Theorem VII permits us to prove that some measures are not quasi-compact. *E. g.* no proper extension of Lebesgue measure to a σ-measure is quasi-compact. In fact, by Theorem III it suffices to prove that no proper extension of Lebesgue measure to a separable σ-measure is compact. By a theorem of Rohlin [9]) such extensions are not isomorphic with the Lebesgue measure and consequently are not compact.

5. Cartesian multiplication. Let μ_t be a σ-measure in a σ-field $\boldsymbol{M}_t$ of subsets of a space X_t (where t runs over any set T of indices). In addition to the terminology of C, we call the *σ-product* of $\{\mu_t\}$ the σ-extension of any product of μ_t.

Théorem VIII. Each σ-product μ of quasi-compact σ-measures $\{\mu_t\}$ is quasi-compact.

By virtue of Theorem III it suffices to prove that $\mu|(\boldsymbol{D})_\beta$ is compact for each denumerable class $\boldsymbol{D}\subset(\boldsymbol{M})_\beta$. Obviously there is a family $\{\boldsymbol{D}_t\}$ of denumerable classes such that

$$\boldsymbol{D}_t\subset\boldsymbol{M}_t, \qquad (\boldsymbol{D})_\beta\subset[\sum_t(\boldsymbol{D}_t)_\beta^*]_\beta.$$

We denote by $\boldsymbol{L}$ the last field.

It follows from C6(vii), that $\mu|\boldsymbol{L}$ is compact, whence, by Theorems I and III, the measure $\mu|(\boldsymbol{D})_\beta$ is compact, q. e. d.

Notice that Theorem VIII can be generalized as follows: Each product of quasi-compact σ-measures has the quasi-compact σ-extension.

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[9]) Rohlin [7], p. 123.

Undecidability of Some Simple Formalized Theories

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The aim of this paper [1]) is to prove the undecidability of the theory of two equivalence-relations and of some related formalized theories [2]).

With the exception of theorems 2 and 3 in section 2, I consider theories whose logical basis is the functional calculus of the first order with identity. Individual variables $x_1, x_2, \ldots$ are the only variables which occur in those theories [3]).

Negation, conjunction, alternation, implication, and equivalence will be denoted by the symbols $'$, $\cdot$, $+$, $\rightarrow$, $\leftrightarrow$; the quantifiers by the symbols (Ex_j), (x_j). Multiple conjunctions and alternations will be denoted by Greek capitals Π and Σ. The sign $\doteq$ will be used as the symbol of identity within the theory, whereas $=$ denotes the relation of identity in the meta-theory.

When describing a formalized theory I shall enumerate its extra-logical constants and axioms. It is known that those data determine the theory univoquely.

§ 1. The theory T_1 of two equivalence relations. The extra-logical constants of the theory T_1 are two functors R_0, R_1 each with two arguments. The axioms of T_1 are as follows:

(1) $(x_1)x_1R_0x_1$,

(2) $(x_1x_2)(x_1R_0x_2\rightarrow x_2R_0x_1)$,

(3) $(x_1x_2x_3)(x_1R_0x_2\cdot x_2R_0x_3\rightarrow x_1R_0x_3)$,

(4) $(x_1)x_1R_1x_1$,

(5) $(x_1x_2)(x_1R_1x_2\rightarrow x_2R_1x_1)$,

[1]) This paper is a modified version of a paper submitted by the author shortly before his unexpected death (July 1951) to the faculty of Mathematics of the University of Warsaw, to obtain a lower scientific grade in Mathematics. The paper was prepared for print by A. Mostowski with the assistance of A. Grzegorczyk.

[2]) For the notion of decidability see Tarski [6], p. 50. Numbers in brackets refer to the bibliography at the end of the paper.

[3]) In the terminology of Church [1] the theories are based on the applied functional calculus of the first order with identity.

