$\zeta < \alpha$, toute droite située dans P et distincte de chaque droite d_ξ, où $\xi \leqslant \zeta$, contienne au plus deux points de la suite $\{q_\xi\}_{\xi \leqslant \zeta}$. Il en résulte immédiatement que toute droite située dans P et distincte de chaque droite d_ξ, où $\xi < \alpha$, contient au plus deux points de la suite $\{q_\xi\}_{\xi < \alpha}$.

Soit T_α l'ensemble formé de toutes les droites d_ξ, où $\xi < \alpha$, et de toutes les droites du plan P qui passent par deux points au moins de la suite $\{q_\xi\}_{\xi < \alpha}$. Comme $\alpha < \varphi$, l'ensemble T_α est de puissance $< 2^{\aleph_0}$. Les points d'intersection de la droite d_α avec les droites de T_α, distinctes de d_α, forment donc un ensemble E_α de puissance $< 2^{\aleph_0}$ et la droite d_α contient $2^{\aleph_0}$ points qui n'appartiennent ni à E_α ni à la suite $\{q_\xi\}_{\xi < \alpha}$. Si l'ensemble des points de d_α qui sont des termes de la suite $\{q_\xi\}_{\xi < \alpha}$ est de puissance $\mathfrak{m}_{d_\alpha}$, posons $q_\alpha = q_1$; dans le cas contraire, soit q_α le premier terme de la suite $\{p_\xi\}_{\xi < \varphi}$, tel que $q_\alpha \,\bar{\in}\, E_\alpha$ et $q_\alpha \neq q_\xi$ pour $\xi < \alpha$.

On voit sans peine que toute droite d située dans le plan et distincte de chaque droite d_ξ, où $\xi \leqslant \alpha$, contient au plus deux points de la suite $\{q_\xi\}_{\xi \leqslant \alpha}$.

La suite transfinie $\{q_\xi\}_{\xi < \varphi}$ est ainsi définie par l'induction transfinie. Démontrer que l'ensemble S de tous les termes de cette suite satisfait aux conditions du théorème n'offre pas de difficulté.

An Extension of Sperner's Lemma, with Applications to Closed-Set Coverings and Fixed Points

By

F. Bagemihl (Rochester, N. Y.)

I. Introduction. The methods used in this paper are closely patterned after, and intended to enlarge to some extent the range of, those developed by Sperner, Knaster, Kuratowski, and Mazurkiewicz in [4] and [2]. We first introduce below the notion of an *n-dimensional m-plex*, which is, roughly speaking, what one obtains from an n-dimensional simplex by cutting out a finite number, $m-1$, of n-dimensional simplexes. Sperner's Lemma (see [2]; [3], p. 193; [4]) is then sharpened and extended (Lemma 1, Corollary 1) to m-plexes satisfying certain simple conditions (pertaining either to the nature of m or to the orientation of the constituent simplexes). This extension is applied to obtain generalizations (Theorem 1, Corollary 2) of theorems — one of Knaster, Kuratowski, and Mazurkiewicz (see [2]; [3], p. 194), which they used to give a proof of Brouwer's Fixed-Point Theorem, and one of Sperner (see [2]; [3], p. 194; [4]), which he used to give a proof of the invariance of dimension — on closed-set coverings; a fixed-point theorem (Theorem 2) for n-dimensional m-plexes with m odd, derived along the lines of the proof of Brouwer's Theorem given in [2] (or [3], p. 196); a corollary (Corollary 3) on retraction; and a generalization (Theorem 3) of Kakutani's theorem [1] on fixed points.

II. Preliminaries. Let S be an n-dimensional (closed) simplex with vertices $v_0, v_1, \ldots, v_n$; we shall write $S = (v_0 v_1 \ldots v_n)$. Its k-dimensional $(0 \leqslant k \leqslant n)$ face with vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_k}$ will be denoted by $v_{i_0} v_{i_1} \ldots v_{i_k}$. If $n > 0$, we shall denote by $S^+ = +(v_0 v_1 \ldots v_n)$ the oriented simplex obtained from S by giving its vertices the order of succession indicated by the order in which these vertices are written down.

An oriented n-dimensional simplex $+(v_0' v_1' \ldots v_n')$ is said to have the *same orientation* as $+(v_0 v_1 \ldots v_n)$, if, and only if, there exists a continuous deformation $D\{+(v_0' v_1' \ldots v_n')\} = +(v_0 v_1 \ldots v_n)$ such that $D(v_i') = v_i$ $(0 \leqslant i \leqslant n)$. If $+(v_0' v_1' \ldots v_n')$ does not have the same orientation as $+(v_0 v_1 \ldots v_n)$, then it is said to have the *opposite orientation*.

