for every $x \in X$ all the sections $S_x(Z_{k_{p_{k_{-1}}}})$ form a compact class. Thus, for every infinite sequence $(k_1,k_2,\ldots)$, we may apply the fundamental lemma to the sequence of sets $Z_{k_1} \supseteq Z_{k_2} \supseteq \ldots$. Since $P(A \times B) = P$ whenever $B$ is non-void, we obtain

$$P(B) = \sum_{\delta} P(Z_{k_1} \ldots Z_{k_n}) = \sum_{\delta} E_{k_1} E_{k_2} \ldots E_{k_n} = E \quad \text{q.e.d.}$$

**Note.** The assumption in II of the compactness of $F$ may be replaced by the following: there is a compact class $F^*$ such that $F \subseteq F^*$. In fact, if $H^*$ denotes the class of all sets $E \times F$ where $E \in E$ and $F \in F^*$, then it is easily seen that $H^* \supseteq H$, whence, by (4), $H^* = H$. Thus we reduce the generalized form of II from the original one applied to the classes $E,F^*$ and $H^*$.

Remarks on the Compactness and non Direct Products of Measures *)

by

E. Marczewski (Wroclaw) and C. Ryll-Nardzewski (Warsawa)

**Introduction.** The direct product of two normalized measures $\mu$ in $X$ and $\nu$ in $Y$, i.e. the measure $\lambda$ in $X \times Y$ such that

$$\lambda(A \times B) = \mu(A) \cdot \nu(B),$$

has many regularity properties, e.g.

I. If $\mu$ and $\nu$ are countably additive, so is $\lambda$.

II. If $\mu$, $\nu$ and $\lambda$ are $\sigma$-measures, then

$$\lambda(E \times Y) = \mu(E) \quad \text{for } E \subseteq X$$

(where, for every measure $\mu$, the symbol $\mu$ denotes the inner measure induced by $\nu$).

In sections 1 and 2 of this paper we deal with the same propositions for non-direct products, i.e. when the condition (1) is replaced by the weaker ones:

$$\lambda(A \times X) = \mu(A), \quad \lambda(X \times B) = \nu(B).$$

It turns out that the propositions I and II for the non-direct products remain true under the additional assumption that the measure $\nu$ is compact, and that they are false in general.

Sections 3 and 4 are further contributions to the study of compact measures. It is known that the minimal $\sigma$-extension of a compact measure is also compact. Here we show that the converse of this theorem is not true.

We apply theorems on compact measures proved in the paper: Marczewski [3] (quoted below as O), and theorems on projections proved in the paper: Marczewski and Ryll-Nardzewski [4] (quoted as P).

*) Presented to the Polish Mathematical Society, Wroclaw Sect., on April 1, 1932.

1) This follows e.g. from the existence of the direct $\sigma$-product of any two $\sigma$-measures, see e.g. Halmos [2], p. 144, Theorem B.

2) This follows e.g. from the abstract Pelczynski theorem. See ibidem.
Compactness and Products of Measures

Consequently, for every \( C \in L \) and \( \eta > 0 \), there are \( Q \in K \) and \( C^* \in L \), such that
\[
C^* \subseteq C \subseteq C', \quad |C - C'| < \eta.
\]

Hence, we may define by induction a sequence of sets \( D_1 \in L \) and a sequence of sets \( Q_1 \in K \), such that \( D_1 \subseteq C_0 \), \( |D_1| > \alpha^2 \), and
\[
D_1 \supset Q_1 \supset D_2 \supset Q_2 \supset \ldots
\]

Obviously
\[
Q_0 Q_1 \ldots = D_0 D_1 \ldots \subseteq C_1 C_2
\]

It follows from the definition of the class \( K \) that the vertical sections of the sets \( Q_0 Q_1 \ldots \) form a compact class, whence, by a lemma on projections (Fubini fundamental lemma), we have
\[
P(Q_0 Q_1 \ldots) \supset Q (Q_0 Q_1 \ldots) = P(Q_0) \cdot P(Q_1) \ldots \supset P(D_1) \cdot P(D_2) \ldots,
\]

where \( P(Z) \) denotes the vertical projection of the set \( Z \subseteq X \times Y \).

Thus, in view of the relations:
\[
P(D_1) \in M, \quad \mu(P(D_1)) \geq \lambda(D_1) > \frac{a}{2},
\]
and of the countable additivity of \( \mu \), we obtain \( P(D_1) \cdot P(D_2) \ldots \neq \emptyset \), which, together with (5), implies (4).

Next, we shall prove in (i) the assumption of the compactness of \( \nu \) is essential.

(ii) There are two measures \( \mu \) and \( \nu \) and their product \( \lambda \), such that \( \mu \) and \( \nu \) are countably additive and \( \lambda \) is not \( \aleph^+ \).

Let \( m \) and \( m \) denote the Lebesgue measure and the Lebesgue outer measure in the unit interval \( I \). Let \( I \) denote the measure in the unit square \( I \times I \), defined as follows:
\[
l(I) = \frac{m(I)}{m(I)}
\]

for every Borel set \( I \subseteq X \times Y \).