(6) $(x_1 x_2 x_3)(x_1 R_1 x_2 \cdot x_2 R_1 x_3 \to x_1 R_1 x_3)$,

(7) $(x_1 x_2)(x_1 \doteq x_2 \leftrightarrow x_1 R_0 x_2 \cdot x_1 R_1 x_2)$.

The content of these axioms is simply that R_0 and R_1 are equivalence relations whose common part [4]) is the identity-relation.

Theorem 1. T_1 *is an undecidable theory.*

We base the proof of this theorem of the results of Tarski and Mostowski [5]). It follows from those results that in order to prove the undecidability of T_1 it is sufficient to show that the theory T_0 of non-densely ordered rings is consistently interpretable in T_1. In other words, we have to exhibit a self-consistent theory T which is a common extension of both T_0 and T_1 and which has the following property: each constant of T_0 possesses in T a definition in terms of the constants of T_1.

The following abbreviations will be used in the definition of the theory T:

(8) $J(x_n) \overline{\overline{\mathrm{df}}} (E x_{n+1})[(x_{n+1} R_1 x_n)(x_{n+1} \doteq x_n)' \cdot (x_{n+2})(x_{n+2} R_0 x_{n+1} \to x_{n+2} \doteq x_{n+1})]$,

(9) $x_m R_{ij} x_n \overline{\overline{\mathrm{df}}} (E x_{m+n})[(x_m R_i x_{m+n}) \cdot (x_{m+n} R_j x_n)]$,

(10) $$A(x_{n+1} \dots x_{n+r}) \overline{\overline{\mathrm{df}}} \prod_{0<p<q\leqslant r} \{(x_{n+p} \doteq x_{n+q})' \cdot (x_{n+p} R_0 x_{n+q}) \cdot \\ \cdot (x_n)[x_n R_0 x_{n+1} \to \sum_{p=1}^{r} (x_n \doteq x_{n+p})]\}.$$

The formula $A(x_{n+1} \dots x_{n+r})$ is to be read thus: elements $x_{n+1}, \dots, x_{n+r}$ constitute an abstraction-class of R_0 consisting of r elements.

(11) $$S(x_k x_l x_m) \overline{\overline{\mathrm{df}}} J(x_k) \cdot J(x_l) \cdot J(x_m) \cdot (E x_{n+1} \dots x_{n+5}) \\ \cdot [A(x_{n+1} \dots x_{n+5}) \cdot \prod_{p=1}^{3} (x_{n+p} R_{10} x_k) \cdot (x_{n+4} R_{10} x_l) \cdot (x_{n+5} R_{10} x_m)]$$

(in order to avoid a possible collision of variables we put in (11) $n = k+l+m$).

(12) $$P(x_k x_l x_m) \overline{\overline{\mathrm{df}}} J(x_k) \cdot J(x_l) \cdot J(x_m) \cdot (E x_{n+1} \dots x_{n+6}) \\ \cdot [A(x_{n+1} \dots x_{n+6}) \cdot \prod_{p=1}^{4} (x_{n+p} R_{10} x_k) \cdot (x_{n+5} R_{10} x_l) \cdot (x_{n+6} R_{10} x_m)]$$

(where $n = k+l+m$).

We now define the theory T. Its extra-logical constants are R_0, R_1, I, $\oplus$, $\odot$, $<$ and its axioms are as follows:

(i) the axioms (1)-(7) of T_1,

(ii) the axioms of T_0 stating that the set I is an algebraic ring (with respect to the operations $\oplus$ and $\odot$) which is ordered by the relation $<$ in a non-dense type of order;

[4]) In the sense of [4], *. 23.02.

[5]) See [7] and [3].

(iii) axioms securing the definability (within T) of the constants I, $\oplus$, $\odot$, and $<$ in terms of R_0 and R_1; namely

(iii_1) $(x_1)[I(x_1) \leftrightarrow J(x_1)]$,

(iii_2) $(x_1 x_2 x_3)[x_1 \doteq x_2 \oplus x_3 \leftrightarrow S(x_1 x_2 x_3)]$,

(iii_3) $(x_1 x_2 x_3)[x_1 \doteq x_2 \odot x_3 \leftrightarrow P(x_1 x_2 x_3)]$.

