On the Decomposition of a Locally Connected Compactum into Cartesian Product of a Curve and a Manifold

By

K. Borsuk (Warszawa)

1. A space [1]) X is called *topologically prime* if there exist no two spaces Y and Z, each containing at least 2 points, such that the Cartesian product $Y\times Z$ is homeomorphic to X. The factorization into prime factors is in general not unique [2]). However there exist important special cases in which the uniqueness of the factorization holds [3]) and also other important cases in which the problem of uniqueness remains open. In particular the question whether the factorization in the 1-dimensional spaces is unique remains still unsolved.

The purpose of this note is to show that if X_1 and X_2 are locally connected compacta of dimension $\leqslant 1$ and Y is a manifold (closed or with boundary) then the homeomorphism of $X_1\times Y$ with $X_2\times Y$ implies the homeomorphism of X_1 and X_2. In particular it follows that the decomposition of a space in the Cartesian product of a locally connected compactum of dimension $\leqslant 1$ and of a finite number of simple arcs and simple closed curves is unique.

2. Let X be an arbitrary space. A point $x\in X$ is said to be *Euclidean* provided that there exists a neighbourhood U of x in X homeomorphic to a Euclidean space E_n (of an arbitrary dimension n). By $\alpha(X)$ we denote the set consisting of all Euclidean points of X. Evidently $\alpha(X)$ is an open subset of X. The components of $\alpha(X)$ will be called *Euclidean components* of X. Evidently if C is a locally connected curve then the diameters of the Euclidean components of C converge to zero (provided that C contains an infinite number of Euclidean components).

A point $x\in X$ is said to be *semi-Euclidean* provided that it is not Euclidean and there exists a neighbourhood U of x in X homeomorphic with the Euclidean half-space, *i. e.* with the set of all points $(x_1,x_2,\dots,x_n)\in E_n$ with $x_n\geqslant 0$. By $\beta(X)$ we denote the set consisting of all semi-Euclidean points of X. Evidently the set $\beta(X)$ is open in the $X-\alpha(X)$ and it is $\alpha\big(\beta(X)\big)=\beta(X)$.

[1]) Troughout this paper all spaces are metric.

[2]) See [1], p. 825. Also [2], p. 284 and [3].

[3]) See [4] and [5].

By $\gamma(X)$ we denote the set $X-\alpha(X)-\beta(X)$. Thus

$$X=\alpha(X)+\beta(X)+\gamma(X) \tag{1}$$

and the sets $\alpha(X)$, $\beta(X)$, $\gamma(X)$ are disjoint.

Evidently

$$\alpha(X\times Y)\supset\alpha(X)\times\alpha(Y)$$

and

$$\beta(X\times Y)\supset\alpha(X)\times\beta(Y)+\beta(X)\times\alpha(Y)+\beta(X)\times\beta(Y).$$

The problem whether in these formulas the symbol "$\supset$" may be replaced by the symbol "$=$" remains open.

Examples: 1. A (closed) manifold (in the classical sense) is characterized as a continuum X such that $\beta(X)=\gamma(X)=0$. A manifold with a boundary is characterized as a continuum X such that $\gamma(X)=0$. Evidently for a bounded n-dimensional manifold X the set $\beta(X)$ contains a finite number of components and each of them is a closed $(n-1)$-dimensional manifold. In particular a n-dimensional Euclidean cell is a manifold with a boundary.

2. If C is a locally connected curve then $\alpha(C)$ is the same as the interior of the set C_2 composed of all points of order 2 [4]).

In fact, every point $x\in\alpha(C)$ lies evidently in the interior of C_2. On the other hand, if a point x belongs to the interior of C_2 then [5]) there exists a simple arc $L\subset C_2$ such that $x\in\alpha(L)$. In order to prove that $x\in\alpha(C)$ it suffices to show that

$$\text{if}\quad \alpha(L)\subset C_2\quad \text{then the set } \alpha(L) \text{ is open in } C. \tag{2}$$

For if it is not so, then there exists a sequence $\{x_n\}\subset C-L$ convergent to x. Since C is locally connected, there exists, for sufficiently large n, a simple arc $L'\subset C_2$ joining x_n with x and having the diameter less than the distance between x and the end points of L. In L' lies another simple arc L'' containing only one point y of L. Then the order of C in y is $\geqslant 3$, which is incompatible with the relation $y\in L\subset C_2$.

In an analogous manner we show that $\beta(C)$ is the same as the set composed by all points $x\in C$ of order 1 lying in the interior of $C_2+(x)$. It follows that $\gamma(C)$ is the same as the closure of the set composed by all points of C of order $\geqslant 3$.

It follows that the relation

$$\beta(C)+\gamma(C)=0$$

characterises under locally connected curves C the simple closed curves, and the relation

$$\beta(C)\neq 0=\gamma(C),$$

the simple arcs.

[4]) In the sense of Menger-Urysohn. See [6], p. 483 and [7], p. 279.

[5]) [8], p. 577.

3. Let X be an arbitrary space and I the interval $0 \leqslant t \leqslant 1$. A continuous mapping $f(x,t)$ of $X \times I$ into X is called a *homotopic deformation* of X if

$$f(x,0) = x \quad \text{for every} \quad x \in X.$$

A point p of a space X is *homotopically labile* [6]) whenever for every $\varepsilon > 0$ there exists a homotopic deformation of X satisfying the following conditions:

(3) $$\varrho(x, f(x,t)) < \varepsilon \quad \text{for every} \quad (x,t) \in X \times I,$$

(4) $$f(x,1) \neq p \quad \text{for every} \quad x \in X.$$

The points which are not homotopically labile are said to be *homotopically stable*. A point $x_0 \in X$ is said to be *homotopically fixed* in X if for every homotopic deformation $f(x,t)$ of X we have $f(x_0,1) = x_0$.

Examples: 3. In a manifold with a boundary the set of all homotopically labile points is the same as the boundary of the manifold.

4. Let C be a locally connected curve. A point $p \in C$ is homotopically labile if and only if it is of order 1 and there exists a dendrite $D \subset C$ containing p in its interior.

The sufficiency is evident. On the other hand, if $p \in C$ is homotopically labile, then it is of order 1 [7]). If no dendrite $D \subset C$ constitutes a neighbourhood of p then every neighbourhood of p contains a simple closed curve. Then p is not homotopically labile [8]).

5. Let A be a Euclidean component of a locally connected curve C. A point $p \in \bar{A}$ is homotopically labile in C if and only if $p \in \bar{A} \cdot \beta(C)$.

The sufficiency is evident. To prove the necessity let us observe that if $p \in \bar{A}$ is homotopically labile in C, then (by example 4) p is of order 1 in C and there exists a dendrite $D \subset C$ being a neighbourhood of p in C. Let L_1 be a simple arc $\subset D$ such that p is one of its end points. Let q denote the other end point of L_1. If L_1 is not a neighbourhood of p then for every $\varepsilon > 0$ there exists a simple arc $L_2 \subset D$ with the diameter $< \varepsilon$ containing a point $r \in D - L_1$. It follows that there exists a simple arc $L_3 \subset L_2$ such that $L_1 \cdot L_3$ contains only one point s. If $\varepsilon < \varrho(p,q)$ then $s \neq q$. Moreover $s \neq p$, since p is of order 1. Hence s is a point of order $\geqslant 3$ of D and $\varrho(p,s) < \varepsilon$. If ε is sufficiently small, then the diameter of the component G of $D-(s)$ containing p is arbitrarily small. Since D is a neighbourhood of p in C, we infer that for sufficiently small ε the set G constitutes also a component of $C-(s)$ and $\bar{G} \cdot \overline{C-G} = (s)$. Moreover $A \cdot G \neq 0 \neq A - G$. It follows that $s \in A$, which is impossible, because s is of order $\geqslant 3$.