Suppose that $n \geqslant 1$. Let $S_1 = (v_0^{(1)} v_1^{(1)} \ldots v_n^{(1)})$, $S_2 = (v_0^{(2)} v_1^{(2)} \ldots v_n^{(2)})$, ..., $S_m = (v_0^{(m)} v_1^{(m)} \ldots v_n^{(m)})$ $(m \geqslant 1)$ be n-dimensional simplexes satisfying the following conditions:

(I) $S_i \subset S_1$ $(i > 1)$;
(II) the frontier of S_1 contains no point of S_i $(i > 1)$;
(III) $S_i \cap S_j = 0$ $(i, j > 1;\ i \neq j)$.

Denote by $S_1[S_2 S_3 \ldots S_m]$ the (closed) set of points of S_1 obtained by deleting from S_1 all points of S_i $(i > 1)$ which do not belong to the frontier of S_i. We call $S_1[S_2 S_3 \ldots S_m]$ an *n-dimensional m-plex* ($m = 1$, $S_1 =$ *simplex*; $m = 2$, $S_1[S_2] =$ *duplex*; $m = 3$, $S_1[S_2 S_3] =$ *triplex*; etc.). If each S_i is given one of the two possible orientations to form an oriented simplex S_i^+, then we speak of an *oriented m-plex* $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$. The vertices of $S_1, S_2, \ldots, S_m$ are called the *vertices* of the m-plex, and the k-dimensional $(0 \leqslant k < n)$ faces of $S_1, S_2, \ldots, S_m$ are called the *k-dimensional faces* of the m-plex. It is convenient to regard the m-plex itself as its n-dimensional face, and to denote this face by $v_0^{(1)} v_1^{(1)} \ldots v_n^{(1)}$.

We say that an n-dimensional m-plex is *divided simplicially* into subsimplexes, if it is divided into a finite number of n-dimensional simplexes, the intersection of every pair of which is either the empty set or a common k-dimensional face.

Let an n-dimensional m-plex $S_1[S_2 S_3 \ldots S_m]$ (oriented or not) be divided simplicially into subsimplexes. If, with every vertex w of these subsimplexes, there is associated a number $\varphi(w)$ such that

(1) *if w lies on a k-dimensional side $v_{i_0}^{(j)} v_{i_1}^{(j)} \ldots v_{i_k}^{(j)}$ of the m-plex, then $\varphi(w)$ is one of the numbers $i_0, i_1, \ldots, i_k$,*

then $\varphi(w)$ will be called a *vertex function* of this simplicial division of the m-plex.

Relative to a particular vertex function φ of a specific simplicial division of a given m-plex, a subsimplex $(w_0 w_1 \ldots w_n)$ such that $\varphi(w_i) = i$ $(0 \leqslant i \leqslant n)$ will be called a *representative subsimplex*. Let ϱ stand for the number of representative subsimplexes.

Suppose that $\varphi(w)$ is a vertex function of a simplicial division of an n-dimensional oriented m-plex $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$, where $S_1^+ = +(v_0^{(1)} v_1^{(1)} \ldots v_n^{(1)})$, and that $+(w_0 w_1 \ldots w_n)$ is an oriented representative subsimplex $(\varphi(w_i) = i$ for $0 \leqslant i \leqslant n)$. If $+(w_0 w_1 \ldots w_n)$ has the same orientation as $+(v_0^{(1)} v_1^{(1)} \ldots v_n^{(1)})$, we call the former a *positive representative subsimplex*, otherwise, a *negative representative subsimplex*. Denote the number of positive (negative) representative subsimplexes by $\varrho_P(\varrho_N)$. Obviously $\varrho = \varrho_P + \varrho_N$.

Let the number of the simplexes $S_2^+, S_3^+, \ldots, S_m^+$ whose orientation is the same as (opposite of) that of S_1^+ be π (ν), so that we have $\pi + \nu = m - 1$; such a simplex will be referred to as a *π- (ν-) simplex*.

