for every $x \in X$ all the sections $S_x(Z_{k_1 k_2 \ldots k_n})$ form a compact class. Thus, for every infinite sequence $(k_1, k_2, \ldots)$, we may apply the fundamental lemma to the sequence of sets $Z_{k_1} \supset Z_{k_1 k_2} \supset \ldots$ Since $P(A \times B) = A$ whenever B is non-void, we obtain

$$P(Z) = \sum_{\{k_j\}} P(Z_{k_1} Z_{k_1 k_2} \ldots) = \sum_{\{k_j\}} E_{k_1} E_{k_1 k_2} \ldots \in \boldsymbol{E}_{dA} = \boldsymbol{E}_A, \quad \text{q. e. d.}$$

Note. The assumption in II of the compactness of $\boldsymbol{F}$ may be replaced by the following: there is a compact class $\boldsymbol{F}^*$ such that $\boldsymbol{F} \subset \boldsymbol{F}^*_A$. In fact, if $\boldsymbol{H}^*$ denotes the class of all sets $E \times F$ where $E \in \boldsymbol{E}$ and $F \in \boldsymbol{F}^*$, then it is easily seen that $\boldsymbol{H}^*_A \supset \boldsymbol{H}$, whence, by (4), $\boldsymbol{H}^*_A = \boldsymbol{H}_A$. Thus we reduce the generalized form of II from the original one, applied to the classes $\boldsymbol{E}, \boldsymbol{F}^*$ and $\boldsymbol{H}^*$.

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Remarks on the Compactness and non Direct Products of Measures *)

By

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Introduction. The direct product of two normalized measures μ in X and ν in Y, *i. e.* the measure λ in $X \times Y$ such that

(1) $$\lambda(A \times B) = \mu(A) \cdot \nu(B),$$

has many regularity properties, *e. g.*

I. If μ and ν are countably additive, so is λ [1]).

II. If μ, ν and λ are σ-measures, then

$$\lambda_i(E \times Y) = \mu_i(E) \quad \text{for} \quad E \subset X$$

(where, for every measure μ, the symbol μ_i denotes the inner measure induced by μ) [2]).

In sections 1 and 2 of this paper we deal with the same propositions for non-direct products, *i. e.* when the condition (1) is replaced by the weaker ones:

(2) $$\lambda(A \times Y) = \mu(A), \qquad \lambda(X \times B) = \nu(B).$$

It turns out that *the propositions* I *and* II *for the nondirect products remain true under the additional assumption that the measure* ν *is compact, and that they are false in general.*

Sections 3 and 4 are further contributions to the study of compact measures. It is known that the minimal σ-extension of a compact measure is also compact. Here we show that the converse of this theorem is not true.

We apply theorems on compact measures proved in the paper: Marczewski [3] (quoted below as C), and theorems on projections proved in the paper: Marczewski and Ryll-Nardzewski [4] (quoted as P).

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[1]) This follows *e. g.* from the existence of the direct σ-product of any two σ-measures, see *e. g.* Halmos [2], p. 144, Theorem B.

[2]) This follows *e. g.* from the abstract Fubini theorem. See ibidem.

Terminology and notation are those of C and P. In particular, we understand by a *measure* any non-negative, normalized, and additive set function, defined in a field of sets. A measure μ in the field $\boldsymbol{M}$ is *countably additive*, if $\mu(E_1+E_2+\ldots)=\mu(E_1)+\mu(E_2)+\ldots$ for each sequence of sets, such that $E_j \epsilon \boldsymbol{M}$ and $E_1+E_2+\ldots \epsilon \boldsymbol{M}$. Any countably additive measure in a σ-field is called *σ-measure*. If μ is a countably additive measure in a field $\boldsymbol{M}$, then there exists a unique σ-measure ν in the smallest σ-field containing $\boldsymbol{M}$, such that $\mu=\nu|\boldsymbol{M}$. The measure ν is called *minimal σ-extension* of μ.

For any class $\boldsymbol{K}$ of sets, $\boldsymbol{K}_s$, $\boldsymbol{K}_\sigma$ and $\boldsymbol{K}_\delta$ denote respectively the class of all sets of the form $E_1+E_2+\ldots+E_n$, $E_1+E_2+\ldots$ or $E_1E_2\ldots$, where $E_j \epsilon \boldsymbol{K}$.