The constant $<$ possesses in T a definition resulting from the equivalence

(13) $$(x_1 x_2)\{x_1 < x_2 \leftrightarrow (x_1 \doteq x_2)' \cdot (E x_3 x_4 x_5 x_6) \\ [x_2 \doteq x_1 \oplus x_3^2 \oplus x_4^2 \oplus x_5^2 \oplus x_6^2]\}$$ [6])

by an elimination of the constant $\oplus$ and $\odot$ by means of the axioms (iii).

It remains to show that T is self-consistent. This will be done by exhibiting a model in which the axioms of T are satisfied.

The elements of the model will be points (m, n) of the Cartesian plane with integral coordinates such that $n \geqslant 0$. In the sequel letters k, l, m, n with or without indices denote arbitrary integers). Abstraction-classes of a relation R_i will be called briefly *R_i-classes.*

We define the R_1-classes of our model as pairs $\{(2m, n), (2m+1, n)\}$. Thus each R_1-class contains just two points: one point with an even absciss and its right-hand neighbour.

The R_0-classes will either be infinite or will contain one, five, or six points.

The infinite R_0-classes are sets of the form

$$\{(2m_0, 0), \quad (2m_0, 1), \quad (2m_0, 2), \dots\}.$$

Every point $(2m_0+1, 0)$ constitutes a one-element R_0-class.

It follows from the above definitions that points $(2m, 0)$ satisfy the propositional function $J(x_1)$ and according to (iii_1) we shall consider them as the elements of a ring. Subsequent definitions will be arranged so that no other point will satisfy the propositional function $J(x_1)$.

We now define the R_0-classes containing 5 and 6 elements.

First we remark that every pair of the form $\langle (2m+1, n), (2m, 0) \rangle$ satisfies the propositional function $x_1 R_{10} x_2$.

We shall require that for arbitrary k, l, m the equation $k = l+m$ be true if and only if there exists an R_0-class containing just five points three of which have the form $(2k+1, n_1)$ with $n_1 > 0$, and the fourth and fifth the forms $(2l+1, n_2)$ and $(2m+1, n_3)$ where $n_2, n_3 > 0$. If this equivalence is true, then the triple $\langle (2k, 0), (2l, 0), (2m, 0) \rangle$ satisfies the propositional function $S(x_1 x_2 x_3)$ and hence (according to axiom (iii_2)) the propositional function $x_1 \doteq x_2 \oplus x_3$ if, and only if, $k = l+m$.

[6]) x_3^2 stands for $x_3 \odot x_3$.

A similar requirement will be imposed upon the R_0-classes containing six elements: the equation $k=l\cdot m$ has to be equivalent to the existence of an R_0-class containing 4 distinct points of the form $(2k+1,n_1)$ with $n_1>0$, one point of the form $(2l+1,n_2)$ with $n_2>0$ and one of the form $(2m+1,n_3)$ with $n_3>0$. Assuming that this equivalence is true, we find immediately that the triple $\langle(2k,0),(2l,0),(2m,0)\rangle$ satisfies the propositional function $P(x_1x_2x_3)$ and hence (according to axiom (iii_2)) the propositional function $x_1=x_2\odot x_3$ if, and only if, $k=l\cdot m$.