Consider a subsimplex $T = (t_0 t_1 \ldots t_n)$ of a simplicial division, such that $\varphi(t_i) = i$ $(0 \leqslant i \leqslant n-1)$. If the orientation of $+(t_0 t_1 \ldots t_n)$ is the same as (opposite of) that of $+(v_0^{(1)} v_1^{(1)} \ldots v_n^{(1)})$, we say that $+t_0 t_1 \ldots t_{n-1}$ is a *positive (negative) representative face of T*. Let us denote the number of positive (negative) representative faces of the subsimplex T by $a_P(T)(a_N(T))$. When we refer to $+t_0' t_1' \ldots t_{n-1}'$ as a *positive (negative) representative face*, we mean simply that it is a positive (negative) representative face of some subsimplex T'. Because of the assumed simpliciality of the division, if $+t_0'' t_1'' \ldots t_{n-1}''$ is an $(n-1)$-dimensional face of some subsimplex, this face is on the frontier of $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$ if, and only if, it is a face of precisely one subsimplex. On the other hand, it is not on the frontier of $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$ if, and only if, it is a face of precisely two subsimplexes. In this case, if $\varphi(t_i'') = i$ $(0 \leqslant i \leqslant n-1)$, it is a positive representative face of one of these subsimplexes, and a negative representative face of the other one. Let the number of positive (negative) representative faces on the frontier of $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$ be $\sigma_P(\sigma_N)$, and let the number of positive (negative) representative faces on the frontier of S_j^+ $(1 \leqslant j \leqslant m)$ be $\sigma_P^{(j)}(\sigma_N^{(j)})$.

By the term *representative face* we shall mean an $(n-1)$-dimensional face $w_0 w_1 \ldots w_{n-1}$ of some subsimplex of the division, such that $\varphi(w_i) = i$ $(0 \leqslant i \leqslant n-1)$. Let ψ stand for the number of representative faces which are not on the frontier of $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$.

If $w_0 w_1 \ldots w_{n-1}$ is a representative face on the frontier of $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$, then, because of (1), $w_0 w_1 \ldots w_{n-1}$ must lie on $v_0^{(j)} v_1^{(j)} \ldots v_{n-1}^{(j)}$ for some j. If, in addition, $j > 1$ and the orientation of $+w_0 w_1 \ldots w_{n-1}$ is the same as (opposite of) that of $+v_0^{(j)} v_1^{(j)} \ldots v_{n-1}^{(j)}$, then $+w_0 w_1 \ldots w_{n-1}$ is a positive (negative) representative face if S_j^+ is a ν-simplex, but a negative (positive) representative face if S_j^+ is a π-simplex.

By an *incomplete subsimplex* (let the number of such subsimplexes be χ) we mean a subsimplex $T = (t_0 t_1 \ldots t_n)$ such that $\varphi(t_i) = i$ $(0 \leqslant i \leqslant n-1)$ and $\varphi(t_n) \neq n$. Clearly $\varphi(t_n) = i_0$, where $0 \leqslant i_0 \leqslant n-1$. If $+t_0 t_1 \ldots t_{i_0} \ldots t_{n-1}$ is a positive (negative) representative face of T, then $+t_0 t_1 \ldots t_n \ldots t_{n-1}$ is a negative (positive) representative face of T, and these two are the only representative faces of T.

III. ***Lemma* 1.** *Let φ be a vertex function of a simplicial division of an oriented n-dimensional m-plex. Then*

$$\varrho_P + \pi = \varrho_N + \nu + 1. \tag{2}$$

Proof. Let the m-plex in question be $S_1^+[S_2^+ S_3^+ \ldots S_m^+]$. We shall prove the lemma by induction on n, verifying it for $n = 1$, and, simultaneously, for n under the assumption that it is true for some $n - 1 \geqslant 1$.

We have

(3) $\Sigma a_P(T) = \varrho_P + \chi$ and (3′) $\Sigma a_N(T) = \varrho_N + \chi$.

Here, as well as in what follows, the summation is extended over all subsimplexes T of the simplicial division.

We shall prove (3); the proof of (3′) is entirely analogous. Given any T, it is one of the following (the numbers indicate the contribution of T to $\Sigma a_P(T)$, ϱ_P, χ, respectively):

(a) a positive representative subsimplex $(1=1+0)$,

(b) an incomplete subsimplex $(1=0+1)$,

(c) neither (a) nor (b) $(0=0+0)$.

Each T thus contributes the same to the left as to the right of (3), so that the equality holds.

We also have

$$(4)\qquad \Sigma a_P(T)=\sigma_P+\psi \quad \text{and} \quad (4')\quad \Sigma a_N(T)=\sigma_N+\psi.$$

We may again confine ourselves to the first equality. Consider any $(n-1)$-dimensional face of any subsimplex of the simplicial division. This face is one of the following (the numbers referring this time to the terms in (4)):

(a′) a positive representative face on the frontier of the m-plex $(1=1+0)$,

(b′) a negative representative face on the frontier of the m-plex $(0=0+0)$,

(c′) a representative face not on the frontier of the m-plex $(1=0+1)$,

(d′) not a representative face $(0=0+0)$.

Evidently (4) is true.