1. The countable additivity in cartesian products. Let X and Y be two sets, $\boldsymbol{M}$ and $\boldsymbol{N}$ fields of subsets of X and Y, and, finally, μ and ν measures in $\boldsymbol{M}$ and $\boldsymbol{N}$. A measure λ in the smallest field $\boldsymbol{L}$ containing the class of all sets $A\times B$, where $A \epsilon \boldsymbol{M}$ and $B \epsilon \boldsymbol{N}$, is called *product* of μ and ν, if it fullfils the condition (2) for every $A \epsilon \boldsymbol{M}$ and every $B \epsilon \boldsymbol{N}$. Of course, if $C \epsilon \boldsymbol{L}$ and if the projection A of C on X belongs to $\boldsymbol{M}$, then $\lambda(C)\leqslant\mu(A)$.

In this section we deal with the problem of the countable additivity of products of two measures. We recall that every product of compact measures is compact and thus countably additive (C 6(vii)). Now we shall prove that

(i) *A product λ of a countably additive measure μ and of a compact measure ν is countably additive.*

Let us retain the above mentioned symbols $\boldsymbol{M}, \boldsymbol{N}, X$ and Y, and let us denote by $\boldsymbol{F}$ a compact class approximating $\boldsymbol{N}$ with respect to ν. Let $\boldsymbol{L}$ denote the class of all sets

(3) $$(A_1\times B_1)+(A_2\times B_2)+\ldots+(A_n\times B_n),$$

where $A_j \epsilon \boldsymbol{M}$ and $B_j \epsilon \boldsymbol{N}$ and let $\boldsymbol{K}$ denote the class of all sets of the form (3), where $A_j \epsilon \boldsymbol{M}$ and $B_j \epsilon \boldsymbol{F}$.

In order to prove the countable additivity of λ, let us suppose

$$C_1\supset C_2\supset\ldots, \quad \text{where} \quad E_j \epsilon \boldsymbol{L} \text{ and } \lambda(E_j)>\alpha;$$

we have to prove

(4) $$C_1C_2\ldots\neq 0.$$

By hypothesis, for every $B \epsilon \boldsymbol{N}$ and $\eta>0$, there are $P \epsilon \boldsymbol{F}$ and $B^* \epsilon \boldsymbol{N}$, such that

$$B^*\subset P\subset B, \qquad \mu(B-B^*)<\eta,$$

whence, for every $A \epsilon \boldsymbol{M}$,

$$A\times B^*\subset A\times P\subset A\times B, \qquad \lambda(A\times B-A\times B^*)\leqslant\lambda[X\times(B-B^*)]<\eta.$$

Consequently, for every $C \epsilon \boldsymbol{L}$ and $\eta>0$, there are $Q \epsilon \boldsymbol{K}$ and $C^* \epsilon \boldsymbol{L}$, such that

$$C^*\subset Q\subset C, \qquad \lambda(C-C^*)<\eta.$$

Hence, we may define by induction a sequence of sets $D_j \epsilon \boldsymbol{L}$ and a sequence of sets $Q_j \epsilon \boldsymbol{K}$, such that $D_j\subset C_j$, $\lambda(D_j)>a/2$, and

$$D_1\supset Q_1\supset D_2\supset Q_2\supset\ldots$$

Obviously

$$Q_1Q_2\ldots=D_1D_2\ldots\subset C_1C_2.$$

It follows from the definition of the class $\boldsymbol{K}$ that the vertical sections of the sets $Q_1,Q_2,\ldots$ form a compact class, whence, by a lemma on projections (P, fundamental lemma), we have

(5) $$\mathrm{P}(C_1C_2\ldots)\supset\mathrm{P}(Q_1Q_2\ldots)=\mathrm{P}(Q_1)\cdot\mathrm{P}(Q_2)\cdot\ldots\supset\mathrm{P}(D_2)\cdot\mathrm{P}(D_3)\cdot\ldots,$$

where $\mathrm{P}(Z)$ denotes the vertical projection of the set $Z\subset\boldsymbol{X}\times\boldsymbol{Y}$.

