly regular structure.

2. Notations. Let

$$Q \;=\underset{(x,y,z)}{E}\left[0\leqslant x\leqslant 1,\,0\leqslant y\leqslant 1,\,0\leqslant z\leqslant 1\right],$$

$$Q^n_{ijk}=\underset{(x,y,z)}{E}\left[\frac{i-1}{n}\leqslant x\leqslant\frac{i}{n},\frac{j-1}{n}\leqslant y\leqslant\frac{j}{n},\frac{k-1}{n}\leqslant z\leqslant\frac{k}{n}\right] \quad \begin{matrix}\text{for } n=2,4,6,\ldots\\ i,j,k=1,2,\ldots,n,\end{matrix}$$

$$Q \;=\sum_{i,j,k=1}^{n} Q^n_{ijk}.$$

The system of the n^3 sets Q^n_{ijk} for $i,j,k=1,2,3,\ldots,n$ is called the *n-partition of the cube Q*.

Let n denote a natural number. For every natural $q\leqslant n$ we write

$$z_{1q}=\begin{cases} q-\frac{3}{4} & \text{if } q \text{ is odd,}\\ q-\frac{1}{4} & \text{if } q \text{ is even,}\end{cases}$$

$$z_{2q}=\; q-\tfrac{1}{2},$$

$$z_{3q}=\begin{cases} q-\frac{1}{4} & \text{if } q \text{ is odd,}\\ q-\frac{3}{4} & \text{if } q \text{ is even.}\end{cases}$$

Let us write

$$K^r_{lq}(n)=\underset{(x,y,z)}{E}\left[x=0,\left(r-\frac{1}{16}\right)\frac{1}{n}\leqslant y\leqslant\left(r+\frac{1}{16}\right)\frac{1}{n},\left(z_{lq}-\frac{1}{16}\right)\frac{1}{n}\leqslant z\leqslant\left(z_{lq}+\frac{1}{16}\right)\frac{1}{n}\right]$$

for $l=1,2,3$, $r=1,2,\ldots,n-1$ and $q=1,2,\ldots,n$.

For every $l=1,2,3$ let us select from among the squares $K^r_{lq}(n)$, where $r=1,2,\ldots,n-1$; $q=1,2,\ldots,n$ a certain square which will be designated by $K_l(n)$. The system $\{K_1(n),K_2(n),K_3(n)\}$ will be called the *system of three initial squares*. Evidently the initial squares are disjoint.

3. A Lemma. *Let n be a natural number divisible by 4 and let $\{Q^n_{ijk}\}$ be the n-partition of the cube Q for which the initial squares are $K_1(n),K_2(n),K_3(n)$. Then there exist three sets $T_1(n),T_2(n),T_3(n)$ satisfying the conditions:*

1⁰ *$T_l(n)$ is homeomorphic with Q,*

2⁰ $T_l(n)\cdot T_{l'}(n)=0$ *for* $l\neq l'$,

3⁰ $T_l(n)\subset Q$,

4⁰ $T_l(n)\cdot\mathrm{Fr}(Q)=K_l(n)$ [3] *for* $l=1,2,3$,

5⁰ $\overline{Q^n_{ijk}-\sum_{l=1}^{3}T_l(n)}$ *is homeomorphic with Q,*

6⁰ $\dim\left[\overline{Q^n_{ijk}-\sum_{\lambda=1}^{3}T_\lambda(n)}\cdot T_l(n)\right]=2$ *for* $l=1,2,3$; $i,j,k=1,2,\ldots,n$,

and a retraction φ^n of the cube Q to the set $\overline{Q-\sum_{l=1}^{3}T_l(n)}$.