From (3) and (4) we obtain

$$(5)\qquad \varrho_P=\sigma_P+(\psi-\chi),$$

and (3′) and (4′) yield

$$(5')\qquad \varrho_N=\sigma_N+(\psi-\chi).$$

Eliminating $\psi-\chi$ from (5) and (5′), we find that

$$(6)\qquad \varrho_P-\varrho_N=\sigma_P-\sigma_N=\sum_{j=1}^{m}(\sigma_P^{(j)}-\sigma_N^{(j)}).$$

If $n=1$, it is easily seen that $\sigma_P^{(1)}=1$, $\sigma_N^{(1)}=0$, and that $\sigma_N^{(j)}=1$ or 0, and $\sigma_P^{(j)}=0$ or 1, according as S_j^+ $(j>1)$ is a π- or a ν-simplex. Hence, $\sigma_P-\sigma_N=1-\pi+\nu$, and substituting this value in (6), we see that (2) is true for $n=1$.

If $n>1$, the induction hypothesis, applied to the vertex function φ of the simplicial division of $+v_0^{(1)}v_1^{(1)}\dots v_{n-1}^{(1)}$ induced by the given simplicial division of the m-plex, yields $\sigma_P^{(1)}-\sigma_N^{(1)}=1$. The same argument applied to $+v_0^{(j)}v_1^{(j)}\dots v_{n-1}^{(j)}$ $(j>1)$ shows that $\sigma_P^{(j)}-\sigma_N^{(j)}=-1$ or $+1$ according as S_j^+ is a π- or a ν-simplex. Thus again $\sigma_P-\sigma_N=1-\pi+\nu$.

This completes the proof of the lemma.

Corollary 1. *If $\pi\neq\nu+1$, then $\varrho>0$. If m is odd, then (even if the m-plex is not oriented) ϱ is odd (and hence $\varrho>0$).*

For if we add ϱ_N to both sides of (2), we obtain

$$\varrho=2\varrho_N+\nu-\pi+1,$$

which shows that $\varrho>0$ if $\nu\geqslant\pi$. The addition of ϱ_P to both sides of (2) yields

$$\varrho=2\varrho_P+\pi-\nu-1,$$

which implies that $\varrho>0$ if $\pi>\nu+1$. If we combine these two results, we get the first part of Corollary 1, and the second part follows easily from the fact that $m=\pi+\nu+1$.

The assertion that ϱ is odd if $m=1$, is "Sperner's Lemma".

IV. Theorem 1. *Let $C_0, C_1, \dots, C_n$ be closed sets such that every k-dimensional face $v_{i_0}^{(j)}v_{i_1}^{(j)}\dots v_{i_k}^{(j)}$ of the n-dimensional m-plex $S_1^+[S_2^+S_3^+\dots S_m^+]$ is contained in the union $C_{i_0}\cup C_{i_1}\cup\dots\cup C_{i_k}$. If $\pi\neq\nu+1$, or if m is odd (in which case it is not necessary to assume that the m-plex is oriented), then $C_0\cap C_1\cap\dots\cap C_n\neq 0$.*

Proof. For a fixed natural number d, consider a simplicial division of the given m-plex into subsimplexes of diameter less than $1/d$. Let w be an arbitrary vertex of any one of these subsimplexes, and let $v_{i_0}^{(j)}v_{i_1}^{(j)}\dots v_{i_k}^{(j)}$ be that face (of the m-plex) of lowest dimension, which contains w. By hypothesis, the face $v_{i_0}^{(j)}v_{i_1}^{(j)}\dots v_{i_k}^{(j)}$ is contained in the union $C_{i_0}\cup C_{i_1}\cup\dots\cup C_{i_k}$, and consequently there is at least one (specific) i_h $(0\leqslant h\leqslant k)$ such that

$$(7)\qquad w\in C_{i_h}.$$

Put

$$(8)\qquad \varphi(w)=i_h.$$

Then (1) is satisfied, so that φ is a vertex function of the simplicial division of the m-plex. From (7) and (8) we see that

$$(9)\qquad w\in C_{\varphi(w)}.$$

By hypothesis, either $\pi\neq\nu+1$ or m is odd, so that according to Corollary 1, there exists a representative subsimplex, which we shall denote by $(w_0^d\, w_1^d\dots w_n^d)$, where $\varphi(w_i^d)=i$ $(0\leqslant i\leqslant n)$. From (9) we obtain

$$(10)\qquad w_i^d\in C_i \qquad (0\leqslant i\leqslant n).$$

Now let d tend to infinity. We may assume that $y=\lim_{d\to\infty} w_0^d$ exists, and since the diameters, $1/d$, of the subsimplexes tend to 0, we have

$$(11)\qquad y=\lim_{d\to\infty} w_i^d \qquad (0\leqslant i\leqslant n).$$

Since, by assumption, the sets C_i are closed, it follows from (10) and (11) that $y\in C_0\cap C_1\cap\dots\cap C_n$, and the theorem is proved.

