Projections in Abstract Sets

By

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1. Problems and results. Let \( X \) and \( Y \) be two fixed abstract sets. For every subset \( Z \) of the Cartesian product \( X \times Y \) let us denote by \( P(Z) \) the projection of \( Z \) on \( X \); in other terms \( z \in P(Z) \) if and only if there exists \( y \in Y \) such that \( (x,y) \in Z \).

The operation \( P \) is absolutely additive:

\[
P(\bigcup Z) = \sum P(Z_i)
\]

but it is not multiplicative. We have

\[
P(\bigcap Z) \subseteq \bigcap P(Z_i)
\]

while the converse inclusion is true only under some additional assumptions, e.g. in the fundamental lemmas (Section 2) and in the following obvious proposition:

(3) If \( A \subseteq B \subseteq C \subseteq Y \) and \( B \neq C \), then \( P(A \times B_i)|(A \times B) \ldots = A \times A \).

For every class \( Q \) of sets let us denote respectively

<table>
<thead>
<tr>
<th>( Q_1 )</th>
<th>( Q_2 )</th>
<th>( Q_3 )</th>
<th>( Q_4 )</th>
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| the class of all \( n \)-sets of the form \( (x, y) \) a class of subsets of \( X \), \( F \) a class of subsets of \( Y \), and \( H \) the class of all sets \( E \times F \), where \( E \in E \) and \( F \in F \). It follows easily from (1) and (3) that

\[
\begin{array}{c|cccc|ccc}
\text{If } 0 \neq Z & H & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\
\text{then } P(Z) & E & E_1 & E_2 & E_3 & E_4 & E_5 & E_6
\end{array}
\]

Thus the following problem arises: what can be said of the projection of a set \( Z \in H \)? The answer is simple: nothing in general. If \( X = Y = \) the unit interval, \( I \) is the class of all closed subintervals of \( X \) and \( I^2 \) is the class of all subsets of \( Y \), then for every \( A \in X \) there is a set \( C \in H \) such that \( P(C) = A \).

In fact, denoting by \( D \) the diagonal, i.e. the set of points \( (x,y) \), where \( x \in X \), and setting

\[
C(A \times A) \cdot D, \quad F_m \cdot \frac{m-1}{m}, \quad \text{for } m = 1, 2, \ldots, n; \quad n = 1, 2, \ldots
\]

and

\[
C_n = [I^1 \times (A F^1)] + [I^2 \times (A F^2)] + \ldots + [I^n \times (A F^n)]
\]

we obtain

\[
C_1 \in H_1, \quad C = C_1 C_2 \ldots C_n, \quad P(C) = A.
\]

The purpose of this paper 1) is to prove that, nevertheless, under some assumptions relating to \( F \), the projections of sets belonging to \( H \), etc., belong to some classes determined by the class \( E \) only.

The notion which plays an essential role in what follows is that of a compact class of sets 2). A class \( F \) is compact, if, for every sequence \( F_n \subseteq F \), we have \( F_1 \cap F_2 \cap \ldots = 0 \) whenever \( F_1, F_2, \ldots \neq 0 \) for \( n = 1, 2, \ldots \). Consequently a multiplicative class \( F \) (i.e. such that \( P(Z) \cap F \) is compact if and only if every descending sequence of non-void products \( F_n \) has a non-void product.

Next, denote by \( Q \) the class of all sets of the form \( X_0 Q_1 Q_2 \ldots \), where \( Q_1, Q_2, \ldots \subseteq Q \) and the summation extends over all infinite sequence \( k_1 k_2, \ldots \) of natural numbers. It is well known that this operation, termed operation \( (A) \), has the following properties 3):

4) \[
Q_1 \leq Q_2 \leq Q_3, \quad Q_{A_1} = Q_{A_2}.
\]

The result of this paper is

Theorem. If the class \( F \) is compact, then

\[
\begin{array}{c|ccc|ccc}
\text{if } 0 \neq Z & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\
\text{then } P(Z) & E_1 & E_2 & E_3 & E_4 & E_5 & E_6
\end{array}
\]

\[
1) \text{Presented to the Polish Mathematical Society, Wrocław Section, on April 1, 1952.}
\]

\[
2) \text{Cf. E. Marczewski, "On compact measures, Fundamenta Mathematicae 40 (1953), p. 113-124, especially p. 116.}
\]

\[
\]

Fundamenta Mathematicae T. XL.
Notice that the implication II' (which is an easy consequence of II and (4)) cannot be strengthened. Indeed, in case $E=F$ the class of all closed sub-intervals of the unit interval $I$, the projections of all sets belonging to $H_0$ constitute the class of all analytical subsets of $I$, or, in other words, the whole class $E_1$.