Proof. For every system of natural indices l,s,t such that $l\leqslant 3$, $s,t\leqslant n$ consider the point

$$a_{lst}=(x_{lst},\,y_{lst},\,z_{lst})$$

where

$$x_{lst}=\begin{cases}\dfrac{t}{n} & \text{if } s \text{ and } t \text{ are odd,}\\[2mm] \dfrac{n-t}{n} & \text{if } s \text{ is even and } t \text{ is odd,}\\[2mm] \dfrac{t-1}{n} & \text{if } s \text{ is odd and } t \text{ is even,}\\[2mm] \dfrac{n-t+1}{n} & \text{if } s \text{ and } t \text{ are even,}\end{cases}$$

$$y_{lst}=\begin{cases}\dfrac{1}{n} & \text{if } t=4k \text{ or } t=4k+1,\\[2mm] 1-\dfrac{1}{n} & \text{if } t=4k+2 \text{ or } t=4k+3,\end{cases}$$

$$z_{1st}=\begin{cases}\dfrac{4s-3}{4n} & \text{if } s \text{ is odd,}\\[2mm] \dfrac{4s-1}{4n} & \text{if } s \text{ is even,}\end{cases}$$

$$z_{2st}=\frac{2s-1}{2n}$$

$$z_{3st}=\begin{cases}\dfrac{4s-1}{4n} & \text{if } s \text{ is odd,}\\[2mm] \dfrac{4s-3}{4n} & \text{if } s \text{ is even.}\end{cases}$$

(See, for $n=8$ and $s=1$, fig. 1).

[3] The symbol $\mathrm{Fr}(X)$, where X is a subset of the space E_3 denotes the boundary of the set X, *i. e.* the set $\overline{X}\cdot\overline{E_3-X}$.

Designating by $\overline{a_{lst}\,a_{l,s,t+1}}$ the segment with the end points a_{lst} and $a_{l,s,t+1}$, we write

$$L_{ls}=\sum_{t=1}^{n-1}\overline{a_{lst}\,a_{l,s,t+1}}.$$

Let us designate by $J_{l\bar{s}}$, $l=1,2,3$; $s=1,2,\ldots,n-1$ a polygonal line, lying in the plane which contains the points $a_{l,\bar{s},n-1}$, $a_{l\bar{s}n}$, $a_{l,\bar{s}+1,1}$, and joining the point $a_{l\bar{s}n}$ with the point $a_{l,\bar{s}+1,1}$ as shown in fig. 2.

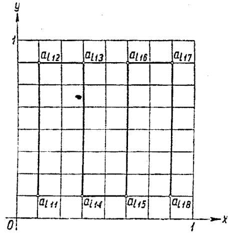

Fig. 1.

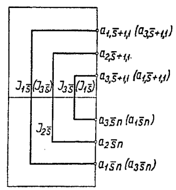

Fig. 2.

Let us write

$$L_l=L_{l1}+J_{l1}+L_{l2}+J_{l2}+\ldots+L_{l,n-1}+J_{l,n-1}+L_{ln}.$$

The arc L_l has as its end points $p_{l0}=a_{l11}$ and $p_{l,n^2-1}=a_{lnn}$. Moreover L_l contains n^2-2 points at which it makes a turn of a right angle. Let $p_{l1},p_{l2},\ldots,p_{l,n^2-2}$ be those points ordered in conformity with the orientation of L_l from p_{l0} to p_{l,n^2-1} and let $Q_{l\tau}$ denote, for $\tau=0,1,\ldots,n^2-1$, a closed cube with the center $p_{l\tau}$, with length of the edge $=1/8n$, the edges being parallel to the edges of Q. Now let us consider the minimal convex set $R_{l\tau}$ containing $Q_{l\tau}$ and $Q_{l,\tau+1}$ for $\tau=0,1,\ldots,n^2-2$.

Evidently the set $T_l'(n)=\sum_{\tau=0}^{n^2-2}R_{l\tau}$ is homeomorphic with Q.

From the construction of the set $T_l'(n)$ it follows that for every point $p=(0,y,z)\in K_l(n)$ the point $p'=(15/16n,y,z)$ is the next point of the set $T_l'(n)$. Let us designate by $T_l(n)$ the sum of the set $T_l'(n)$ and of all the segments $\overline{pp'}$ where $p\in K_l(n)$.

It is easy to see that the set $T_l(n)$ fulfils all the conditions 1^0—6^0. Moreover, since the boundary of $T_l(n)$ is homeomorphic with the surface of a sphere and the interior of the square $K_l(n)$ is homeomorphic with an open circle, therefore [8] the set $A_l(n)=\overline{\mathrm{Fr}(T_l(n))-K_l(n)}$ is homeomorphic with a closed circle. Hence there exists a retraction φ_l of the set $T_l(n)$ to the set $A_l(n)$.