Corollary 2. Let the n-dimensional m-plex $S_1^+[S_2^+S_3^+\ldots S_m^+]$ be contained in the union of the closed sets $C_0, C_1, \ldots, C_n$, and let the intersection of the face $s_i^{(j)} = v_0^{(j)}v_1^{(j)}\ldots v_{i-1}^{(j)}v_{i+1}^{(j)}\ldots v_n^{(j)}$ with C_i $(0\leqslant i\leqslant n;\ 1\leqslant j\leqslant m)$ be empty. If $\varkappa\neq\nu+1$, or if m is odd (in which case it is not necessary to assume that the m-plex is oriented), then $C_0\cap C_1\cap\ldots\cap C_n\neq 0$.

Proof. An arbitrary face $v_{i_0}^{(j)}v_{i_1}^{(j)}\ldots v_{i_k}^{(j)}$ of the m-plex is contained in any face $s_i^{(j)}$ (with the same j) for which i is none of the numbers $i_0, i_1, \ldots, i_k$; and therefore, since, by the hypothesis of Corollary 2, $v_{i_0}^{(j)}v_{i_1}^{(j)}\ldots v_{i_k}^{(j)}$ contains no point of any C_i with such an index i, this face must be contained in the union of the remaining closed sets. Thus, the hypothesis of Theorem 1 is satisfied, which implies the conclusion of Corollary 2.

V. ***Theorem 2.*** *Let $S_1[S_2S_3\ldots S_m]$ be an n-dimensional m-plex, with m odd, and let f be a continuous mapping of this m-plex into the n-dimensional Euclidean space containing it, such that the frontier of the simplex S_j $(1\leqslant j\leqslant m)$ is mapped into the simplex S_j. Then the m-plex has at least one fixed point under the mapping f.*

Proof. Let $S_j = (v_0^{(j)}v_1^{(j)}\ldots v_n^{(j)})$ $(1\leqslant j\leqslant m)$.

Assume first that every k-dimensional face $v_{i_0}^{(j)}v_{i_1}^{(j)}\ldots v_{i_k}^{(j)}$ of S_j $(j>1)$ is parallel to the corresponding face $v_{i_0}^{(1)}v_{i_1}^{(1)}\ldots v_{i_k}^{(1)}$ of S_1. Regard the vertices $v_i^{(1)}$ $(0\leqslant i\leqslant n)$ as vectors of n-dimensional space, and let the points x of the given m-plex be represented barycentrically in the form

$$x = b_0v_0^{(1)} + b_1v_1^{(1)} + \ldots + b_nv_n^{(1)}, \tag{12}$$

where every $b_i\geqslant 0$, and $b_0+b_1+\ldots+b_n=1$. Let $x'=f(x)$. Then x' has a unique barycentric representation:

$$x' = b_0'v_0^{(1)} + b_1'v_1^{(1)} + \ldots + b_n'v_n^{(1)}.$$

Because of the assumed special position of the simplexes S_j relative to S_1, there exist fixed, nonnegative numbers $a_i^{(j)}$ $(0\leqslant i\leqslant n;\ 1\leqslant j\leqslant m)$ with the following property: If x is on the $(n-1)$-dimensional face $v_0^{(j)}v_1^{(j)}\ldots v_{h-1}^{(j)}v_{h+1}^{(j)}\ldots v_n^{(j)}$ of S_j, then, for the h-th coordinate of x in (12), we have

$$b_h = a_h^{(j)}.$$

(In particular, if $j=1$, $a_i^{(j)}=0$ for every i). Since, by hypothesis, the frontier of S_j is mapped into S_j,

$$b_h' \geqslant a_h^{(j)}.$$

Consequently, if x is on the k-dimensional face $v_{i_0}^{(j)}v_{i_1}^{(j)}\ldots v_{i_k}^{(j)}$ of S_j, and if $v_{i_1'}^{(j)}, v_{i_2'}^{(j)}, \ldots, v_{i_{n-k}'}^{(j)}$ are the remaining vertices of S_j, then

$$b_{i_1'} = a_{i_1'}^{(j)},\ b_{i_2'} = a_{i_2'}^{(j)}, \ldots, b_{i_{n-k}'} = a_{i_{n-k}'}^{(j)}, \tag{13}$$
$$b_{i_0} + b_{i_1} + \ldots + b_{i_k} = 1 - (a_{i_1'}^{(j)} + a_{i_2'}^{(j)} + \ldots + a_{i_{n-k}'}^{(j)}),$$

and

$$b_{i_1'}' \geqslant a_{i_1'}^{(j)},\ b_{i_2'}' \geqslant a_{i_2'}^{(j)}, \ldots, b_{i_{n-k}'}' \geqslant a_{i_{n-k}'}^{(j)}. \tag{14}$$

Let C_i be the set of points x of the m-plex, for which $b_i'\leqslant b_i$. Due to the continuity of f, the sets $C_0, C_1, \ldots, C_n$ are closed.