The theorem will be applied in a forthcoming paper on measures in product spaces.

**2. Fundamental lemma.** For $Z \subseteq X \times Y$ and $x \in X$ denote by $S_x(Z)$ the vertical section of $Z$ corresponding to $x$; in other terms, $y \in S_x(Z)$ if and only if $(x, y) \in Z$. We shall prove that

If $Z_1 \supseteq Z_2 \supseteq \ldots$, where $Z_j \subseteq X \times Y$ for $j = 1, 2, \ldots$, and if for every $x \in X$ the sequence $S_x(Z_j)$ forms a compact class, then $P(Z_1, Z_2, \ldots) = P(Z_1) \cdot P(Z_2) \cdots$.

In virtue of (2), it suffices to prove that

$$P(Z_1, Z_2, \ldots) \supseteq P(Z_1) \cdot P(Z_2) \cdots.$$

Let $x \in P(Z_1) \cdot P(Z_2) \cdots$. Consequently the sets

$$S_x(Z_1) \supseteq S_x(Z_2) \supseteq \ldots$$

are non-void and, since they form a compact class, there is an $y$ such that

$$y \in S_x(Z_1) \cdot S_x(Z_2) \cdots$$

or, in other words, $(x, y) \in Z_1 \supseteq Z_2 \supseteq \ldots$ whence $x \in P(Z_1, Z_2, \ldots)$, q. e. d.

**3. The operation (4).** For any system of sets $3 = (Z_{k_1, k_2, \ldots})$ where $(k_1, k_2, \ldots, k_n)$ runs over all the set of finite sequences of natural numbers, let us set

$$A(3) = \sum_{k_1} Z_{k_1, k_2, \ldots},$$

where the summation extends over all infinite sequences $(k_i)$. For every class $Q$ of sets, $A_Q$ is by definition the class of all sets $A(3)$, where $3$ runs over all systems of sets belonging to $Q$.

A system $(Z_{k_1, k_2, \ldots})$ of sets will be called monotone if we always have

$$Z_{k_1, k_2, \ldots} \supseteq Z_{k_1, k_2, \ldots, 1},$$

The following lemma is essential for projection properties:

If $0 \neq Z \in Q_1$, then there exists a monotone system $3 = (Z_{k_1, k_2, \ldots})$ of non-void sets belonging to $Q_2$ such that $Z = A(3)$.

I shall outline the proof of this lemma.

Since $Z \in Q_1$, there is a system $\hat{A} = (A_{k_1, k_2, \ldots})$ of sets belonging to $Q$ such that $Z = A(\hat{A})$.

**Projections in Abstract Sets**

Let us set

$$B_{k_1, k_2, \ldots} = A_{k_1, k_2, \ldots} \cdot A_{k_1, k_2, \ldots, 1};$$

obviously $B = (B_{k_1, k_2, \ldots})$ is a monotone system of sets belonging to $Q_2$ and we have $Z = A(B)$.

Next, denote by $M$ the set of all infinite sequences $(k_i)$ of natural numbers such that

$$R_{k_1, k_2, \ldots} B_{k_1, k_2, \ldots, 1};$$

Hence

$$Z = \sum_{k_1, k_2, \ldots} R_{k_1, k_2, \ldots},$$

where the summation extends only over sequences $(k_i)$ belonging to $M$.

By a suitable new numeration of the sets $B_{k_1, k_2, \ldots}$ appearing in this sum, namely by repeating some sets satisfying (5), we obtain the required system $3$.

**4. Proof of Theorem. I.** Suppose $0 \neq Z \in H_2$. Thus we may write $Z = Z_{k_1, k_2, \ldots}$, where $Z_1 \supseteq Z_2 \supseteq \ldots$, and $Z_1 \in H_1$, whence for every $x \in X$ we have

$$S_x(Z_1) \cdot F_1 \quad \text{or} \quad S_x(Z_1) = 0.$$

Since $F$ is compact by hypothesis, the class $F_1$ is also compact; by the fundamental lemma

$$P(Z) = P(Z_1) \cdot P(Z_2) \cdots \cdot F_{k_1, k_2, \ldots} = F_{k_1, k_2, \ldots},$$

q. e. d.