Writing

$$\varphi^n(p)=\varphi_l(p)\quad\text{for}\quad p\in T_l(n),$$

$$\varphi^n(p)=p\quad\text{for}\quad p\in\overline{Q-\sum_{l=1}^{3}T_l(n)},$$

we obtain a retraction of the cube Q to the set $\overline{Q-\sum_{l=1}^{3}T_l(n)}$.

This completes the proof of our lemma.

4. The construction of the set W will be done by induction.

a) *The initial situation.* Let

$$C_1=\mathop{E}_{(x,y,z)}\left[-\frac{1}{2}\leqslant x\leqslant 0,\,0\leqslant y\leqslant\frac{1}{2},\,0\leqslant z\leqslant\frac{1}{2}\right],$$

$$C_2=\mathop{E}_{(x,y,z)}\left[0\leqslant x\leqslant\frac{1}{2},\,0\leqslant y\leqslant\frac{1}{2},\,0\leqslant z\leqslant\frac{1}{2}\right],$$

$$\Gamma_l^0=\mathop{E}_{(x,y,z)}\left[-\frac{1}{32}<x<\frac{1}{32},\,\frac{1}{4}-\frac{1}{32}<y<\frac{1}{4}+\frac{1}{32},\,\frac{1+l}{8}-\frac{1}{32}<z<\frac{1+l}{8}+\frac{1}{32}\right]$$

for $l=1,2,3$,

$$W_0=(C_1+C_2)-(\Gamma_1^0+\Gamma_2^0+\Gamma_3^0),\qquad \Gamma_3^{0'}=[E_3-(C_1+C_2)]+\Gamma_3^0.$$

We see at once that:

1) W_0 disconnects the space E_3 into three regions $\Gamma_1^0,\Gamma_2^0,\Gamma_3^{0'}$, *i. e.* $E_3-W_0=\Gamma_1^0+\Gamma_2^0+\Gamma_3^{0'}$, $\Gamma_1^0\cdot\Gamma_2^0=\Gamma_1^0\cdot\Gamma_3^0=\Gamma_2^0\cdot\Gamma_3^{0'}=0$,

2) W_0 is a polytope,

3) $W_0=C_1^0+C_2^0$, where $C_{l'}^0=C_{l'}-(\Gamma_1^0+\Gamma_2^0+\Gamma_3^0)$ $(l'=1,2)$ is homeomorphic with the cube Q, $\operatorname{diam}(C_{l'}^0)\leqslant 1$[4]) for $l'=1,2$,

4) $\dim(C_{l'}^0\cdot\Gamma_l^0)=2$ for $l'=1,2$; $l=1,2,3$.

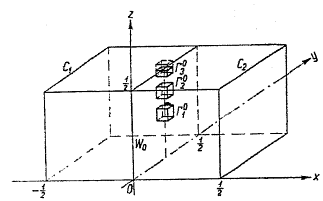

Fig. 3.

b) *The first stage of construction.* Let us consider the set $C_{l'}^0$ $(l'=1,2)$. By 3) there exists a homeomorphism $h_{0l'}$ so that $h_{0l'}(C_{l'}^0)=Q$. We can

[4]) The symbol diam (X) denotes the diameter of X. Cf. S. Lefschetz, *Introduction to Topology*, Princeton 1949, p. 29.

suppose that $h_{0l'}(C^0_{l'}\cdot\overline{\Gamma^0_l})\subset \underset{(x,y,z)}{E}[x=0, 0\leqslant y\leqslant 1, 0\leqslant z\leqslant 1]$. Let n_0 be such a natural number divisible by 4 that $\operatorname{diam}[h^{-1}_{0l'}(Q^{n_0}_{ijk})]\leqslant \frac{1}{2}$ and that the set $\{h_{0l}(C^0_l\cdot\overline{\Gamma^0_l})\}\subset \operatorname{Fr}(Q)$ contains a certain system of three initial squares $K_{1l'}(n_0), K_{2l'}(n_0), K_{3l'}(n_0)$.