We shall show that they satisfy the hypothesis of Theorem 1. Indeed, suppose that there were a point x of a k-dimensional face $v_{i_0}^{(j)}v_{i_1}^{(j)}\ldots v_{i_k}^{(j)}$ of the m-plex, which did not belong to the union $C_{i_0}\cup C_{i_1}\cup\ldots\cup C_{i_k}$. Then we should have

$$b_{i_0}' > b_{i_0},\ b_{i_1}' > b_{i_1}, \ldots, b_{i_k}' > b_{i_k},$$

and hence, by (13),

$$b_{i_0}' + b_{i_1}' + \ldots + b_{i_k}' > 1 - (a_{i_1'}^{(j)} + a_{i_2'}^{(j)} + \ldots + a_{i_{n-k}'}^{(j)}). \tag{15}$$

On the other hand, by (14),

$$b_{i_1'}' + b_{i_2'}' + \ldots + b_{i_{n-k}'}' \geqslant a_{i_1'}^{(j)} + a_{i_2'}^{(j)} + \ldots + a_{i_{n-k}'}^{(j)}. \tag{16}$$

Combining (15) and (16), we should obtain

$$b_0' + b_1' + \ldots + b_n' > 1,$$

which is impossible.

Let $y\in C_0\cap C_1\cap\ldots\cap C_n$; y exists according to Theorem 1. By the definition of C_i, we have, for $x=y$:

$$b_0'\leqslant b_0,\ b_1'\leqslant b_1, \ldots, b_n'\leqslant b_n,$$

and hence

$$1 = b_0' + b_1' + \ldots + b_n' \leqslant b_0 + b_1 + \ldots + b_n = 1.$$

Consequently $b_i'=b_i$, and therefore $y'=y$, *i. e.*, y is a fixed point under the mapping f.

Now let us remove from our m-plex $S_1[S_2S_3\ldots S_m]$ the special assumption made at the beginning of the proof. Let $S_1^*[S_2^*S_3^*\ldots S_m^*]$ be an auxiliary m-plex, however, for which the special assumption does hold, and which is the image of $S_1[S_2S_3\ldots S_m]$ under a homeomorphism g of S_1 onto S_1^* such that $g(S_j)=S_j^*$ $(1\leqslant j\leqslant m)$. Then, according to the first part of our proof, the mapping gfg^{-1} has a fixed point $x^*\in S_1^*[S_2^*S_3^*\ldots S_m^*]$, so that

$$gfg^{-1}(x^*) = x^*,$$

and hence

$$fg^{-1}(x^*) = g^{-1}(x^*),$$

which means that the point $g^{-1}(x^*)\in S_1[S_2S_3\ldots S_m]$ is fixed under f.

This completes the proof of Theorem 2.

VI. Remarks. The following example shows that if $\pi=\nu+1$ or if m is even, then the conclusion of Theorem 1 or of Corollary 1 need not hold.

Let ν be a nonnegative integer, $S_1^+=+(v_0^{(1)}v_1^{(1)}\dots v_n^{(1)})$ be an n-dimensional oriented simplex, and w be the barycenter of the $(n-1)$-dimensional face, $v_1^{(1)}v_2^{(1)}\dots v_n^{(1)}$, of S_1^+. (If $n=1$, then $w=v_1^{(1)}$). On the open segment $v_0^{(1)}w$, choose $2\nu+1$ points, $z_0, y_1, z_1, y_2, z_2, \dots, y_\nu, z_\nu$ in the order $v_0^{(1)}, z_0, y_1, z_1, y_2, z_2, \dots, y_\nu, z_\nu, w$. It will be convenient to denote $v_0^{(1)}$ by y_0.

Call the n-dimensional simplex

$$(y_i\, z_i\, v_1^{(1)} v_2^{(1)}\dots v_{j-1}^{(1)} v_{j+1}^{(1)}\dots v_n^{(1)}) \qquad (0\leqslant i\leqslant \nu;\ 1\leqslant j\leqslant n)$$

$A_{i\tau(j)}$, where $\tau(j)=j+1$ for $j=1,2,\dots,n-1$, and $\tau(n)=0$. Put

$$C'_k=\bigcup_{0\leqslant i\leqslant \nu} A_{ik} \qquad (0\leqslant k\leqslant n;\ k\neq 1)$$

and

$$C'_1=\overline{S_1-\bigcup_{\substack{0\leqslant k\leqslant n\\ k\neq 1}} C'_k}$$

