Projections in Abstract Sets

By

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1. Problems and results. Let X and Y be two fixed abstract sets. For every subset Z of the Cartesian product $X\times Y$ let us denote by $\mathrm{P}(Z)$ the *projection* of Z on X; in other terms $x\in\mathrm{P}(Z)$ if and only if there exists $y\in Y$ such that $(x,y)\in Z$.

The operation P is absolutely additive:

$$\mathrm{P}\Big(\sum_t Z_t\Big)=\sum_t \mathrm{P}(Z_t) \tag{1}$$

but it is not multiplicate. We have

$$\mathrm{P}\Big(\prod_t Z_t\Big)\subset\prod_t \mathrm{P}(Z_t), \tag{2}$$

while the converse inclusion is true only under some additional assumptions, *e. g.* in the fundamental lemma (Section 2) and in the following obvious proposition:

(3) If $A_j\subset X$, $B_j\subset Y$ and $B_1B_2\ldots\neq 0$, then $\mathrm{P}[(A_1\times B_1)(A_2\times B_2)\ldots]=A_1A_2\ldots$

For every class $\boldsymbol{Q}$ of sets let us denote respectively

by	$\boldsymbol{Q}_s$	$\boldsymbol{Q}_d$	$\boldsymbol{Q}_\sigma$	$\boldsymbol{Q}_\delta$
the class of all sets of the form	$Q_1+Q_2+\ldots+Q_n$	$Q_1Q_2\ldots Q_n$	$Q_1+Q_2+\ldots$	$Q_1Q_2\ldots$

where $n=1,2,\ldots$ and $Q_j\in\boldsymbol{Q}$.

Let $\boldsymbol{E}$ denote a class of subsets of X, $\boldsymbol{F}$ a class of subsets of Y, and $\boldsymbol{H}$ the class of all sets $E\times F$, where $E\in\boldsymbol{E}$ and $F\in\boldsymbol{F}$.

It follows easily from (1) and (3) that

If $0\neq Z\in$	$\boldsymbol{H}$	$\boldsymbol{H}_s$	$\boldsymbol{H}_d$	$\boldsymbol{H}_\sigma$	$\boldsymbol{H}_\delta$	$\boldsymbol{H}_{sd}=\boldsymbol{H}_{ds}$	$\boldsymbol{H}_{\delta s}$	$\boldsymbol{H}_{d\sigma}$
then $\mathrm{P}(Z)\in$	$\boldsymbol{E}$	$\boldsymbol{E}_s$	$\boldsymbol{E}_d$	$\boldsymbol{E}_\sigma$	$\boldsymbol{E}_\delta$	$\boldsymbol{E}_{sd}=\boldsymbol{E}_{ds}$	$\boldsymbol{E}_{\delta s}$	$\boldsymbol{H}_{d\sigma}$

Thus the following problem arises: what can be said of the projection of a set $Z\in\boldsymbol{H}_{s\delta}$? The answer is simple: nothing in general. If $X=Y=$ the unit interval, if $\boldsymbol{E}$ is the class of all closed subintervals of X and if $\boldsymbol{F}$ is the class of all subsets of Y, then for every $A\subset X$ there is a set $C\in\boldsymbol{H}_{s\delta}$ such that $\mathrm{P}(C)=A$.

In fact, denoting by D the diagonal, *i. e.* the set of points (x,x), where $x\in X$, and setting

$$C(A\times A)\cdot D,\qquad I_m^n=\Big\langle\frac{m-1}{n},\frac{m}{n}\Big\rangle\qquad\text{for}\quad m=1,2,\ldots,n;\quad n=1,2,\ldots$$

and

$$C_n=[I_1^n\times(AI_1^n)]+[I_2^n\times(AI_2^n)]+\ldots+[I_n^n\times(AI_n^n)]$$

we obtain

$$C_n\in\boldsymbol{H}_s,\qquad C=C_1C_2\ldots\in\boldsymbol{H}_{s\delta},\qquad \mathrm{P}(C)=A.$$

The purpose of this paper[1]) is to prove that, nevertheless, under some assumptions relating to $\boldsymbol{F}$, the projections of sets belonging to $\boldsymbol{H}_{s\delta}$, $\boldsymbol{H}_{\sigma\delta}$, etc. belong to some classes determined by the class $\boldsymbol{E}$ only.