Applying lemma 3 to the cube Q we infer the existence of a function $\varphi^0_{l'}=h^{-1}_{0l'}\varphi^{n_0}h_{0l'}$ which is a retraction of the set $C^0_{l'}$ to the set $\overline{C^0_{l'}-h^{-1}_{0l'}\Big(\sum_{l=1}^{3}T_{ll'}(n_0)\Big)}$, where $T_{ll'}(n_0)$ is the set $T_l(n_0)$ from lemma 3 for the initial squares $K_{1l'}(n_0)$, $K_{2l'}(n_0)$, $K_{3l'}(n_0)$.

The conditions 1^0—5^0 of the lemma give $\overline{C^0_{l'}-h^{-1}_{0l'}\Big(\sum_{l=1}^{3}T_{ll'}(n_0)\Big)}$ $=\sum_{i,j,k=1}^{n_0}h^{-1}_{0l'}\Big(\overline{Q^{n_0}_{ijk}-\sum_{l=1}^{3}T_{ll'}(n_0)}\Big)$, where $\overline{Q^{n_0}_{ijk}-\sum_{l=1}^{3}T_{ll'}(n_0)}$ is homeomorphic with Q, and the condition 6^0 gives $\dim\Big[h^{-1}_{0l'}\Big(\overline{Q^{n_0}_{ijk}-\sum_{l=1}^{3}T_{ll'}(n_0)}\Big)\cdot h^{-1}_{0l'}(T_{ll'}(n_0))\Big]=2$.

Let us write

$$U^1_l=\sum_{l'=1}^{2}\Big[C^0_{l'}-\overline{C^0_{l'}-h_{0l'}^{-1}(T_{ll'}(n_0))}\Big],\quad P_1=U^1_1+U^1_2+U^1_3.$$

If we set

$$f_1(p)=\varphi^0_{l'}(p)\quad \text{for}\quad p\in C^0_{l'}$$

then

1_1) f_1 is a retraction of W_0 to $W_1=W_0-P_1$,

2_1) W_1 disconnects the space E_3 into three regions; indeed, the sets $\Gamma^1_1=\Gamma^0_1+U^1_1$, $\Gamma^1_2=\Gamma^0_2+U^1_2$, $\Gamma^1_3=\Gamma^{0'}_3+U^1_3$ are regions such that $E_3-W_1=\Gamma^1_1+\Gamma^1_2+\Gamma^1_3$ and $\Gamma^1_1\cdot\Gamma^1_2=\Gamma^1_1\cdot\Gamma^1_3=\Gamma^1_2\cdot\Gamma^1_3=0$,

3_1) $W_1=\sum_{\nu=1}^{\nu_1}C^1_\nu$, where $C^1_\nu=h^{-1}_{0l'}\Big(\overline{Q^{n_0}_{ijk}-\sum_{l=1}^{3}T_{ll'}(n_0)}\Big)$ is homeomorphic with the cube Q, $\operatorname{diam}(C^1_\nu)\leqslant\frac{1}{2}$, $\dim(C^1_\nu\cdot\overline{\Gamma^1_l})=2$, $(l'=1,2;\ l=1,2,3)$, and ν_1 is a natural number,

4_1) $f_1(C^0_{l'})\subset C^0_{l'}$ and $\operatorname{diam}(C^0_{l'})\leqslant 1$ for $l'=1,2$.

c) *The induction.* Let us suppose that a set W_m has been constructed, possesing the following properties:

1_m) There is a function f_m which is a retraction of the set W_{m-1} to the set W_m,

2_m) W_m disconnects the space E_3 into three regions $\Gamma^m_1, \Gamma^m_2, \Gamma^m_3$, *i. e.* $E_3-W_m=\Gamma^m_1+\Gamma^m_2+\Gamma^m_3$ where $\Gamma^m_1\cdot\Gamma^m_2=\Gamma^m_1\cdot\Gamma^m_3=\Gamma^m_2\cdot\Gamma^m_3=0$,

3_m) $W_m=\sum_{\nu=1}^{\nu_m}C^m_\nu$, and C^m_ν is homeomorphic with Q, $\operatorname{diam}(C^m_\nu)\leqslant 1/(m+1)$ $\dim(C^m_\nu\cdot\overline{\Gamma^m_l})=2$ for $l=1,2,3$, and ν_m is a natural number,

4_m) $f_m(C^{m-1}_\nu)\subset C^{m-1}_\nu$ and $\operatorname{diam}(C^{m-1}_\nu)\leqslant 1/m$ for $\nu=1,2,\ldots,\nu_{m-1}$.