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[6]) [9], p. 160.

[7]) [9], p. 168, Corollary 3.

[8]) [9], p. 175, Corollary 2.

6. In a locally connected curve C the homotopically fixed points are the same as the points in which C is not locally a dendrite.

In fact, if there exists a dendrite $D \subset C$ which is a neighbourhood of a point x_0 in C and x_1 is another point of D, then setting

$$f(x,t) = x \quad \text{for every} \quad (x,t) \in C \times (0) + \overline{C-D} \times I,$$
$$f(x_0,1) = x_1,$$

we obtain a continuous function $f(x,t)$ mapping the compactum

$$Z = C \times (0) + \overline{C-D} \times I + (x_0) \times (1) \subset C \times I$$

onto a subset of C. The values of $f(x,t)$ in the set $Z \cdot (D \times I)$ belong to the set D which is an absolute retract. It follows that $f(x,t)$ can be extended to a homotopic deformation of C carrying x_0 in x_1. Hence x_0 is not homotopically fixed.

If, however, there exists no dendrite $D \subset C$ which is a neighbourhood of x_0 in C, then for every $n = 1, 2, \ldots$ there exists a simple closed curve $\Omega_n \subset C$ such that the distance $\varrho(x_0, \Omega_n)$ and also the diameter of Ω_n converge to 0. Then for every homotopic deformation $f(x,t)$ of C we have [8])

$$\Omega_n \subset f(\Omega_n, 1).$$

In every curve Ω_n let us choose a point x_n. Then $x_n \to x_0$ and $\varrho(x_n, f(x_n,1)) \to 0$. Hence

$$f(x_0,1) = \lim_{n \to \infty} f(x_n, 1) = \lim_{n \to \infty} x_n = x_0,$$

i. e. the point x_0 is homotopically fixed.

4. ***Lemma.*** *Let C be a locally connected curve, M a manifold (closed or not), $x \in C$ and $y \in M$. Then*

(5) *(x,y) is homotopically labile in $C \times M$ if and only if x is homotopically labile in C or y is homotopically labile in M.*

(6) *$(x,y) \in \alpha(C \times M)$ if and only if $x \in \alpha(C)$ and $y \in \alpha(M)$.*

Proof. Since the considered properties are local, we can assume that M is a Euclidean n-dimensional cell Q. In this case statement (5) is proved on another place [9]). Hence it remains to give the proof for statement (6).

It is evident that $x \in \alpha(C)$ and $y \in \alpha(M)$ imply that $(x,y) \in \alpha(C \times M)$. On the other hand, it is known [10]) that if $(x,y) \in \alpha(C \times M)$, *i. e.* if (x,y) has a neighbourhood in $C \times M$ homeomorphic with the Euclidean $(n+1)$-dimensional space, then $x \in \alpha(C)$. Moreover the point $(x,y) \in \alpha(C \times M)$ is not homotopically labile in $C \times M$, hence (by 5) y is not homotopically

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[9]) [9], p. 163 and p. 175, Corollary 4.

[10]) [5], p. 275.

labile in M. It follows, by example 3, that y does not belong to the boundary of M, *i. e.* $y \in a(M)$.

5. A point p of an arbitrary space X is said to be *approximately Euclidean* provided that for every $\varepsilon > 0$ there exists a homotopic deformation $f(x,t)$ of X satisfying the conditions:

(7) $$\varrho\big(f(x,t),x\big) < \varepsilon \quad \textit{for every} \quad (x,t) \in X \times I,$$

(8) $$p \in a\big(f(X,1)\big).$$

(9) *The dimension of $f(X,1)$ at p is equal to the dimension of X at p.*

Examples: 7. Every Euclidean point of X is also approximately Euclidean.

8. Let Q_n denote the 2-dimensional Euclidean cell defined in E_2 by the inequality

$$\left(x + \frac{3}{2^{n+1}}\right)^2 + y^2 \leqslant \frac{1}{4^{n+1}} \quad \text{for} \quad n = 1, 2, \ldots$$

It will easily be seen that in the set $X = \sum_{n=1}^{\infty} Q_n + I \times (0)$ the point $(0,0)$ is approximately Euclidean, but not Euclidean.

9. In a manifold (closed or not) the approximately Euclidean points are the same as the Euclidean points.

Remark. For an $\varepsilon > 0$ consider the continuous function $\lambda(u)$ defined for $u \geqslant 0$ by the formulas:

$$\lambda(u) = \begin{cases} 1 & \text{for} \quad 0 \leqslant u \leqslant 2\varepsilon, \\ 3 - \dfrac{u}{\varepsilon} & \text{for} \quad 2\varepsilon \leqslant u \leqslant 3\varepsilon, \\ 0 & \text{for} \quad u \geqslant 3\varepsilon. \end{cases}$$

If $f(x,t)$ is a homotopic deformation of f satisfying the conditions (7)-(9), then setting

$$\varphi(x,t) = f\big(x, \lambda(\varrho(x,p)) \cdot t\big) \quad \text{for} \quad (x,t) \in X \times I$$

we obtain a continuous deformation of X satisfying the conditions:

(10) $$\varrho\big(\varphi(x,t),x\big) < \varepsilon \quad \textit{for every} \quad (x,t) \in X \times I,$$

(11) $$p \in a\big(\varphi(X,1)\big),$$

(12) *the dimension of $\varphi(X,1)$ at p is equal to the dimension of X at p,*

(13) *if $\varrho(x,p) \geqslant 3\varepsilon$ then $\varphi(x,t) = x$ for every $t \in I$.*

Now consider another space Y and suppose that there exists a homeomorphic mapping h of a neighbourhood U of p in X such that $V = h(U)$ constitutes a neighbourhood in Y of the point $q = h(p)$. For every $\eta > 0$ there exists an $\varepsilon > 0$ such that

(14) *If $x \in X$ and $\varrho(x,p) < 3\varepsilon$ then $x \in U$ and $h(x) \in V$,*

(15) *If $x_1, x_2 \in X$ and $\varrho(x_1,p) < \varepsilon$, $\varrho(x_2,p) < \varepsilon$ then $\varrho\big(h(x_1), h(x_2)\big) < \eta$.*

Let $\varphi(x,t)$ be a continuous deformation of X satisfying the condition (10)-(13). Setting

$$\psi(y,t) = h\varphi[h^{-1}(y),t] \quad \text{for every} \quad (y,t) \in V \times I,$$
$$\psi(y,t) = y \quad \text{for every} \quad (y,t) \in (Y - V) \times I,$$

we obtain a continuous deformation of Y satisfying the conditions:

(16) $$\varrho\big(\psi(y,t),y\big) < \eta \quad \textit{for every} \quad (y,t) \in Y \times I,$$

(17) $$q \in a[\psi(V,1)],$$

because $\varphi(U,1)$ constitutes a neighbourhood of p in X and $p \in a[\varphi(U,1)]$.