It is easy to see how the above requirements are to be met: we arrange in an infinite sequence

$$k_1=l_1+m_1,\qquad k_2=l_2\cdot m_2,\qquad k_3=l_3+m_3,\ldots$$

all the true equations of the form $k=l+m$ or $k=l\cdot m$. Now we take 5 distinct points of the form

$$(2k_1+1,n'),\ (2k_1+1,n''),\ (2k_1+1,n'''),\ (2l_1+1,n^{\mathrm{IV}}),\ (2m_1+1,n^{\mathrm{V}})$$

(where $n',n'',\ldots,n^{\mathrm{V}}$ are >0) and unite them into one R_0-class. Then we take 6 other points

$$(2k_2+1,q'),\ (2k_2+1,q''),\ (2k_2+1,q'''),\ (2k_2+1,q^{\mathrm{IV}}),\ (2l_2+1,q^{\mathrm{V}}),\ (2m_2+1,q^{\mathrm{VI}})$$

(where $q',q'',\ldots,q^{\mathrm{VI}}$ are >0) and unite them into one R_0-class. Continuing this process, we obtain R_0-classes as required.

It is evident that suitably selecting points used in the above process we can include every point of the form $(2j+1,n)$, where $n\neq 0$, into an R_0-class containing 5 or 6 elements. Points $(2k,0)$ are then the only ones which satisfy the propositional function $J(x_1)$.

In this way, we obtain a ring C of points $(2k,0)$ which is isomorphic to the ring C_1 of integers. The isomorphic mapping of C_1 onto C is effectuated by the function $f(k)=(2k,0)$.

It follows that axioms (ii) are true of the model (under the assumption that the less-than relation $<$ has been defined by (13)). Axioms (i) and (iii) are also true. Hence the consistency of T is proved and Theorem 1 is demonstrated.

§ 2. Corollaries to Theorem 1. First we remark that the role of identity $=$ is not essential for the validity of Theorem 1.

Let T_2 be a theory based on the functional calculus of the first order without identity and containing just two extra-logical terms R_0 and R_1. Let (1)-(6) be axioms of T_2.

Define $=$ by the formula (7). It is then easy to see that provable theorems of T_1 and provable theorems of T_2 are identical. Since the defined term $=$ can be eliminated, we obtain the following

Theorem 2. *Theory* T_2 *is undecidable.*

From this we obtain a result concerning the monadic functional calculus of the 2-nd order (*i. e.* one in which only variables for individuals, classes of individuals, and classes of such classes are allowed).

It is well known that the monadic functional calculus of the 1-st order is decidable [7]). Skolem [8]) has shown that the calculus remains decidable even if quantification of class-variables is introduced into the system.

We now enlarge this system and introduce monadic variables of the next higher type (no quantification of these variables being allowed). Let us call S the resulting system.

Theorem 3. *The system* S *is undecidable.*

Proof. Let Φ_0 and Φ_1 be two variables of the type of a class of classes and X a variable of the type of a class. We let an expression $\overline{a}$ of S correspond to every expression a of the theory T_2, replacing in a every atomic expression $x_m R_j x_n$ by

$$(EX)[\Phi_j(X)\cdot X(x_m)\cdot X(x_n)].$$

Let C be the conjunction of the axioms of T_2 and K the class of all theorems of S having the form $\overline{C}\to\overline{a}$ where a is an expression of T_2.

We shall show that the class K is undecidable. Indeed, if a is provable in T_2, then $C\to a$ is provable within the functional calculus, and hence $\overline{C}\to\overline{a}$ is provable in S. Conversely, if a is not provable in T_2, then $C\to a$ is not provable in the functional calculus, and hence by Gödel's completness theorem [9]) there is a model for the conjunction $C\cdot a'$. Since C is true of the model, R_0 and R_1 are interpreted in the model as two equivalence relations. These relations define two decompositions of the class of elements of the model into mutually disjoint sets. We now let the class of sets of the first decomposition correspond to Φ_0 and the class of sets of the second correspond to Φ_1, and obtain a model for the conjunction $\overline{C}\cdot\overline{a}'$. Hence $\overline{C}\to\overline{a}$ is not provable in S.

We have thus proved that a is provable in T_2 if and only if $C\to a$ is in K. It follows that if S were decidable, we should have a method allowing us to decide whether an arbitrary expression of T_2 is provable or unprovable in T_2. Since this contradicts Theorem 2, the proof of Theorem 3 is complete.