It is clear that the set W_1 fulfils the above conditions for $m=1$.

Supposing the set W_m has been given, the construction of the set W_{m+1} runs as follows:

We consider the set C^m_ν $(\nu=1,2,\ldots,\nu_m)$. By 3_m) there is a homeomorphism $h_{m\nu}$ so that $h_{m\nu}(C^m_\nu)=Q$. We can suppose that $h_{m\nu}(C^m_\nu\cdot\overline{\Gamma^m_l})\subset \underset{(x,y,z)}{E}[x=0, 0\leqslant y\leqslant 1, 0\leqslant z\leqslant 1]$. Let n_m be a natural number divisible by 4 and such that for every $\nu=1,2,\ldots,\nu_m$ it is $\operatorname{diam}[h^{-1}_{m\nu}(Q^{n_m}_{ijk})]\leqslant 1/(m+2)$ and that the set $\{h_{m\nu}(C^m_\nu\cdot\overline{\Gamma^m_l})\}\subset\operatorname{Fr}(Q)$ contains a certain system of three initial squares $K_{1\nu}(n_m), K_{2\nu}(n_m), K_{3\nu}(n_m)$. One can consequently apply lemma 3 to Q.

First of all we infer the existence of a retraction φ^m_ν $=h^{-1}_{m\nu}\varphi^{n_m}h_{m\nu}$ of the set C^m_ν to the set $\overline{C^m_\nu-h^{-1}_{m\nu}\Big(\sum_{l=1}^{3}T_{l\nu}(n_m)\Big)}$, where $T_{l\nu}(n_m)$ is the set $T_l(n_m)$ from lemma 3 constructed for the initial squares $K_{1\nu}(n_m)$, $K_{2\nu}(n_m)$, $K_{3\nu}(n_m)$.

The conditions 1^0—5^0 of the lemma give

$$\overline{C^m_\nu-h^{-1}_{m\nu}\Big(\sum_{l=1}^{3}T_{l\nu}(n_m)\Big)}=\sum_{i,j,k=1}^{n_m}h^{-1}_{m\nu}\Big(\overline{Q^{n_m}_{ijk}-\sum_{l=1}^{3}T_{l\nu}(n_m)}\Big),$$

where $\overline{Q^{n_m}_{ijk}-\sum_{l=1}^{3}T_{l\nu}(n_m)}$ is homeomorphic with Q. The condition 6^0 gives

$$\dim\Big[h^{-1}_{m\nu}\Big(\overline{Q^{n_m}_{ijk}-\sum_{l=1}^{3}T_{l\nu}(n_m)}\Big)\cdot h^{-1}_{m\nu}(T_{l\nu}(n_m))\Big]=2.$$

Let us write

$$U^{m+1}_l=\sum_{\nu=1}^{\nu_m}\Big[C^m_\nu-\overline{C^m_\nu-h^{-1}_{m\nu}(T_{l\nu}(n_m))}\Big],\qquad P_{m+1}=\sum_{l=1}^{3}U^{m+1}_l.$$

Since

$$\varphi^m_{\nu_1}(p)=\varphi^m_{\nu_2}(p)=p\qquad\text{for}\quad p\in C^m_{\nu_1}\cdot C^m_{\nu_2},\quad \nu_1\neq\nu_2,$$

we infer that setting

$$f_{m+1}(p)=\varphi^m_\nu(p)\qquad\text{for}\quad p\in C^m_\nu$$

we obtain a retraction of W_m to the set $W_{m+1}=W_m-P_{m+1}$.