(18) *The dimension of $\psi(Y,1)$ at q is equal to the dimension of Y at q,*

because the dimension of $\psi(Y,1)$ at q is equal to the dimension of $h^{-1}\psi(Y,1)$ at p, that is to the dimension of $\varphi(U,1)$ at p. But this last dimension is equal to the dimension of X at p, hence also to the dimension of Y at q.

Thus we see that *the property of a point being approximately Euclidean is a local one.*

6. *Lemma.* *If p is an approximately Euclidean point of an arbitrary space X, then X is locally contractible at p.*

Proof. Let f be a homotopic deformation satisfying (7) and (8). By (8) there exists a neighbourhood V of p in $f(X,1)$ and a homotopic deformation $g(y,t)$ of V in $f(X,1)$ such that

$$\varrho\big(g(y,t),y\big) < \varepsilon \quad \text{for every} \quad (y,t) \in V \times I$$

and

$$g(y,1) = p \quad \text{for every} \quad y \in V.$$

The set $U = f^{-1}(V,1)$ constitutes a neighbourhood of p in X. Setting

$$\varphi(x,t) = f(x,2t) \quad \text{for every} \quad x \in U \quad \text{and} \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\varphi(x,t) = g[f(x,1), 2t-1] \quad \text{for every} \quad x \in U \quad \text{and} \quad \tfrac{1}{2} \leqslant t \leqslant 1,$$

we obtain a homotopic deformation of U satisfying the following conditions:

$$\varrho\big(\varphi(x,t),p\big) < 2\varepsilon \quad \text{for every} \quad (x,t) \in U \times I,$$
$$\varphi(x,1) = p \quad \text{for every} \quad x \in U.$$

Hence X is locally contractible at p.

7. Theorem. *Let C be a locally connected curve. A necessary and sufficient condition that a point $p \in C$ be approximately Euclidean is that C is locally contractible at p and p is of order 2.*

Proof. To prove the necessity suppose that p is approximately Euclidean. By the lemma of No. 6 the curve C is locally contractible at p. Hence there exists a dendrite $D \subset C$ which is a neighbourhood of p in C. By (8) there exists a simple arc $L \subset C$ such that $p \in a(L)$. Hence p is of order $\geqslant 2$. Suppose, contrary to our condition, that p is of order >2. Then $D-(p)$ contains at least 3 components $\Gamma_1, \Gamma_2, \Gamma_3$. Let us choose a point $p_i \in \Gamma_i$ for $i=1,2,3$ and let L_i denote the simple arc joining in D the points p and p_i. It is easy to see that there exists a positive ε such that

(19) $$\varrho(x, C-D) > \varepsilon \quad \text{for every} \quad x \in L_1+L_2+L_3,$$

(20) For every component $\Gamma \neq \Gamma_i$ of $D-(p)$ it is $\varrho(p_i, \Gamma) > \varepsilon$ for $i=1,2,3$.

Let $f(x,t)$ be a homotopic deformation of $X=C$ satisfying (7) and (9). Then, by (19), it is $f(L_i,1) \subset D$ and, by (20), $f(p_i,1) \in \Gamma_i$ for $i=1,2,3$. It follows that $L_i \subset f(L_i,1)$ for $i=1,2,3$, hence $L_1+L_2+L_3 \subset f(C,1)$. Consequently p is of order $\geqslant 3$ in $f(C,1)$, *i. e.* condition (8) fails.

To prove the sufficiency let us observe that the local contractibility of C at p implies that for every $\varepsilon > 0$ there exists a dendrite D of diameter less than ε constituting a neighbourhood of p in C. Since p is of order 2, there exists a simple arc $L \subset D$ such that $p \in a(L)$. Moreover we can assume that the diameter of L is less than an arbitrarily given $\eta > 0$. Let p_1, p_2 be the end points of L and let D_0 denote the closure of the component of $D-(p_1)-(p_2)$ containing p. Evidently, for η sufficiently small, D_0 is a dendrite such that $L \subset D_0 \subset D - \overline{C-D}$ and consequently $D_0 \cdot \overline{C-D_0} = D_0 \cdot \overline{D-D_0} = (p_1)+(p_2)$. It is easy to observe that there exists a homotopic deformation f of D_0 satisfying the conditions:

$$f(x,t)=x \quad \text{for every} \quad x \in L, \ 0 \leqslant t \leqslant 1,$$
$$f(D_0,1)=L.$$

Setting $f(x,t)=x$ for every $(x,t) \in (C-D_0) \times I$ we obtain a homotopic deformation f of C satisfying the conditions (7), (8) and (9).

8. Let us denote by T_n the polytope made up of three n-dimensional simplexes $\Delta_1^n, \Delta_2^n, \Delta_3^n$ having exactly one $(n-1)$-dimensional face Δ^{n-1} in common. In particular T_1 is homeomorphic to the sum of three simple arcs disjoint except for one of their and points, and T_n is homeomorphic to the Cartesian product of T_1 and $n-1$ simple arcs. Let a_n denote the barycenter of Δ^{n-1} and Δ the polytope made up of all $(n-1)$-dimensional (closed) faces of $\Delta_1^n, \Delta_2^n, \Delta_3^n$, distinct from Δ^{n-1}.

Lemma. *If h is a homeomorphism mapping T_n into a n-dimensional space X then the point $h(a_n)$ is not approximately Euclidean in X.*

Proof. Suppose, on the contrary, that $h(a_n)$ is approximately Euclidean in X. By lemma 6 there exists a neighbourhood U_0 of $h(a_n)$ (in the space X) contractible in X. Hence every true cycle (in the sense of Vietoris) lying in a compact subset of U_0 is homologous to zero in X.

Since, for every $\varepsilon > 0$, there exists a homeomorphism mapping T_n in its subset with diameter $<\varepsilon$, in such a manner that a_n remains fixed, we can assume, without loss of generality, that

$$\delta[h(T_n)] < \tfrac{1}{2}\varrho\big(h(a_n), X-U_0\big),$$

where δ denotes the diameter. Moreover we can assume that h is the identical mapping. Hence $T_n \subset U_0$.

Let us choose an orientation in the simplexe Δ^{n-1} and assign to each of the simplexes Δ_ν^n, $\nu=1,2,3$ an orientation such that the boundary of Δ_ν^n contains Δ^{n-1} with the coefficient 1. Then the chain

$$\varkappa = \Delta_1^n + \Delta_2^n + \Delta_3^n,$$

in which we regard the coefficients as the rests modulo 3, has as its boundary a $(n-1)$-dimensional cycle (mod 3) γ lying in Δ.

Let γ denote the sequence $\{\gamma_k\}$ made up of the successive barycentric subdivisions of γ. Evidently γ is a true $(n-1)$-dimensional convergent cycle mod 3 (in the sense of Vietoris) and it is homologous to zero in T_n, but not homologous to zero in any closed proper subset of T_n. Moreover, if P is a compact subset of U_0 such that γ is homologous to zero in P, then $T_n \subset P$. In fact, if $\lambda = \{\lambda_n\}$ denotes a true chain (mod 3) in P such that

(21) $$\partial \lambda_k = \gamma_k \quad \text{for} \quad k=1,2,\ldots$$

and if $\varkappa = \{\varkappa_k\}$ denotes the sequence of the successive barycentric subdivisions of $\varkappa$, then

(22) $$\partial \varkappa = \partial \lambda = \gamma.$$

Hence $\varkappa - \lambda$ is an n-dimensional true cycle (mod 3) lying in $P+T_n \subset U_0$. But every cycle lying in U_0 is homologous to zero in X. In particular the n-dimensional cycle $\varkappa-\lambda$ is homologous to zero in X, and since $\dim X \leqslant n$ we infer, that $\varkappa-\lambda$ is homologous to zero in $P+T_n$. It follows, by the well-known theorem of Phragmen-Brouwer, that γ is homologous to zero in the set $P \cdot T_n$ and consequently $T_n \subset P$.