Thus, in view of the relations:

$$\mathrm{P}(D_j) \epsilon M, \qquad \mu[\mathrm{P}(D_j)]\geqslant\lambda(D_j)>\frac{a}{2},$$

and of the countable additivity of μ, we obtain $\mathrm{P}(D_2)\cdot\mathrm{P}(D_3)\cdot\ldots\neq 0$, which, together with (5), implies (4).

Next, we shall prove in (i) the assumption of the compactness of ν is essential.

(ii) *There are two measures μ and ν and their product λ, such that μ and ν are countably additive and λ is not*[3]).

Let m and m_e denote the Lebesgue measure and the Lebesgue outer measure in the unit interval I. Let l denote the measure in the unit square $I\times I$, defined as follows:

$$l(C)=m\{\underset{x}{E}[(X,X) \epsilon C]\} \quad \text{for every Borel set } C\subset I\times I.$$

Let us denote by D the set of all points $(x,x) \epsilon I\times I$ and let us consider a decomposition $I=Z+Z'$, where

(6) $$m_e(Z)=m_e(Z')=1 \qquad \text{and} \qquad ZZ'=0.$$

Further, we denote by $\boldsymbol{M}$ and $\boldsymbol{N}$ the classes of all sets of the form BZ and BZ', where B runs over the field of Borel subsets of I, and by μ and ν, the measures in $\boldsymbol{M}$ and $\boldsymbol{N}$ defined by the formulae

$$\mu(BZ)=m(B), \qquad \nu(BZ')=m(B).$$

[3]) The idea of the proof is that of Sparre-Andersen and Jessen [1]. Cf. also Halmos [2], p. 214, (3).

For every set $C(Z\times Z')$, where C is of the form $C=(A_1\times B_1)+\dots+(A_n\times B_n)$ and A_j and B_j are Borel subsets of I, we set

(7) $$\lambda[C(Z\times Z')]=l(C).$$

It follows easily from (6) that the measures μ, ν and λ are uniquely defined and that λ is a product of μ and ν. Of course, μ and ν are countably additive.

Let us suppose λ countably additive and denote by $\varkappa$ the minimal σ-extension of λ. It follows easily from (7) and from the σ-additivity of $\varkappa$ and l that the class of all sets $C\subset I\times I$, such that

$$\varkappa[C(Z\times Z')]=l(C),$$

contains all Borel subsets of $I\times I$, which leads to the contradiction:

$$0=\varkappa(0)=\varkappa[D(Z\times Z')]=l(D)=1.$$

2. Inner measure in Cartesian products. The minimal σ-extension of any product of two measures μ and ν is called *σ-product* of μ and ν.

(i) *If λ is a σ-product of a σ-measure μ in X and a compact σ-measure ν in Y, then, for every $E\subset X$,*

$$\lambda_i(E\times Y)=\mu_i(E).$$

Let us denote by $\boldsymbol{M}$, $\boldsymbol{N}$ and $\boldsymbol{L}$ the σ-fields in which the σ-measures μ, ν and λ are defined, and by $\boldsymbol{F}$ a compact subclass of $\boldsymbol{N}$ which approximates $\boldsymbol{N}$ with respect to ν (such a class exists in virtue of C 4 (iii)). Further let $\boldsymbol{G}$ denote the class of all sets $A\times B$ where $A\in\boldsymbol{M}$ and $B\in\boldsymbol{N}$, and let $\boldsymbol{H}$ denote the class of all sets $A\times P$ where $A\in\boldsymbol{M}$ and $P\in\boldsymbol{F}$.