The notion which plays an essential role in what follows is that of a *compact class of sets*[2]). A class $\boldsymbol{F}$ is *compact*, if, for every sequence $F_n\in\boldsymbol{F}$, we have $F_1F_2\ldots\neq 0$ whenever $F_1F_2\ldots F_n\neq 0$ for $n=1,2,\ldots$ Consequently a multiplicative class $\boldsymbol{F}$ (*i. e.* such that $\boldsymbol{F}_d=\boldsymbol{F}$) is compact if and only if every descending sequence of non-void sets $F_n\in\boldsymbol{F}$ has a non-void product.

Next, denote by $\boldsymbol{Q}_A$ the class of all sets of the form $\Sigma Q_{k_1}Q_{k_1k_2}\ldots$, where $Q_{k_1k_2\ldots k_n}\in\boldsymbol{Q}$ and the summation extends over all infinite sequence $k_1,k_2,\ldots$ of natural numbers. It is well known that this operation, termed *operation* (A), has the following properties[3]):

$$\boldsymbol{Q}_A\supset\boldsymbol{Q}_\sigma+\boldsymbol{Q}_\delta,\qquad \boldsymbol{Q}_{AA}=\boldsymbol{Q}_A. \tag{4}$$

The result of this paper is

Theorem. *If the class $\boldsymbol{F}$ is compact, then*

	I	II′	II
if $0\neq Z\in$	$\boldsymbol{H}_{s\delta}$	$\boldsymbol{H}_{\sigma\delta}$	$\boldsymbol{H}_A$
then $\mathrm{P}(Z)\in$	$\boldsymbol{E}_{s\delta}$	$\boldsymbol{E}_A$	$\boldsymbol{E}_A$

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[1]) Presented to the Polish Mathematical Society, Wrocław Section, on April 1, 1952.

[2]) Cf. E. Marczewski, *On compact measures*, Fundamenta Mathematicae **40** (1953), p. 113-124, especially p. 115.

[3]) Cf. F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1935, p. 90-93.

Notice that the implication II′ (which is an easy consequence of II and (4)) cannot be strengthened. Indeed, in case $\boldsymbol{E}=\boldsymbol{F}=$ the class of all closed subintervals of the unit interval I, the projections of all sets belonging to $\boldsymbol{H}_{\sigma\delta}$ constitute the class of all analytical subsets of I, or, in other words, the whole class $\boldsymbol{E}_A$.

The theorem will be applied in a forthcoming paper on measures in product spaces.

2. Fundamental lemma. For $Z\subset X\times Y$ and $x\in X$ denote by $S_x(Z)$ the vertical section of Z corresponding to x; in other terms, $y\in S_x(Z)$ if and only if $(x,y)\in Z$. We shall prove that

If $Z_1\supset Z_2\supset\ldots$, where $Z_j\subset X\times Y$ for $j=1,2,\ldots$, and if for every $x\in X$ the sequence $S_x(Z_j)$ forms a compact class, then $P(Z_1Z_2\ldots)=P(Z_1)\cdot P(Z_2)\cdot\ldots$

In virtue of (2), it suffices to prove that

$$P(Z_1Z_2\ldots)\supset P(Z_1)\cdot P(Z_2)\cdot\ldots$$

Let $x\in P(Z_1)\cdot P(Z_2)\cdot\ldots$ Consequently the sets

$$S_x(Z_1)\supset S_x(Z_2)\supset\ldots$$

are non-void and, since they form a compact class, there is an y such that

$$y\in S_x(Z_1)\cdot S_x(Z_2)\cdot\ldots$$

or, in other words, $(x,y)\in Z_1Z_2\ldots$ whence $x\in P(Z_1Z_2\ldots)$, q. e. d.