By our assumption a_n is approximately Euclidean in X, consequently there exists, for every $\varepsilon > 0$, a homotopic deformation $f(x,t)$ of X such that

$$\varrho\big(f(x,t), x\big) < \varepsilon \quad \text{for every} \quad (x,t) \in X \times I,$$
$$a_n \in a[f(x,1)].$$

If ε is sufficiently small, then $f(x,1)$ carries the true cycle γ into the true cycle $f(\gamma,1)$ lying in the set $X-U_\varepsilon$, where U_ε denotes the set composed of all points $p\in X$ with $\varrho(p,a_n)<\varepsilon$. Moreover, if ε is sufficiently small, the set $f(T_n,I)$ lies in U_0.

But the set $U_\varepsilon\cdot T_n$ contains three n-dimensional simplexes having exactly one $(n-1)$-dimensional face in common. Hence $U_\varepsilon\cdot T_n$ is not a subset of $a[f(X,1)]$. We infer that for ε sufficiently small there exists a point

(23) $$b\in T_n-f(A,I)-f(T_n,1).$$

But

$$\gamma\sim f(\gamma,1) \quad \text{in } f(A,I),$$
$$f(\gamma,1)=\partial f(\varkappa,1)\sim 0 \quad \text{in } f(T_n,1).$$

It follows that γ is homologous to zero in the compact set $f(A,I)+f(T_n,1)\subset U_0$, which implies, as we have shown, that $T_n\subset f(A,I)+f(T_n,1)$, contrary to (23). Thus the lemma is proved.

9. *Theorem.* *Let $C_1,C_2,\ldots,C_k$ be locally connected curves and M a manifold (closed or not). In order that a point $p=(p_1,p_2,\ldots,p_k,q)\in X=C_1\times C_2\times\ldots\times C_k\times M$ be approximately Euclidean in X it is necessary and sufficient that p_i be approximately Euclidean in C_i for every $i=1,2,\ldots,k$ and q be Euclidean in M.*

Proof. The sufficiency is evident. To prove the necessity let us observe that if $p=(p_1,p_2,\ldots,p_k,q)$ is approximately Euclidean in X then, by lemma 6, X is locally contractible at p, hence also C_i is locally contractible at p_i for $i=1,2,\ldots,k$. It follows that there exists a dendrite lying in C_i and constituting a neighbourhood of p_i in C_i. Since the property of a point being approximately Euclidean is a local one we can suppose that C_i is a dendrite for $i=1,2,\ldots,k$ and that M is an Euclidean cell, *i. e.* M is of the form $C_{k+1}\times\ldots\times C_n$, where $C_{k+1},\ldots,C_n$ are simple arcs and $n=\dim X$.

The point p, as an approximately Euclidean one, lies in the interior of an n-dimensional cell $\subset X$. We infer [11]) that p is homotopically stable in X. It follows [12]) that p_i is homotopically stable in C_i, hence the order of p_i in C_i is $\geqslant 2$ for $i=1,2,\ldots,n$.

By the theorem of No. 7 it remains to prove that p_i is of order $\leqslant 2$ in C_i for $i=1,2,\ldots,n$.

Suppose, on the contrary, that for an index i the point p_i is of order $\geqslant 3$ in C_i. We can assume that $i=1$. Then there exists three simple arcs $L_1^{(1)},L_1^{(2)},L_1^{(3)}$ starting from p_1 and satisfying the condition

$$L_1^{(1)}L_1^{(2)}=L_1^{(1)}L_1^{(3)}=L_1^{(2)}L_1^{(3)}=(p_1).$$

[11]) [9], p. 168, Corollary 1.

[12]) [9], p. 163.

Since p_i is of order $\geqslant 2$ for $i\geqslant 2$, there exists a simple arc $L_i\subset C_i$ such that $p_i\in a(L_i)$ for $i=1,2,\ldots,n$. Evidently the set $(L_1^{(1)}+L_1^{(2)}+L_1^{(3)})\times L_2\times\ldots\times L_n$ can be topologically mapped into a sum of three n-dimensional simplexes $\Delta_1^n,\Delta_2^n,\Delta_3^n$, having an $(n-1)$-dimensional face Δ^{n-1} in common in such a manner, that the point p is carried into the barycenter a_n of Δ^{n-1}. Applying the lemma of No. 8 we infer that p is not approximately Euclidean in X, contrary to our supposition.

10. Let X be an arbitrary space. By an *isotopic deformation in X* we mean a homotopic deformation $f(x,t)$ of X such that for every $t_0\in I$ the mapping $f(x,t_0)$ maps X into itself topologically.

A point $p\in X$ is said to be *isotopically labile* if for every $\varepsilon>0$ there exists an isotopic deformation $f(x,t)$ in X satisfying the following conditions:

$$\varrho(f(x,t),x)<\varepsilon \quad \text{for every} \quad (x,t)\in X\times I,$$
$$f(x,1)\neq p \quad \text{for every} \quad x\in X.$$

The points which are not isotopically labile are said to be *isotopically stable.*

Evidently every isotopically labile point is also homotopically labile, but not vice versa (see example 11).

If X and X' are two spaces and p is isotopically labile in X and p' is an arbitrary point of X', then the point (p,p') is isotopically labile in $X\times X'$.

Examples: 10. Every semi-Euclidean point is isotopically labile. In particular in an n-dimensional manifold M the isotopically labile points are the same as the points lying on the boundary of M.

11. If C is a locally connected curve, then the isotopically labile points are the same as semi-Euclidean points.

Since every semi-Euclidean point is isotopically labile and every Euclidean point is homotopically stable, it remains to show that every point $p\in\gamma(C)$ is isotopically stable. By example 4 it suffices to prove this in the case when there exists a dendrite D containing p in its interior and when p is of order 1. Suppose that there exists a homotopic deformation $f(x,t)$ in C such that $p\in C-f(C,1)$. Since $p\in\gamma(C)$ there exists in every neighbourhood of p a point p' of order $\geqslant 3$. If we choose p' in a sufficiently small neighbourhood of p then $p'\in D-f(C,1)$. Then the continuum $f(p',I)$ contains a point $p''\in D-f(C,1)$ of order 2 [13]). By theorem 7 the point p'' is approximately Euclidean in C. But there exists a number t_0 such that the homeomorphism $f(x,t_0)$ satisfies the condition

$$f(p',t_0)=p''.$$

[13]) [10], p. 223.

Since p' is of order $\geqslant 3$ there exist in C three simple arcs $L_1^{(1)}, L_1^{(2)}, L_1^{(3)}$ having only the point p' in common. By the lemma of No. 8 the point p'' cannot be approximately Euclidean.