Let us denote by \( D \) the set of all points \((a, b) \in I \times I \) and let us consider a decomposition \( I = B + B' \), where
\[
m(B) = m(B), \quad \nu(B') = m(B).
\]

Further, we denote by \( M \) and \( N \) the classes of all sets of the form \( B \subseteq X \), where \( B \cap \subseteq X \), and by \( \mu \) and \( \nu \), the measures in \( M \) and \( N \) defined by the formulae
\[
\mu(B) = m(B), \quad \nu(B') = m(B).
\]

The idea of the proof is that of Sparre Andersen and Jensen [1]. (Cf. also Halmos [2], p. 214, (3).)
Compactness and Products of Measures

The class $G$ is a field and, obviously, $\lambda$ is the minimal $\sigma$-extension of $\lambda|G_\lambda$ whence for every $C \in L$ we have

$$\lambda(C) = \inf \lambda(D), \quad \text{where } D \supset C, \ D \in G_\lambda.$$

Since the complement of any set belonging to $G_\alpha = G_\alpha'$ belongs to $G_\lambda$, we obtain for every $C \in L$:

$$\lambda(C) = \sup \lambda(D), \quad \text{where } D \supset C, \ D \in G_\lambda$$

whence, by (9),

$$\lambda(C) = \sup \lambda(D), \quad \text{where } D \supset C, \ D \in H_\lambda.$$

Thus, for every $\eta > 0$, there is $D \in H_\lambda$ such that

$$\lambda(D) > \lambda(X \times Y) - \eta.$$

Since the class $F$ is compact, the projection $A$ of $D$ on $X$ belongs to the class $M_\mu = M$ (in virtue of theorem proved in $F$ and the relations (10) and $\mu(A) < \lambda(D)$ imply (8), q. e. d.

Next, we shall prove that in (i) the assumption of compactness in (i) is essential.

(ii) There are two $\sigma$-measures $\mu$ and $\nu$ in $Y$, a $\sigma$-product $\lambda$ of $\mu$ and $\nu$, and a set $E \times Y$ such that $\lambda(E 	imes Y) = 0$ and $\lambda(E 	imes Y) = 1$.

Let $X$ denote the unit interval and let $\mu$ and $\nu$ the Lebesgue measure in the field of Borel subsets of $X$. Let $Y$ denote a subset of $X$ such that the outer Lebesgue measure of $Y$ and $X - Y$ is equal to 1. Let $\nu$ denote the outer Lebesgue measure in the field of relatively Borel subsets of $Y$. For each relatively borel subset $Z$ of $X \times Y$ we set

$$\lambda(Z) = \nu \{ (y, y) \in Z \}.$$

It is easy to see that $\mu$ and $\nu$ are $\sigma$-measures and that $\lambda$ is a $\sigma$-product of $\mu$ and $\nu$. Setting $D = (X \times Y) \cap E \times Y$, we obtain

$$\lambda(Y) < \lambda(D) = \nu(Y) = 1,$$

while, on the other hand, $\mu(E) = 0$.

3. Non-atomic and purely atomic measures. If $\mu$ is a measure in a field $M$, then $A \in M$ is called an atom of $\mu$ if $\mu(A)$ and if for each $B \in M$ such that $B \supset A$ we have either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. If there is no atom, then $\mu$ is called non-atomic. If the space is the sum of a sequence, at most denumerable, of atoms, then $\mu$ is said to be purely atomic.

(i) If $\mu$ is a non-atomic compact measure in a field $M$, then every set $E \in M$ with $\mu(E) > 0$ contains such a set $N$ of the power of the continuum that $\mu_{\mu}(N) = 0$.

* This is a generalization of theorem C 7 (iii).
Sur une propriété des ensembles analytiques linéaires (solution d'un problème de E. Marczewski) 

Par 

W. Sierpiński (Warszawa)

Soit $F$ la famille de tous les ensembles plans qui sont de la forme $H_nH_{n+1}$... où $H_n (n=1, 2, ...)$ est une somme d'un nombre fini de rectangles aux côtés parallèles aux axes des coordonnées. Le but de cette note est de démontrer un théorème qui représente la solution d'un problème que m'a posé E. Marczewski. Le voici:

**Théorème.** Tout ensemble analytique linéaire borné est la projection orthogonale d'un ensemble plan de la famille $F$.

**Démonstration.** Soit $E$ un ensemble analytique linéaire borné; il est donc situé à l'intérieur d'un intervalle fini $(a, b)$. Comme on le sait, il existe une fonction $f(x)$ définie et continue dans l'ensemble $X$ de tous les nombres irrationnels de l'intervalle $(0, 1)$. Soit $I$ l'image géométrique de la fonction $f$; c'est-à-dire l'ensemble de tous les points $(x, f(x))$ du plan, où $x$ est un nombre irrationnel de l'intervalle $(0, 1)$. Soient $I$ la fermeture de l'ensemble $I$, et $a$ un nombre naturel. Puisqu'il correspond à chaque point $p$ de $I$ l'intérieur d'un carré dont $p$ est le centre et dont le côté est égal à $1/a$. D'après le théorème de Borel, il existe un nombre fini de tels carrés dont la somme $P_a$ recouvre l'ensemble (fermé et borné) $I$. Comme on peut le démontrer sans peine, on a $I = P_aP_{a+1}$.

Soit $r_1, r_2, ...$ une suite inﬁnie formée de tous les nombres rationnels intérieurs à l'intervalle $(0, 1)$. Soient $Q_n$ l'intérieur du rectangle aux côtés situés sur les droites $x = 0$, $x = r_n$, $y = a$, $y = b$, et $R_n$ l'intérieur du rectangle formé par les droites $x = r_n$, $x = 1$, $y = a$, $y = b$.

La fonction $f(x)$ étant continue dans l'ensemble $X$, on démontre sans peine que

$$I = \bigcap_{n=1}^{\infty} P_n(Q_n + R_n);$$

l'ensemble $I$ est donc un ensemble de la famille $F$. L'ensemble $E$ étant la projection orthogonale de $I$ sur l'axe d'ordonnées, le théorème se trouve démontré.

Le théorème inverse est vrai aussi, vu que la projection d'un ensemble $G$ est un ensemble analytique. Donc, pour qu'un ensemble linéaire borné soit analytique, il faut et il suffit qu'il soit la projection d'un ensemble plan de la famille $F$.