It is easy to prove that $\lambda_i(E\times Y)\geqslant\mu_i(E)$ and in order to establish the opposite inequality it is enough to prove that for every $\eta>0$ there is a set $A\in\boldsymbol{M}$ such that

(8) $$A\subset E,\qquad \mu(A)>\lambda_i(E\times Y)-\eta.$$

It follows from the hypotheses that for every $B\in\boldsymbol{N}$ and $\eta>0$ there exists $P\in\boldsymbol{F}$ such that

$$P\subset B,\qquad \mu(B-P)<\eta$$

whence, for every $A\in\boldsymbol{M}$,

$$A\times P\subset A\times B,\qquad \lambda(A\times B-A\times P)\leqslant\lambda[X\times(B-P)]<\eta.$$

Consequently, for every $C\in\boldsymbol{G}_{s\delta}$ we have

(9) $$\lambda(C)=\sup\lambda(D),\qquad\text{where}\quad D\subset C,\ D\in\boldsymbol{H}_{s\delta}.$$

The class $\boldsymbol{G}_s$ is a field and, obviously, λ is the minimal σ-extension of $\lambda|\boldsymbol{G}_s$, whence for every $C\in\boldsymbol{L}$ we have

$$\lambda(C)=\inf\lambda(D),\qquad\text{where}\quad D\supset C,\ D\in\boldsymbol{G}_\sigma.$$

Since the complement of any set belonging to $\boldsymbol{G}_\sigma=\boldsymbol{G}_{s\sigma}$ belongs to $\boldsymbol{G}_{s\delta}$, we obtain for every $C\in\boldsymbol{L}$:

$$\lambda(C)=\sup\lambda(D),\qquad\text{where}\quad D\subset C,\ D\in\boldsymbol{G}_{s\delta}$$

whence, by (9),

$$\lambda(C)=\sup\lambda(D),\qquad\text{where}\quad D\subset C,\ D\in\boldsymbol{H}_{s\delta}.$$

Thus, for every $\eta>0$, there is $D\in\boldsymbol{H}_{s\delta}$, such that

(10) $$D\subset E\times Y,\qquad \lambda(D)>\lambda_i(E\times Y)-\eta.$$

Since the class $\boldsymbol{F}$ is compact, the projection A of D on X belongs to the class $\boldsymbol{M}_{s\delta}=\boldsymbol{M}$ (in virtue of theorem proved in P) and the relations (10) and $\mu(A)\geqslant\lambda(D)$ imply (8), q. e. d.

Next, we shall prove that in (i) the assumption of compactness in (i) is essential.

(ii) *There are two σ-measures μ in X and ν in Y, a σ-product λ of μ and ν, and a set $E\subset X$ such that $\mu_i(E)=0$ and $\lambda_i(E\times Y)=1$.*

Let X denote the unit interval and μ the Lebesgue measure in the field of Borel subsets of X. Let Y denote a subset of X such that the outer Lebesgue measure of Y and $X-Y$ is equal to 1. Let ν denote the outer Lebesgue measure in the field of relatively borelian subsets of Y. For each relatively borelian subset Z of $X\times Y$ we set

$$\lambda(Z)=\nu\mathop{E}_{y}[(y,y)\in Z)].$$

It is easy to see that μ and ν are σ-measures and that λ is a σ-product of μ and ν. Setting $D=(X\times Y)\cdot\mathop{E}_{(x,y)}[x=y]$, we obtain

$$\lambda_i(Y\times Y)\geqslant\lambda(D)=\nu(Y)=1,$$

while, on the other hand, $\mu_i(Y)=0$.

3. Non-atomic and purely atomic measures. If μ is a measure in a field $\boldsymbol{M}$, then $A\in\boldsymbol{M}$ is called an *atom* of μ, if $\mu(A)$ and if for each $B\in\boldsymbol{M}$ such that $B\subset A$ we have either $\mu(A)=\mu(B)$ or $\mu(B)=0$. If there is no atom, then μ is called *non-atomic*. If the space is the sum of a sequence, at most denumerable, of atoms, then μ is said to be *purely atomic*.

(i) *If μ is a non-atomic compact measure in a field $\boldsymbol{M}$, then every set $E\in\boldsymbol{M}$ with $\mu(E)>0$ contains such a set N of the power of the continuum that $\mu_e(N)=0$* [4].

[4]) This is a generalization of theorem C 7 (iii).

The proof is based upon the following lemma:

Let F be a compact class which approximates M with respect to μ. Then, for every $E \in M$ with $\mu(E) > 0$ there exist sets $P_0, P_1 \in F$ and $E_0, E_1 \in M$ such that

$$E_0 \subset P_0 \subset E, \qquad E_1 \subset P_1 \subset E, \qquad P_0 P_1 = 0,$$
$$\mu(E_0) > 0, \qquad \mu(E_1) > 0, \qquad \mu(E_0 + E_1) < \tfrac{7}{8}\mu(E).$$

This lemma permits us to obtain the required set N by the known dyadic construction. The compactness of F guarantees that the power of the obtained set is really that of the continuum.