11. *Lemma.* *If C is a locally connected curve and M a manifold then the set of all isotopically labile points of $C\times M$ is the same as the set $C\times\beta(M)+\beta(C)\times M$.*

Proof. Evidently every point $(x_0,y_0)\in C\times\beta(M)+\beta(C)\times M$ is isotopically labile. Moreover we know that every point $(x_0,y_0)\in\alpha(C)\times\alpha(M)$ is homotopically stable, and homotopically stable is also every point (x_0,y_0) such that in every neighbourhood of x_0 in C there exist simple closed curves and $y_0\in\alpha(M)$. Consequently it remains to show that if $x_0\in\gamma(C)$ and there exists a dendrite $D\subset C$ containing x_0 in its interior and if $y_0\in\alpha(M)$, then (x_0,y_0) is isotopically stable.

Suppose, on the contrary, that for every $\varepsilon>0$ there exists an isotopic deformation $f((x,y),t)=(\varphi(x,y,t),\psi(x,y,t))$ of $C\times M$ such that

$$\varrho\big(f((x,y),t),(x,y)\big)<\varepsilon \quad \text{for every} \quad (x,y)\in C\times M, \tag{24}$$
$$(x_0,y_0)\in C\times M-f(X\times Y,1).$$

Let η be a positive number so small that for every $x\in C$ with $\varrho(x,x_0)<2\eta$ we have $x\in D$ and for every $y\in M$ with $\varrho(y,y_0)<2\eta$ we have $y\in\alpha(M)$. Let V_0 denote the neighbourhood of y_0 in M composed of all points y satisfying the inequality $\varrho(y,y_0)<\eta$. If for every $y\in V_0$ it is $\varphi(x_0,y,1)=x_0$ then $\psi(x_0,y,1)$ maps V_0 into $\alpha(M)$ and it is

$$\varrho\big(\psi(x_0,y,1),y\big)<\varepsilon \quad \text{for every} \quad y\in V_0, \quad y_0\in V_0-\psi(x_0,V_0,1). \tag{25}$$

But this is impossible for sufficiently small ε, because the point $a\in\alpha(M)$ is homotopically stable in M [14]).

Hence there exists an isotopic deformation f satisfying (24) and a point $y_0'\in V_0$ such that

$$\varphi(x_0,y_0',1)\neq x_0. \tag{26}$$

Moreover, we can assume that $\varepsilon<\eta$. Then $\varrho\big(\varphi(x_0,y_0',t),x_0\big)<\eta$, hence

$$\varphi(x_0,y_0',t) \text{ lies in the interior of } D \text{ for every } t\in I,$$

and

$$\varrho\big(\psi(x_0,y_0',t),y_0\big)\leqslant\varrho\big(\psi(x_0,y_0',t),y_0'\big)+\varrho(y_0',y_0)<2\eta,$$

hence

$$\psi(x_0,y_0',t)\in\alpha(M) \quad \text{for every} \quad t\in I.$$

[14]) [9], p. 168.

By the continuity of φ and ψ there exists a positive $\varepsilon'>0$ such that if $\varrho(x_0,x_0')<\varepsilon'$ then

$$\varphi(x_0',y_0',t) \text{ lies in the interior of } D \text{ for every } t\in I,$$
$$\varphi(x_0',y_0',1)\neq x_0',$$
$$\psi(x_0',y_0',t)\in\alpha(M) \quad \text{for every} \quad t\in I.$$

Since $x_0\in\gamma(C)$, there exists a point x_0' of order $\geqslant 3$ satisfying the inequality $\varrho(x_0,x_0')<\varepsilon'$. Evidently there exists a homeomorphism mapping a polytope T_n (where $n-1$ denotes the dimension of M) made up of three n-dimensional simplexes $\Delta_1^n,\Delta_2^n,\Delta_3^n$ having one $(n-1)$-dimensional face Δ^{n-1} in common, into $C\times M$ in such a manner that the barycenter of Δ^{n-1} is mapped onto (x_0',y_0'). Applying the lemma of No. 8 we infer that for every $t\in I$ the point $f(x_0',y_0',t)$ is not approximately Euclidean in $C\times M$. But the set $\varphi(x_0',y_0',I)$ is a continuum joining, in the interior of D, the point x_0' with the point $\varphi(x_0',y_0',1)\neq x_0'$. It follows that for some $t_0\in I$ the point $\varphi(x_0',y_0',t_0)$ is of order 2. By the theorem of No. 9 we infer that $f(x_0',y_0',t_0)$ is approximately Euclidean. Thus our supposition that the point (x_0,y_0) is isotopically labile leads to a contradiction.

12. By an *isotopic deformation on* X we understand an isotopic deformation $f(x,t)$ in X satisfying, for every $t_0\in I$, the condition

$$f(X,t_0)=X.$$

Two points $p,q\in X$ are said to be *isotopic on* X if there exists an isotopic deformation $f(x,t)$ on X such that $f(p,1)=q$. Evidently the relation of isotopy is reflexive. Let us show that it is also symmetrical and transitive. Let $f^{-1}(x,t_0)$ denote for every $t_0\in I$ the inverse of the mapping $f(x,t_0)$. It will easily be seen that $f^{-1}(x,t)$ constitutes an isotopic deformation on X and that $f^{-1}(q,1)=p$. Hence isotopy is a symmetrical relation. Moreover if $f(x,t)$ and $g(x,t)$ are two isotopic deformations on X, then setting

$$\varphi(x,t)=g[f(x,t),t] \quad \text{for every} \quad (x,t)\in X\times I$$

we obtain an isotopic deformation on X such that $\varphi(p,1)=g(q,1)$. It follows that the isotopy of p with $q=f(p,1)$ and of q with $r=g(q,1)$ implies the isotopy of p with r, *i. e.* the relation of isotopy is transitive.

We infer that the space X decomposes into disjoint sets of isotopic points. It is clear that these sets are connected (even arcwise connected); we call them *isotopy components* of X. Evidently if p and q are two points belonging to one isotopy component of X, then X is locally homeomorphic in p and in q.

Moreover let us observe that if X and X' are two spaces and $p,q\in X$ are isotopic on X and p',q' are isotopic on X', then the points (p,p') $(q,q')\in X\times X'$ are isotopic on $X\times X'$.

Examples: 12. Let X be the closure of the subset A of the Euclidean plane E_2 composed of all points of the form $(x, \sin\pi/x)$ with $0<x<1$. Then X has 5 isotopy components: A, three 0-dimensional components, each containing one of the points $a_0=(1,0)$, $a_1=(0,1)$ and $a_1=(0,-1)$ respectively, and the interior of the segment $\overline{a_1 a_{-1}}$.

13. Every Euclidean component of an arbitrary space X is an isotopy component of X. In particular the interior of a manifold M is an isotopy component of M. Evidently the other isotopy components of M are identical with the components of the boundary N of M.

14. Let C be a locally connected curve. The isotopy components of C containing at least 2 points are identical with the Euclidean components of C.