(where $\overline{X}$ denotes the closure of the set X). The sets C'_k $(0\leqslant k\leqslant n)$ are obviously closed, and

$$\bigcap_{0\leqslant k\leqslant n} C'_k=\{z_0, y_1, z_1, y_2, z_2, \dots, y_\nu, z_\nu\}. \tag{17}$$

On the open segment $z_\nu w$, choose a point $u_{\nu 1}$. If $\nu>0$, then on the open segment $z_r y_{r+1}$ $(0\leqslant r\leqslant \nu-1)$ choose two points, $u_{r1}, u'_{r+1,1}$, in the order $z_r, u_{r1}, u'_{r+1,1}, y_{r+1}$. The points chosen all belong to C'_1.

On the open segment $y_0 z_0$, take a point h_0, and let H_0 be the $(n-1)$-dimensional hyperplane which contains h_0 and is perpendicular to the segment $y_0 w$. If $\nu>0$, then on the open segment $y_r z_r$ $(1\leqslant r\leqslant \nu)$ take two points, h'_r, h_r, in the order y_r, h'_r, h_r, z_r. Let H'_r, H_r be the $(n-1)$-dimensional hyperplane which contains h'_r, h_r, respectively, and is perpendicular to the segment $y_0 w$.

In the interior of the simplex A_{0k} $(0\leqslant k\leqslant n;\ k\neq 1)$, select a point u_{0k} which belongs to H_0. If $\nu>0$, then in the interior of the simplex A_{rk} $(1\leqslant r\leqslant \nu;\ 0\leqslant k\leqslant n;\ k\neq 1)$, select two points, u'_{rk}, u_{rk}, which belong to H'_r, H_r, respectively.

Consider the following $2\nu+1$ oriented n-dimensional simplexes:

$$R_r^+=+(u_{r0}\, u_{r1}\dots u_{rn}) \qquad (0\leqslant r\leqslant \nu), \tag{18}$$

$$R_{r'}^+=+(u'_{r0}\, u'_{r1}\dots u'_{rn}) \qquad (1\leqslant r\leqslant \nu); \tag{19}$$

(the simplexes (19) are defined only if $\nu>0$). These simplexes are mutually exclusive and lie in the interior of S_1^+, so that they may be regarded as constituent simplexes of an oriented n-dimensional m-plex $R_1^+[R_0^+R_1^+\dots R_\nu^+R_{1'}^+R_{2'}^+\dots R_{\nu'}^+]$, where $m=2\nu+2$, an even number. The $\nu+1$ simplexes (18) are π-simplexes, the ν simplexes (19) are ν-simplexes, and consequently $\pi=\nu+1$.

Let C_i $(0\leqslant i\leqslant n)$ be the intersection of C'_i with the m-plex just defined; obviously C_i is closed. These closed sets and our m-plex satisfy the condition expressed in the first sentence of Theorem 1. From the construction of the simplexes (18) and (19), it is clear that the m-plex does not contain the points $z_0, y_1, z_1, y_2, z_2, \dots, y_\nu, z_\nu$. This means, if we bear (17) in mind, that

$$C_0\cap C_1\cap\dots\cap C_n=0.$$

Thus, the conclusion of Theorem 1 is false. This implies, in view of the proof of Theorem 1, that $\varrho=0$ for some vertex function of some simplicial division of our m-plex, so that the conclusion of Corollary 1 is false too.

It is also possible to give an example which shows that Corollary 2 may fail to hold if $\pi=\nu+1$ or if m is even.

Theorem 2 is obviously false if $m=2$, for there is no fixed point under the mapping of a duplex $S_1[S_2]$ into the barycenter of S_2.

Theorem 2 has the following corollary (cf. [3], p. 197):

Corollary 3. *The frontier of an n-dimensional m-plex $S_1[S_2S_3\dots S_m]$, with m odd, is not a retract of the m-plex.*

Proof. Let p be a point on the frontier of the m-plex; p, then, is on the frontier of precisely one S_j. By $\tilde{p}$ we mean the point of intersection of the frontier of S_j with the ray emanating from p and passing through the barycenter of S_j. Clearly $\tilde{p}\neq p$, and $\tilde{\tilde{p}}=p$.

Now suppose that there existed a retraction, f, of the m-plex onto its frontier. Then, by definition (see [3], p. 75),

$$f(x)=x \text{ for every } x \text{ on the frontier of the } m\text{-plex}. \tag{20}$$

According to Theorem 2, the function $\widetilde{f(x)}$ has a fixed point, *i. e.*, there exists a point x_0 (naturally on the frontier of the m-plex) such that $\widetilde{f(x_0)}=x_0$. Hence, $f(x_0)=\tilde{x}_0\neq x_0$, contradicting (20).