(ii) *A purely atomic σ-measure μ is compact.*

Let $X = A_1 + A_2 + \ldots$ be the decomposition of X into disjoint atoms. It is easy to see that the sets of the form $A_{k_1} + A_{k_2} + \ldots + A_{k_n} - Z$, where $\mu(Z) = 0$, constitute a compact class F which approximates M with respect to μ.

4. Minimal σ-extensions. The minimal σ-extension of a compact measure is also compact (C 4 (ii)). We shall prove that

(i) *There is a non-atomic and non compact measure the minimal σ-extension of which is purely atomic and consequently compact.*

Let $X = (r_1, r_2, \ldots)$ be the set of all rational numbers of the interval $(\sqrt{2}, 1+\sqrt{2})$, and let M denote the field spanned on the class of all intervals of the set X with irrational extremities. For every $E \in M$ put

$$\mu(E) = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \ldots \tag{11}$$

where $r_{n_1}, r_{n_2}, \ldots$ is the sequence of all $r_n \in E$.

It follows from 3(i) that μ is non compact. On the other hand, μ has the σ-extension to M_β, namely the measure defined by the formula (11) for every $E \subset X$. Obviously this σ-extension is purely atomic (with the atoms $(r_1), (r_2), \ldots$) and, by 3(ii), compact.

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Sur une propriété des ensembles analytiques linéaires (solution d'un problème de E. Marczewski)

Par

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Soit F la famille de tous les ensembles plans qui sont de la forme $H_1 H_2 \ldots$, où H_n $(n = 1, 2, \ldots)$ est une somme d'un nombre fini de rectangles aux côtés parallèles aux axes des coordonnées. Le but de cette note est de démontrer un théorème qui représente la solution d'un problème que m'a posé E. Marczewski. Le voici:

Théorème. *Tout ensemble analytique linéaire borné est la projection orthogonale d'un ensemble plan de la famille F.*

Démonstration. Soit E un ensemble analytique linéaire borné; il est donc situé à l'intérieur d'un intervalle fini (a, b). Comme on le sait, il existe une fonction $f(x)$ définie et continue dans l'ensemble N de tous les nombres irrationnels de l'intervalle $(0,1)$. Soit I l'image géométrique de la fonction f, c'est-à-dire l'ensemble de tous les points $(x, f(x))$ du plan, où x est un nombre irrationnel de l'intervalle $(0,1)$. Soient $\bar{I}$ la fermeture de l'ensemble I, et n un nombre naturel. Faisons correspondre à chaque point p de $\bar{I}$ l'intérieur d'un carré dont p en est le centre et dont le côté est égal à $1/n$. D'après le théorème de Borel, il existe un nombre fini de tels carrés dont la somme P_n recouvre l'ensemble (fermé et borné) $\bar{I}$. Comme on peut le démontrer sans peine, on a $\bar{I} = P_1 P_2 \ldots$

Soit $r_1, r_2, \ldots$ une suite infinie formée de tous les nombres rationnels intérieurs à l'intervalle $(0,1)$. Soient Q_n l'intérieur du rectangle aux côtés situés sur les droites $x=0$, $x=r_n$, $y=a$, $y=b$, et R_n l'intérieur du rectangle formé par les droites $x=r_n$, $x=1$, $y=a$, $y=b$.

La fonction $f(x)$ étant continue dans l'ensemble N, on démontre sans peine que

$$I = \prod_{n=1}^{\infty} P_n(Q_n + R_n);$$

l'ensemble I est donc un ensemble de la famille F. L'ensemble E étant la projection orthogonale de I sur l'axe d'ordonnées, le théorème se trouve démontré.

Le théorème inverse est vrai aussi, vu que la projection d'un ensemble G_δ est un ensemble analytique. Donc, *pour qu'un ensemble linéaire borné soit analytique, il faut et il suffit qu'il soit la projection d'un ensemble plan de la famille F.*