II. Suppose $0 \neq Z \in H_2$. By lemma 3 there is a system $3$ of non-void sets

$$Z_{k_1, k_2, \ldots} = B_{k_1, k_2, \ldots, 1} \cdot F_{k_1, k_2, \ldots},$$

such that

$$E_{k_1, k_2, \ldots} \in F_{k_1, k_2, \ldots} \quad \text{and} \quad E_{k_1, k_2, \ldots} \cdot F_{k_1, k_2, \ldots, 1} = C_{k_1, k_2, \ldots, 1} \cdot F_{k_1, k_2, \ldots},$$

where

$$Z = A(3).$$

Since, for every sequence $(k_1, k_2, \ldots)$, we have

$$S_x(Z_{k_1, k_2, \ldots}) = F_{k_1, k_2, \ldots} \quad \text{or} \quad S_x(Z_{k_1, k_2, \ldots}) = 0,$$

the second part of the lemma (i. e. the statement that $E_{k_1, k_2, \ldots}$ are non-void) is superfluous in the case when the empty set belongs to $F$. Then it suffices to consider a monotone system $B$, such that the sets $B_{k_1, k_2, \ldots}$ and $F_{k_1, k_2, \ldots}$ are both void or both non-void.
for every \( x \in X \) all the sections \( S_\alpha(Z_{k_1} \ldots k_n) \) form a compact class. Thus, for every infinite sequence \( (k_1, k_2, \ldots) \), we may apply the fundamental lemma to the sequence of sets \( Z_{k_1} \cup Z_{k_2} \cup \ldots \). Since \( P(\varnothing \times B) = 0 \) whenever \( B \) is non-void, we obtain

\[
P(B) = \sum_{\alpha \in \mathbb{P}} P(Z_{k_1} \cup Z_{k_2} \cup \ldots) = \sum_{\alpha \in \mathbb{P}} E_{k_1} E_{k_2} \ldots \epsilon E_{k_\alpha} = E_A, \quad q.e.d.
\]

**Note.** The assumption in II of the compactness of \( F \) may be replaced by the following: there is a compact class \( F^* \) such that \( F \subseteq F^* \). In fact, if \( H^* \) denotes the class of all sets \( E \times F \) where \( E \in E \) and \( F \in F^* \), then it is easily seen that \( H^* \subseteq H \), whence, by (4), \( H^* = H_e \). Thus we reduce the generalized form of II from the original one applied to the classes \( E, F^* \) and \( H_e \).

**Remarks on the Compactness and non Direct Products of Measures */

by

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**Introduction.** The direct product of two normalized measures \( \mu \) in \( X \) and \( \nu \) in \( Y \), i.e. the measure \( \lambda \) in \( X \times Y \) such that

\[
\lambda(A \times B) = \mu(A) \cdot \nu(B),
\]

has many regularity properties, e.g.

I. If \( \mu \) and \( \nu \) are countably additive, so is \( \lambda \).

II. If \( \mu, \nu \) and \( \lambda \) are \( \sigma \)-measures, then

\[
\lambda(A \times X) = \mu(A) \quad \text{for } A \subseteq X
\]

(where, for every measure \( \mu \), the symbol \( \mu(A) \) denotes the inner measure induced by \( \mu \)).

In sections 1 and 2 of this paper we deal with the same propositions for non-direct products, i.e. when the condition (1) is replaced by the weaker ones:

\[
\lambda(A \times Y) = \mu(A), \quad \lambda(X \times B) = \nu(B).
\]

It turns out that the propositions I and II for the non-direct products remain true under the additional assumption that the measure \( \nu \) is compact, and that they are false in general.

Sections 3 and 4 are further contributions to the study of compact measures. It is known that the minimal \( \sigma \)-extension of a compact measure is also compact. Here we show that the converse of this theorem is not true.

We apply theorems on compact measures proved in the paper: Marczewski [3] (quoted below as O), and theorems on projections proved in the paper: Marczewski and Ryll-Nardzewski [4] (quoted as P).

*) Presented to the Polish Mathematical Society, Wrocław Sect., on April 1, 1932.

1) This follows e.g. from the existence of the direct \( \sigma \)-product of any two \( \sigma \)-measures, see e.g. Halmos [2], p. 144, Theorem B.

2) This follows e.g. from the abstract Pahgini theorem. See ibidem.