Writing $\Gamma_l^{m+1} = \Gamma_l^m + U_l^{m+1}$ and $C_\nu^{m+1} = h_{m\nu}^{-1}\left(\overline{Q_{ijk}^{\nu m} - \sum_{l=1}^{3} T_{l\nu}(n_m)}\right)$ (for all systems (i,j,k), $\nu = 1,2,\dots,\nu_m$, $l = 1,2,3$) we infer that:

1_{m+1}) There exists a retraction f_{m+1} of W_m to W_{m+1},

2_{m+1}) the set W_{m+1} disconnects the space E_3 into three regions $\Gamma_1^{m+1}, \Gamma_2^{m+1}, \Gamma_3^{m+1}$,

3_{m+1}) $W_{m+1} = \sum_{\nu=1}^{\nu_{m+1}} C_\nu^{m+1}$, C_ν^{m+1} is homeomorphic with Q,

$$\operatorname{diam}(C_\nu^{m+1}) \leqslant 1/(m+2),\ \dim(C_\nu^{m+1} \cdot \overline{\Gamma_l^{m+1}}) = 2,$$

and ν_{m+1} is a natural number,

4_{m+1}) $f_{m+1}(C_\nu^m) \subset C_\nu^m$, $\operatorname{diam}(C_\nu^m) \leqslant 1/(m+1)$, for $\nu = 1,2,\dots,\nu_m$.

d) *The limit set W*. In this way we obtain the sequence of the sets $W_0, W_1, W_2, \dots$ and the sequence of continuous functions $f_1, f_2, f_3, \dots$ so that $f_m(W_{m-1}) = W_m$, $f_m(p) = p$ for $p \in W_m$, $m = 1,2,\dots$

Let $W = W_0 \cdot W_1 \cdot W_2 \cdot \dots$ and

(1) $$r_m(p) = f_m f_{m-1} \dots f_1(p) \quad \text{for} \quad p \in W_0, \quad m = 1,2,3,\dots$$

Then $r_m(W_0) = W_m$, $r_m(p) = p$ for $p \in W_m$, *i. e.* r_m is a retraction of the set W_0 to the set W_m.

By (1) and 4_m) we have

$$\varrho[r_{m+k}(p), r_m(p)] \leqslant \frac{1}{m} \quad \text{for} \quad p \in W_0, \quad k, m = 1,2,3,\dots$$

For every $\varepsilon > 0$ there exists a natural number n^* so that $n^* > 1/\varepsilon$. Then for every $p \in W_0$ and $n' > n'' \geqslant n^*$ there is

$$\varrho[r_{n'}(p), r_{n''}(p)] \leqslant \frac{1}{n''} \leqslant \frac{1}{n^*} < \varepsilon.$$

Also the sequence of the functions $\{r_m\}$ is uniformly convergent. Writing

$$r(p) = \lim_{m\to\infty} r_m(p) \quad \text{for every } p \in W_0$$

we obtain a continuous function r which maps the set W_0 onto the set $W = W_0 \cdot W_1 \cdot W_2 \cdot \dots$ and satisfies the condition $r(p) = p$ for $p \in W$. Thus the function r is a retraction of W_0 to the set W.

5. The properties of the set *W*. Let $\Gamma_1 = \sum_{n=0}^{\infty} \Gamma_1^n$, $\Gamma_2 = \sum_{n=0}^{\infty} \Gamma_2^n$, $\Gamma_3 = \Gamma_3^{0'} + \sum_{n=1}^{\infty} \Gamma_3^n$. Since $E_3 - W = \Gamma_1 + \Gamma_2 + \Gamma_3$, where W is closed, Γ_l $(l = 1,2,3)$ is a region and $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cdot \Gamma_3 = \Gamma_2 \cdot \Gamma_3 = 0$, hence

1*) The set W disconnects the space E_3 into three regions $\Gamma_1, \Gamma_2, \Gamma_3$.

The condition 3_m) gives $\varrho(p, \Gamma_1) = \varrho(p, \Gamma_2) = \varrho(p, \Gamma_3) = 0$ for $p \in W$. Evidently no point of the set W belongs to any of the regions $\Gamma_1, \Gamma_2, \Gamma_3$. Consequently:

2*) The set W is the common boundary of the regions $\Gamma_1, \Gamma_2, \Gamma_3$.

Since W is a retract of W_0, and W_0 is an absolute neighbourhood retract, therefore [9]:

3*) W is an absolute neighbourhood retract.