In fact, if p and q belong to one Euclidean component of C, then they are isotopic. On the other hand let p and q be two different points of an isotopy component A of C. Let $f(x,t)$ denote the isotopic deformation on C satisfying the condition $f(p,1)=q$. Then there exists a neighbourhood U of p in C such that

$$f(U,1)\cdot U=0.$$

We infer[15]) that U does not contain any simple closed curve. It follows that C is a local dendrite in every point $p\in A$. Since A is arcwise connected, there exists a simple arc L joining the points p and q in A. By the local homeomorphism all points of A have the same order $\geqslant 2$ in C. Since the set of all points of order $\geqslant 3$ of a dendrite is finite or countable[16]), we infer that L is a subset of the set C_2 composed of all points of order 2 of C. By (2) the set $\alpha(L)$ is open in C. It follows that in every point of $\alpha(L)$, hence also in every point of A the curve C is locally homeomorphic with the Euclidean 1-dimensional space. Consequently A is a subset of a Euclidean component of C.

13. ***Lemma.*** *Let C be a locally connected curve and M a manifold. Two points $(x_0,y_0)\in\gamma(C)\times\alpha(M)$ and $(x_1,y_1)\in C\times M$ are isotopic on $C\times M$ if and only if $x_0=x_1$ and $y_1\in\alpha(M)$.*

Proof. It is evident that $x_0=x_1$ and $y_1\in\alpha(M)$ imply the isotopy of (x_0,y_0) and (x_1,y_1).

Let us assume that (x_0,y_0) and (x_1,y_1) are isotopic. By lemma **11** the point (x_0,y_0) is isotopically stable in $C\times M$. Hence also (x_1,y_1) is isotopically stable. We infer, by example 10, that $x_1\in\alpha(C)+\gamma(C)$ and $y_1\in\alpha(M)$. By the lemma of No. 4 the point (x_0,y_0) is not Euclidean in $C\times M$, hence x_1 does not belong to $\alpha(C)$.

It remains to prove that if $(x_0,y_0)\in\gamma(C)\times\alpha(M)$ and $(x_1,y_1)\in\gamma(C)\times\alpha(M)$ are isotopic on $C\times M$, then $x_0=x_1$. If it is not so, then there exists an isotopic deformation $f((x,y),t)=(\varphi(x,y,t),\psi(x,y,t))$ on $C\times M$ such that $x_1=\varphi(x_0,y_0,1)\neq x_0$. For every $t\in I$ the point $(\varphi(x,y,t),\psi(x,y,t))$ is isotopic to (x_0,y_0), hence $\varphi(x,y,t)\in\gamma(C)$ and $\psi(x,y,t)\in\alpha(M)$. The mapping $\varphi(x,y_0,t)$ is a homotopic deformation of C. Since $\varphi(x_0,y_0,1)\neq x_0$ it follows[17]) that not every neighbourhood of x_0 in C contains simple closed curves, *i. e.* x_0 has a neighbourhood in C which is a dendrite. It follows that for some $t_0'\in I$ the point $\varphi(x_0,y_0,t_0')$ is of order 2 and for some other $t_0''\in I$ the point $\varphi(x_0,y_0,t_0'')$ is of order $\geqslant 3$. By the theorem of No. **9** the point $f(x_0,y_0,t_0')=(\varphi(x_0,y_0,t_0'),\psi(x_0,y_0,t_0'))$ is approximately Euclidean in $C\times M$, and the point $f(x_0,y_0,t_0'')=(\varphi(x_0,y_0,t_0''),\psi(x_0,y_0,t_0''))$ is not approximately Euclidean in $C\times M$, which is impossible, because these points are isotopic.

14. Let A be a Euclidean component of a locally connected curve C. We shall say that:

1. A is of the *first type* if $\bar{A}\cdot[\beta(C)+\gamma(C)]=0$ (*i. e.* if $C=A$ is a simple closed curve).

2. A is of the *second type* if $\bar{A}\cdot\beta(C)=0$ and $\bar{A}\cdot\gamma(C)$ contains exactly one point (*i. e.* if $\bar{A}\neq A$ and $\bar{A}$ is a simple closed curve).

3. A is of the *third type* if $\bar{A}\cdot\beta(C)=0$ and $\bar{A}\cdot\gamma(C)$ contains exactly two points.

4. A is of the *fourth type* if $\bar{A}\cdot\beta(C)\neq 0$.

Remark. Only in the case of C being a simple arc there exists a Euclidean component A such that $\bar{A}\cdot\beta(C)$ contains exactly two points. In any other case $\bar{A}\cdot\beta(C)$ contains at most one point.

Lemma. *Let C be a locally connected curve and M a manifold (closed or not). The isotopy components of $C\times M$ are identical with the sets of the following 8 types:*

1⁰ $A\times\alpha(M)$, where A is a Euclidean component of C of the first type,

2⁰ $A\times\alpha(M)$, where A is a Euclidean component of C of the second type,

3⁰ $A\times\alpha(M)$, where A is a Euclidean component of C of the third type,

4⁰ $A\times\alpha(M)$, where A is a Euclidean component of C of the fourth type,

5⁰ $A\times N_\mu$, where A is a Euclidean component of C of the first, second or third type and N_μ is a component of $\beta(M)$,

6⁰ $[\bar{A}\cdot\beta(C)]\times M+A\times\beta(M)$, where A is a Euclidean component of C of the fourth type,

7⁰ $(x_0)\times\alpha(M)$, where $x_0\in\gamma(C)$,

8⁰ $(x_0)\times N_\mu$, where $x_0\in\gamma(C)$ and N_μ is a component of $\beta(M)$.

Moreover, if a homeomorphism h maps $C\times M$ onto the Cartesian product $C'\times M$, where C' is another locally connected curve, then h maps every isotopy component of $C\times M$ onto an isotopy component of $C'\times M$ of the same type.

15) [9], p. 174.

16) [10], p. 227.

17) [9], p. 174.

Proof. It is clear that each of the sets 1^0-8^0 lies in one isotopy component of $C\times M$ and that every point of $C\times M$ belongs to exactly one of the sets 1^0-8^0.

By the lemma of No. 4 the sets 1^0-4^0 are the same as the Euclidean components of $C\times M$. By example 9 they are isotopy components of $C\times M$ and h maps every of them onto an isotopy component of $C'\times M$ belonging to one of the types 1^0-4^0. In order to prove that h maps each of them onto a set of the same type it suffices to indicate some topological properties distinguishing each of the types 1^0-4^0.

To do it let us observe that:

If A is of the first type, then $\overline{A\times\alpha(M)}=C\times M$.

If A is of the second type, then $\overline{A\times\alpha(M)}$ is homeomorphic to $S_1\times M$ (where S_1 is a simple closed curve) and $\overline{A\times\alpha(M)}\cdot\overline{(C\times M)-\overline{A\times\alpha(M)}}$ is connected (homeomorphic with M).

If A is of the third type, then $\overline{A\times\alpha(M)}$ is homeomorphic to $I\times M$ and $\overline{A\times\alpha(M)}\cdot\overline{(C\times M)-\overline{A\times\alpha(M)}}$ is not connected (homeomorphic to $\beta(I)\times M$).

If A is of the fourth type, then $\overline{A\times\alpha(M)}$ is homeomorphic to $I\times M$ and $\overline{A\times\alpha(M)}\cdot\overline{(C\times M)-\overline{A\times\alpha(M)}}$ is connected (homeomorphic with M).