Theorem 2 can be used to extend Kakutani's generalization of the fixed-point theorem of Brouwer. The proof of this extension is formally so analogous to Kakutani's proof of his Theorem 1 in [1], that, after making a few necessary remarks, we may refer the reader to Kakutani's paper for the details of the proof. With this in mind, our notation and terminology will be chosen as close to Kakutani's as possible.

Let $S=S_1[S_2S_3\dots S_m]$ be an r-dimensional m-plex with m odd, and R be a closed, bounded region (in r-dimensional Euclidean space) containing S. Denote by $\mathfrak{K}(R)$ the family of all nonempty, closed, convex subsets of R.

A point-to-set mapping $x \to \Phi(x) \in \mathfrak{K}(R)$ of S into $\mathfrak{K}(R)$ is called *upper semi-continuous*, if $\lim x_n = x_0$, $y_n \in \Phi(x_n)$, and $\lim y_n = y_0$ imply that $y_0 \in \Phi(x_0)$. Our extension of Kakutani's theorem may be stated as follows:

Theorem 3. *Let $x \to \Phi(x)$ be an upper semi-continuous point-to-set mapping of an r-dimensional m-plex $S = S_1[S_2 S_3 \dots S_m]$, with m odd, into $\mathfrak{K}(R)$, such that if x is on the frontier of S_j $(1 \leqslant j \leqslant m)$, then $\Phi(x)$ is a subset of the simplex S_j. Then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

By the *n-th barycentric simplicial subdivision* of the m-plex S, we mean the simplicial division of S determin d by the n-th barycentric subdivision of every subsimplex of some fixed simplicial division of S. If, now, in Kakutani's proof, we replace the appeal to Brouwer's Theorem by an appeal to our Theorem 2 proved above, and make several obvious minor modifications, we obtain a proof of our Theorem 3.

Bibliography

[1] Kakutani, S., *A generalization of Brouwer's fixed point theorem*, Duke Math. J **8** (1941), p. 457-459.

[2] Knaster, B., Kuratowski, C. and Mazurkiewicz, S., *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. **14** (1929), p. 132-137.

[3] Kuratowski. C., *Topologie I*, 2d ed., Warszawa-Wrocław 1948.

[4] Sperner, E., *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abh. math. Semin. Hamburg. Univ. 6 (1928), p. 265-272.

Sur la caractérisation topologique de l'ensemble des bouts d'une courbe

Par

B. Knaster et M. Reichbach (Wrocław)

Généralités. Nous entendons ici par *courbe* tout continu (ensemble compact et connexe dans un espace séparable) de dimension 1 au sens de Menger [1]), par l'*ordre* du point p d'un ensemble E, en symbole: $\operatorname{ord}(p, E)$, le plus petit nombre cardinal pour lequel il existe dans tout entourage de p (ensemble ouvert contenant p) un entourage du même point dont la frontière a exactement ce nombre des points communs avec E, enfin par *bout* (extrémité) de E — tout point pour lequel $\operatorname{ord}(p, E) = 1$ [2]). Cette égalité implique que $p \in E$ lorsque E est fermé.

Le livre — déjà classique — de Menger [3]), auquel ces notions sont empruntées, contient des théorèmes dont il résulte en particulier que l'ensemble C^1 des bouts d'une courbe C quelconque est un G_δ de dimension 0 [4]). La question s'impose, si les deux dernières propriétés nécessaires sont déjà caractéristiques, c'est-à-dire à la fois suffisantes pour qu'un ensemble soit celui des bouts d'une courbe. Les considérations qui suivent donnent réponse à cette question.

Il y a d'abord lieu de fixer ce qu'il y est à caractériser *topologiquement*. Etant donné un ensemble B, l'existence d'une courbe C telle que $B = C^1$ n'est un invariant de l'homéomorphie de B ni dans des espaces topologiques fort pauvres (tels, en particulier, que la dendrite Δ [5]) dont nous ferons, à la fin, un usage essentiel pour la solution du problème), ni dans des espaces très vastes (celui de Hilbert par exemple). Envisageons en effet les deux exemples.

---

1) Voir C. Kuratowski, *Topologie I*, Monografie Matematyczne. Warszawa-Wrocław 1948, deuxième édition, p. 162.

2) K. Menger, *Kurventheorie*, Leipzig-Berlin 1932, p. 97.

3) Op. cit., p. 99.

4) Ibidem, p. 105 et 112.

5) Cf. H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics 27 (1925-1926), p. 42 et 43, où une construction analogue est employée à un but différent. La partie de Δ située au-dessous de l'axe des x est homéomorphe à celle de la courbe de Gehman située au-dessus de cet axe.