$\varepsilon > 0$ being given, there exists a natural number m so that $\varepsilon > 1/m$. Let us consider the set W_m. By 3_m) the set W_m is a finite sum of absolute retracts whose diameters are $< \varepsilon$.

Let C_ν^m be one of such absolute retracts.

Since $r(C_\nu^m) = W \cdot C_\nu^m = W_{m\nu}$ and $r(p) = p$ for $p \in W_{m\nu}$, hence $W_{m\nu}$ is an absolute retract. Consequently:

4*) For every $\varepsilon > 0$ the set W can be decomposed into a finite sum of absolute retracts with diameters $< \varepsilon$.

The properties 1*), 2*), 3*) and 4*) show that the set W fulfils the conditions required. Thus the theorem is proved.

6. As we have proved, for every $\varepsilon > 0$ the set W may be decomposed into a finite sum of the sets $W_{m\nu}$ that are absolute retracts. Obviously the common parts of such two sets are not absolute retracts generally. According to K. Borsuk [10], let us call the *regular decomposition* of a space A a finite sequence of absolute retracts $A_1, A_2, \dots, A_k$ which fulfil the following two conditions:

1. $A = \sum_{i=1}^{k} A_i$.

2. Each of the sets $A_{i_1} \cdot A_{i_2} \cdot \dots \cdot A_{i_l}$ either is empty or is an absolute retract.

The sets which possess regular decompositions constitute a larger class of spaces than the class of polytopes, but they constitute a narrower class than the class of absolute neighbourhood retracts. It seems wortwhile to investigate how much their structure differs from the topological structure of polytopes. Particularly interesting may be the following

Problem. *Can a set having a regular decomposition show the phenomenon of Brouwer?*

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A Solution of a Problem of R. Sikorski

By

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C. Kuratowski showed in his paper [1]) that there exist two 1-dimensional compact sets (on the plane) which are not homeomorphic to each other, although each of them is homeomorphic to a relatively open subset of the other. In this note we construct two 0-dimensional compact sets satisfying the same condition [2]), which give the answer to a problem of R. Sikorski [3]).

Let P be a given 0-dimensional perfect compact set and let p be a given point of P. Let Q_a be a countable compact set such that the a-th derivative $Q_a^{(a)}$ of Q_a consists of a single point q_a $(a<2\omega)$. Let $R_a = P\times q_a + p\times Q_a$ in the product space $P\times Q_a$.

Consider the set of points $p_n = 2^{(-2^{-n})}$, $n=0, \pm 1, \pm 2, \ldots$, and put $a_{mn}=(1/m, p_n)$, $m=\pm 1, \pm 2, \ldots$, on the plane.

We construct the sets $D_{mn} \ni a_{mn}$ on the plane as follows:

If m is positive and $m+n>0$, let D_{mn} be a topological image of R_{m+n}, where a_{mn} corresponds to (p, q_{m+n}), the diameter being less than

$$\frac{1}{2}\,\mathrm{Min}\left(\frac{1}{m}-\frac{1}{m+1},\, p_{|n|+1}-p_{|n|}\right).$$

If m is positive and $m+n\leqslant 0$, put $D_{mn}=a_{mn}$. If on the other hand m is negative and $|m|+n>0$, let D_{mn} be a topological image of $R_{\omega+|m|+n}$, where a_{mn} corresponds to $(p, q_{\omega+|m|+n})$, the diameter being less than

$$\frac{1}{2}\,\mathrm{Min}\left(\frac{1}{|m|}-\frac{1}{|m|+1},\, p_{|n|+1}-p_{|n|}\right).$$

If m is negative and $|m|+n\leqslant 0$, put $D_{mn}=a_{mn}$.

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[1]) C. Kuratowski, *On a Topological Problem Connected with the Cantor-Bernstein Theorem*, Fund. Math. **37** (1950), p. 213-216.

[2]) The construction of this example is essentially analogous to that of C. Kuratowski, which are 1-dimensional, for the proposition in his paper that two sets are not homeomorphic can be proved by the same method as in this paper.

[3]) Coll. Math. **1** (1947-48), p. 242. See also C. Kuratowski, loc. cit.