Evidently every point lying on a set of the form 5^0 or 6^0 belongs to $\beta(C\times M)$, while by lemmas **11** and **13** every point lying on a set of the form 7^0 belongs to $\gamma(C\times M)$. Since every set of the form 8^0 lies on the boundary of a set of the form 7^0 we infer that also every set of the form 8^0 lies in $\gamma(C\times M)$. Consequently $\beta(C\times M)$ is the sum of the sets 5^0 and 6^0.

It follows that the sets 5^0 and 6^0 are components of $\beta(C\times M)$, hence each of them is an isotopy component of $C\times M$. Each of them lies on the boundary of exactly one Euclidean component of $C\times M$, namely a component of the type 5^0 on the boundary of a Euclidean component of the type 2^0 or 3^0, and a component of the type 6^0 — on the boundary of an Euclidean component of the type 4^0. It follows that h maps the isotopy components of the type 5^0 and 6^0 onto the components of the same type respectively.

By lemma **13** the sets of the form 7^0 are isotopy components of $C\times M$ and the sets of the type 8^0 constitute the components of the boundaries of the sets of the type 7^0. It follows that also the sets 7^0 and 8^0 are isotopy components of $C\times M$ and that h maps every of them onto a set of the same type respectively. This completes the proof of our lemma.

15. Let C be a locally connected curve. For every point $(x,y)\in C\times C$ let us denote by $\nu_C(x,y)$ the number (finite or not) of the Euclidean components A of C such that the boundary of A contains only the points x,y.

Examples. If C is a simple closed curve, then $\nu_C(x,y)=0$. If $C=I$, then $\nu_C(0,1)=\nu_C(1,0)=1$ and $\nu_C(x,y)=0$ for $(x,y)\neq(0,1)$ and $(x,y)\neq(1,0)$. If C is locally contractible, then every value of $\nu_C(x,y)$ is finite. If $x\neq y$, then $\nu_C(x,y)$ is finite for every locally connected curve C. If $C=\sum_{n=1}^{\infty}C_n$, where C_n denotes the circle lying in the Euclidean plane E_2 with centre $(1/n,0)$ and radius $1/n$, then $\nu_C\big((0,0),(0,0)\big)=\infty$ and $\nu_C(x,y)=0$ for all others pairs (x,y).

Lemma. *Let C and C' be two locally connected curves. In order that a homeomorphism h, mapping $\beta(C)+\gamma(C)$ onto $\beta(C')+\gamma(C')$, be extendable to a homeomorphism of C onto C' it is necessary and sufficient that*

$$\nu_C(x,y)=\nu_{C'}\big(h(x),h(y)\big)$$

for all points $x,y\in\beta(C)+\gamma(C)$.

Proof. The necessity of the condition is evident. To prove the sufficiency let us consider for all points $x,y\in\beta(C)+\gamma(C)$ all Euclidean components $A_1,A_2,\ldots$ of C with endpoints x,y. Since $\nu_C(x,y)=\nu_{C'}\big(h(x),h(y)\big)$ we can assign to them in a one-to-one manner all Euclidean components $A_1',A_2',\ldots$ of C' with endpoints $h(x),h(y)$. Let us extend the homeomorphism h, defined in the points x,y, to a homeomorphism of A_i onto A_i'. Since the diameters of Euclidean components of C and C' tend to zero, we see at once that the mapping defined in such a manner is a homeomorphism of C onto C'.

16. ***Theorem.*** *Let C and C' be two locally connected curves and M a manifold (closed or not). A necessary and sufficient condition that $C\times M$ be homeomorphic with $C'\times M$ is that C be homeomorphic with C'.*

Proof. The sufficiency of the condition is evident. To prove the necessity we consider first the case of $\gamma(C\times M)=0$. Then $\gamma(C)=0$; hence C is a simple arc or a simple curve. In the first case the 1-dimensional Betti number of $C\times M$ is equal to the 1-dimensional Betti number of M, in the second case the 1-dimensional Betti number of $C\times M$ is larger than the 1-dimensional Betti number of M. Hence in this case the topological structure of C is completely determined by the topological structure of M and $C\times M$.

Now let us assume that $\gamma(C\times M)\neq 0$. By the lemma of No. **14** there exists a one-to-one correspondence between the points $x\in\gamma(C)$ and the isotopic components of $C\times M$ of the form $(x)\times\alpha(M)$. Setting

$$\varphi(x)=(x)\times M \quad \text{for every} \quad x\in\gamma(C)$$

we obtain a one-to-one correspondence between the points $x\in\gamma(C)$ and the closed sets of the form $(x)\times M$. Evidently φ is a homeomorphism mapping $\gamma(C)$ onto a subset of the space $2^{C\times M}$.

Moreover to every point $x \in \beta(C)$ corresponds exactly one Euclidean component A_x of C such that $x \in \bar{A}_x$. Evidently $\bar{A}_x \in 2^C$ depends continuously on x. Since $\gamma(C) \neq 0$, to different points $x, x' \in \beta(C)$ always correspond different Euclidean components A_x and $A_{x'}$. Setting

$$\varphi(x) = (x) \times M + \bar{A}_x \times \beta(M) \quad \text{for every} \quad x \in \beta(C)$$

we obtain a continuous one-to-one mapping of $\beta(C)$ into $2^{C\times M}$. Moreover if $x_n \in \beta(C)$ and $x_n \to x_0 \in \gamma(C)$, then the diameters of $\bar{A}_{x_n}$ converge to zero, and we infer that

$$\varphi(x_n) \to (x_0) \times M = \varphi(x_0).$$

Hence φ is a homeomorphism mapping the compact set $\beta(C)+\gamma(C)$ onto a subset of $2^{C\times M}$. Moreover, for every two points $x, y \in \beta(C)+\gamma(C)$ the number $\nu_C(x,y)$ is equal to the number of Euclidean components of $C \times M$ for which the boundary contains both sets $\varphi(x)$ and $\varphi(y)$ and does not contain any other of the sets $\varphi(z)$.

Let φ' denote the homeomorphism of $\beta(C')+\gamma(C')$ into $2^{C'\times M}$ analogous to φ. Then $\nu_{C'}(x', y')$ is equal to the number of Euclidean components of $C' \times M$ for which the boundary contains both sets $\varphi'(x')$ and $\varphi'(y')$ and does not contain any other of the sets $\varphi'(z')$.

Consider now a homeomorphism h mapping $C \times M$ onto $C' \times M$. By lemma **14**, h maps each isotopy component of $C \times M$ onto an isotopy component of $C' \times M$ of the same type. It follows that h maps each of the sets $\varphi(x)$, where $x \in \beta(C)+\gamma(C)$, onto a set of the form $\varphi'(x')$. Since h induces a homeomorphism of $2^{C\times M}$ onto $2^{C'\times M}$ we infer that the mapping $\varphi(x) \to \varphi'(x')$ is a homeomorphism. It follows that setting

$$x' = \psi(x)$$

we obtain a homeomorphism mapping $\beta(C)+\gamma(C)$ onto $\beta(C')+\gamma(C')$. Moreover for every $x, y \in \beta(C)+\gamma(C)$ the Euclidean components of $C \times M$ for which the boundary contains both sets $\varphi(x)$ and $\varphi(y)$ and does not contain any other of the sets $\varphi(z)$ are mapped by h onto Euclidean components of $C' \times M$ for which the boundary contains both sets $\varphi'(x')$ and $\varphi'(y')$ and does not contain any other of the sets $\varphi'(z')$.