3. The operation (A). For any system of sets $\mathfrak{Z}=\{Z_{k_1k_2\ldots k_n}\}$ where $(k_1,k_2,\ldots,k_n)$ runs over the set of all finite sequences of natural numbers let us set

$$A(\mathfrak{Z})=\sum_{\{k_j\}}Z_{k_1}Z_{k_1k_2}\ldots,$$

where the summation extends over all infinite sequences $\{k_j\}$.

For every class $\boldsymbol{Q}$ of sets, $\boldsymbol{Q}_A$ is by definition the class of all sets $A(\mathfrak{Z})$, where $\mathfrak{Z}$ runs over all systems of sets belonging to $\boldsymbol{Q}$.

A system $\{Z_{k_1k_2\ldots k_n}\}$ of sets will be called *monotone*, if we always have

$$Z_{k_1k_2\ldots k_nk_{n+1}}\subset Z_{k_1k_2\ldots k_n}.$$

The following lemma is essential for projection properties:

If $0\neq Z\in\boldsymbol{Q}_A$, then there exists a m o n o t o n e *system $\mathfrak{Z}=\{Z_{k_1k_2\ldots k_n}\}$ of non-void sets belonging to $\boldsymbol{Q}_d$ such that $Z=A(\mathfrak{Z})$.*

I shall outline the proof of this lemma.

Since $Z\in\boldsymbol{Q}_A$, there is a system $\mathfrak{A}=\{A_{k_1k_2\ldots k_n}\}$ of sets belonging to $\boldsymbol{Q}$ such that $Z=A(\mathfrak{A})$.

Let us set

$$B_{k_1k_2\ldots k_n}=A_{k_1}A_{k_1k_2}\ldots A_{k_1k_2\ldots k_n};$$

obviously $\mathfrak{B}=\{B_{k_1k_2\ldots k_n}\}$ is a monotone system of sets belonging to $\boldsymbol{Q}_d$ and we have $Z=A(\mathfrak{B})$.

Next, denote by M the set of all infinite sequences $\{k_j\}$ of natural numbers such that

$$B_{k_1}B_{k_1k_2}B_{k_1k_2k_3}\ldots\neq 0. \tag{5}$$

Hence

$$Z=\sum_{\{k_j\}\in M}B_{k_1}B_{k_1k_2}\ldots,$$

where the summation extends only over sequences $\{k_j\}$ belonging to M.

By a suitable new numeration of the sets $B_{k_1k_2\ldots k_n}$ appearing in this sum, namely by repeating some sets satisfying (5), we obtain the required system $\mathfrak{Z}$.

4. Proof of Theorem. I. Suppose $0\neq Z\in\boldsymbol{H}_{s\sigma}$. Thus we may write $Z=Z_1Z_2\ldots$, where $Z_1\supset Z_2\supset\ldots$ and $Z_j\in\boldsymbol{H}_{ds}$, whence for every $x\in X$ we have

$$S_x(Z_j)\in\boldsymbol{F}_{ds}\qquad\text{or}\qquad S_x(Z_j)=0.$$

Since $\boldsymbol{F}$ is compact by hypothesis, the class $\boldsymbol{F}_{ds}$ is also compact [4]), and by the fundamental lemma

$$P(Z)=P(Z_1)\cdot P(Z_2)\cdot\ldots\in\boldsymbol{E}_{ds\delta}=\boldsymbol{E}_{s\delta},\qquad\text{q. e. d.}$$

II. Suppose $0\neq Z\in\boldsymbol{H}_A$. By lemma 3 there is a system $\mathfrak{Z}$ of non-void sets [5])

$$Z_{k_1k_2\ldots k_n}=E_{k_1k_2\ldots k_n}\times F_{k_1k_2\ldots k_n}$$

such that

$$E_{k_1k_2\ldots k_n}\in\boldsymbol{E}_d,\qquad F_{k_1k_2\ldots k_n}\in\boldsymbol{F}_d,$$

$$E_{k_1k_2\ldots k_nk_{n+1}}\subset E_{k_1k_2\ldots k_n},\qquad F_{k_1k_2\ldots k_nk_{n+1}}\subset F_{k_1k_2\ldots k_n},$$

$$Z=A(\mathfrak{Z}).$$

Since, for every sequence $(k_1,k_2,\ldots,k_n)$ we have

$$S_x(Z_{k_1k_2\ldots k_n})=F_{k_1k_2\ldots k_n}\qquad\text{or}\qquad S_x(Z_{k_1k_2\ldots k_n})=0,$$

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[4]) Cf. E. Marczewski, l. c. [2]), theorems 2 (ii) and 2 (iii).