It follows that $\nu_C(x,y) = \nu_{C'}(\psi(x), \psi(y))$ for every $x, y \in \beta(C)+\gamma(C)$. By lemma **15** we infer that ψ can be extended to a homeomorphism of C onto C'. Hence C and C' are homeomorphic and our theorem is proved.

17. A space Y will ba said to be *topologically divisible by a natural number* n if Y is homeomorphic with the Cartesian product $Y_1 \times Y_2$, where Y_2 contains exactly n points.

Theorem. *A locally connected compactum Z with $\gamma(Z) \neq 0$ can be decomposed, in at most one manner, into a Cartesian product $X \times Y$, where X is 1-dimensional and not divisible by any natural number >1, and $\gamma(Y) = 0$.*

Proof. Suppose first that Z is connected. Then X is a locally connected curve C and Y a manifold M. Since $\gamma(Z) \neq 0$ and $\gamma(M) = 0$, we infer that $\gamma(C) \neq 0$. By lemma **14** there exists in Z an isotopy component of the form 7^0. The closure of this component is homeomorphic with M. Hence the topological structure of the manifold Y is uniquelly determined by the topological structure of Z. Applying theorem **16** we infer that also the topological structure of X is uniquely determined.

Before investigating the general case let us introduce some general notions:

For every compactum X let us denote by $T(X)$ the topological type of X. The topological type of the empty set will be denoted by 0 and the topological type of the space containing exactly one point — by 1.

If Z decomposes into a sum of two disjoint compacta X and Y, then the topological type of Z will also be denoted by $T(X)+T(Y)$. If $X_1, X_2, \ldots, X_n$ are disjoint compacta of the same topological type, then the type $T(X_1+X_2+\ldots+X_n) = T(X_1)+T(X_2)+\ldots+T(X_n)$ will also be denoted by $n \cdot T(X_1)$.

The topological type of the Cartesian product $X \times Y$ will be denoted by $T(X) \cdot T(Y)$.

Let us assume now that

$$X = C_1 + C_2 + \ldots + C_k, \tag{27}$$

where C_i are disjoint locally connected continua of dimension $\leqslant 1$ and that X is topologically prime. We can assume that the topological types $T(C_1), \ldots, T(C_p)$ are distinct from one another and the remaining types, $T(C_{p+1}), \ldots, T(C_k)$, appear already among them. Let a_μ denote, for every $\mu = 1, 2, \ldots, p$, the number of the sets of the type $T(C_\mu)$ among $C_1, C_2, \ldots, C_k$. Then

$$T(X) = a_1 T(C_1) + \ldots + a_p T(C_p). \tag{28}$$

Moreover, we can assume that

$$\gamma(C_\mu) \neq 0 \quad \text{for} \quad 1 \leqslant \mu \leqslant r \quad \text{and} \quad \gamma(C_\mu) = 0 \quad \text{for} \quad r < \mu \leqslant p. \tag{29}$$

Since every continuum $C \neq 0$ with $\gamma(C) = 0$ and $\dim C \leqslant 1$ is either a simple closed curve, or a simple arc, or it contains only one point, we can assume that $p = r+3$ and that a_{r+1}, a_{r+2}, a_{r+3} denote respectively the number of simple closed curves, of simple arcs, and of separate points among the components of X.

Let X^* denote the sum of all C_μ with $\gamma(C_\mu) \neq 0$. Then

$$T(X^*) = a_1 T(C_1) + \ldots + a_r T(C_r). \tag{30}$$

Similarly $Y = M_1 + M_2 + \ldots + M_l$ where M_i are disjoint manifolds, and we can assume that $T(M_1), \ldots, T(M_q)$ are different from one another and the other types, $T(M_{q+1}), \ldots, T(M_l)$, appear already among them.

Let b_ν denote, for $\nu=1,2,\dots,q$ the number of the sets of the type $T(M_\nu)$ appearing among $M_1, M_2, \dots, M_l$. Then

$$T(Y)=b_1T(M_1)+\dots+b_qT(M_q), \tag{31}$$

and the natural coefficients $b_1,\dots,b_q$ have no common factor >1 (because Y is not divisible by any natural >1).

Since $Z=X\times Y$, we have

$$T(Z)=\sum_{\mu=1}^{p}\sum_{\nu=1}^{q}a_\mu b_\nu T(C_\mu)T(M_\nu). \tag{32}$$

Let Z^* denote the subset of Z made up of all components $C_\mu\times M_\nu$ with $\gamma(C_\mu\times M_\nu)\neq 0$. By (29) and (30)

$$T(Z^*)=\sum_{\mu=1}^{r}\sum_{\nu=1}^{q}a_\mu b_\nu T(C_\mu)T(M_\nu)=T(X^*)T(Y). \tag{33}$$

According to the case already examined the decomposition of every component $C_\mu\times M_\nu$ ($\mu=1,2,\dots,r$; $\nu=1,2,\dots,q$) of Z^* into the Cartesian product of C_μ and M_ν is unique. It follows by (33) that the topological types $T(C_1),\dots,T(C_r)$ and $T(M_1),\dots,T(M_q)$ are uniquely determined by Z^*, hence also by Z. Moreover let us observe that the coefficients $b_1,\dots,b_q$ are proportional to the numbers of components of Z^* topologically divisible by M_ν. Since $b_1,\dots,b_q$ have no common factor >1, we infer that they are uniquely determined by Z^*, hence also by Z. Moreover, if d_μ denotes the number of components of Z^* topologically divisible by C_μ, then by (33) there is

$$a_\mu=\frac{d_\mu}{\sum_{\nu=1}^{q}b_\nu}\qquad\text{for}\quad \mu=1,2,\dots,r.$$

Hence the coefficients $a_1,\dots,a_r$ are uniquely determined by Z.

It remains to show that every one of the numbers a_{r+1}, a_{r+2}, a_{r+3} is uniquely determined by Z. Let s denote the smallest integer among the dimensions of the components of Y and let m_s denote the number of components of Y of the dimension s. Then Z contains $a_{r+3}\cdot m_s$ of components of the dimension s. Thus a_{r+3} is uniquely determined by Z. Moreover let t denote the greatest integer among the 1-dimensional Betti numbers of the components of Y and let n_t denote the number of components of Y with the 1-dimensional Betti number equal to t. It is easy to observe that Z contains $a_{r+1}\cdot n_t$ components for which the 1-dimensional Betti number is equal to $t+1$ and that the 1-dimensional Betti number of other components of Z is $\leqslant t$. It follows that a_{r+1} is uniquely determined by Z. Finally let us observe that $(a_{r+1}+a_{r+2}+a_{r+3})\cdot\sum_{\nu=1}^{q}b_\nu$ is equal to the number of components of $Z-Z^*$.

Hence $a_{r+1}+a_{r+2}+a_{r+3}$, and consequently also a_{r+2} is uniquely determined by Z.

Thus the proof of the theorem is completed.

Corollary. *A locally connected compactum Z has at most one decomposition into a Cartesian product $X_0\times X_1\times\dots\times X_k$, where* $\dim X_0\leqslant 1$ *and each of the factors X_i, $i=1,2,\dots,k$ is either a simple arc or a simple closed curve.*

It is enough to combine the last theorem with the theorem[18]) which says that a connected polytope can have at most one decomposition into a Cartesian product of 1-dimensional factors.

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