[5]) The second part of the lemma (*i. e.* the statement that $Z_{k_1k_2\ldots k_n}$ are non-void) is superflous in the case when the empty set belongs to $\boldsymbol{E}$. Then it suffices to consider a monotone system $\mathfrak{Z}$, such that the sets $E_{k_1k_2\ldots k_n}$ and $F_{k_1k_2\ldots k_n}$ are both void or both non-void.

for every $x \in X$ all the sections $S_x(Z_{k_1k_2...k_n})$ form a compact class. Thus, for every infinite sequence $(k_1, k_2, ...)$, we may apply the fundamental lemma to the sequence of sets $Z_{k_1} \supset Z_{k_1k_2} \supset ...$ Since $P(A \times B) = A$ whenever B is non-void, we obtain

$$P(Z) = \sum_{\{k_j\}} P(Z_{k_1} Z_{k_1k_2} ...) = \sum_{\{k_j\}} E_{k_1} E_{k_1k_2} ... \in \boldsymbol{E}_{dA} = \boldsymbol{E}_A, \quad \text{q. e. d.}$$

Note. The assumption in II of the compactness of $\boldsymbol{F}$ may be replaced by the following: there is a compact class $\boldsymbol{F}^*$ such that $\boldsymbol{F} \subset \boldsymbol{F}^*_A$. In fact, if $\boldsymbol{H}^*$ denotes the class of all sets $E \times F$ where $E \in \boldsymbol{E}$ and $F \in \boldsymbol{F}^*$, then it is easily seen that $\boldsymbol{H}^*_A \supset \boldsymbol{H}$, whence, by (4), $\boldsymbol{H}^*_A = \boldsymbol{H}_A$. Thus we reduce the generalized form of II from the original one, applied to the classes $\boldsymbol{E}, \boldsymbol{F}^*$ and $\boldsymbol{H}^*$.

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Remarks on the Compactness and non Direct Products of Measures *)

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Introduction. The direct product of two normalized measures μ in X and ν in Y, *i. e.* the measure λ in $X \times Y$ such that

(1) $$\lambda(A \times B) = \mu(A) \cdot \nu(B),$$

has many regularity properties, *e. g.*

I. If μ and ν are countably additive, so is λ [1]).

II. If μ, ν and λ are σ-measures, then

$$\lambda_i(E \times Y) = \mu_i(E) \quad \text{for} \quad E \subset X$$

(where, for every measure μ, the symbol μ_i denotes the inner measure induced by μ) [2]).

In sections 1 and 2 of this paper we deal with the same propositions for non-direct products, *i. e.* when the condition (1) is replaced by the weaker ones:

(2) $$\lambda(A \times Y) = \mu(A), \qquad \lambda(X \times B) = \nu(B).$$

It turns out that *the propositions* I *and* II *for the nondirect products remain true under the additional assumption that the measure* ν *is compact, and that they are false in general.*

Sections 3 and 4 are further contributions to the study of compact measures. It is known that the minimal σ-extension of a compact measure is also compact. Here we show that the converse of this theorem is not true.

We apply theorems on compact measures proved in the paper: Marczewski [3] (quoted below as C), and theorems on projections proved in the paper: Marczewski and Ryll-Nardzewski [4] (quoted as P).

*) Presented to the Polish Mathematical Society, Wrocław Sect., on April 1, 1952.

[1]) This follows *e. g.* from the existence of the direct σ-product of any two σ-measures, see *e. g.* Halmos [2], p. 144, Theorem B.

[2]) This follows *e. g.* from the abstract Fubini theorem. See ibidem.