§ 3. The theory of one equivalence-relation. The following problem is suggested by Theorems 1 and 2. Is the theory of *one* equivalence relation decidable?

[7]) Cf. *e. g.* Church [1], p. 94.
[8]) [5].
[9]) [2].

We denote this theory by T_3. It is based on the functional calculus of the first order with identity and has just one extra-logical term R_0. The axioms of T_3 are (1)-(3).

It can be shown that the theory T_3 is decidable. To save place we restrict ourselves to formulating the pertinent definitions and lemmata. The proofs of those lemmata is essentially a routine matter.

We introduce the following abbreviations:

(14) $$B(0/n, x_k) \underset{\mathrm{df}}{=} (E x_{k+1} \dots x_{k+n}) \prod_{k+1 \leqslant p < q \leqslant k+n} [(x_p \doteq x_q)' \cdot \prod_{p=k+1}^{k+n} (x_p R_0 x_k)]$$ [10]

(there are at least n elements bearing the relation R_0 to x_k),

(15) $$B(1/n, x_k) \underset{\mathrm{df}}{=} B(0/n, x_k) \cdot B'(0/n+1, x_k)$$

(there are exactly n elements bearing the relation R_0 to x_k),

(16) $$A(0/m, i/n) \underset{\mathrm{df}}{=} (E x_1 \dots x_m)[\prod_{1 \leqslant p < q \leqslant m} (x_p R_0' x_q) \cdot \prod_{p=1}^{m} B(i/n, x_p)],$$

(17) $$A(1/m, i/n) \underset{\mathrm{df}}{=} A(0/m, i/n) \cdot A(1/m+1, i/n),$$

(18) $$A(1/0, i/n) \underset{\mathrm{df}}{=} A'(0/1, i/n).$$

The intuitive content of expressions $A(i/m, j/n)$ is as follows: *there exist at least m (if $i=0$) [exactly m (if $i=1$)] abstraction-classes of R_0 each of which contains at least n elements (if $j=0$) [exactly n elements (if $j=1$)].*

(19) $$E(0/n) \underset{\mathrm{df}}{=} (E x_1 \dots x_n)[(x_1 \doteq x_1) \cdot \prod_{1 \leqslant p < q \leqslant n} (x_p \doteq x_q)']$$

(there exist at least n elements).

(20) $$E(1/n) \underset{\mathrm{df}}{=} E(0/n) \cdot E'(0/n+1)$$

(there exist exactly n elements).

Finally we set

$$V \underset{\mathrm{df}}{=} A(0/1, 0/1).$$

V is evidently provable in T_3.

Expressions (14)-(15) are said to be of *type B*, expressions (16)-(18) of *type A*, and expressions (19)-(20) — of *type E*.

Alternations of conjunctions whose terms are of types A, are called *expressions of type AA*. Alternations of conjunctions whose terms are of types A or E, are called *expressions of types AE*.

[10]) If $n=0$, we assume that $B(0/n, x_k)$ is a void expression Λ, *i. e.* such that $\Phi + \Lambda = \Phi \cdot \Lambda = \Phi$ for any Φ.

Lemma 1. *Let Φ be an expression of the theory T_3 built up from atomic propositional functions $x_i \doteq x_j$, $x_i R_0 x_j$, and from expressions of type B by means of propositional connectives $+$, $\cdot$, and $'$. Then the expression $(Ex)\Phi$ is equivalent to an expression ψ which is an alternation of conjunction of atomic expressions and of expressions of types B, A, E. The variables free in ψ are the same as in $(Ex)\Phi$. The expression ψ can be found explicitly if Φ is explicitly given.*

Lemma 2. *Every sentence Φ of T_3 involving no free variables is equivalent to an expression ψ of type AA; the expression ψ can be found explicitly once Φ is explicitly given* [11]).

Definition. Expressions $A(i_1/n_1, j_1/k_1)$ and $A(i_2/n_2, j_2/k_2)$ are *disjoint* if either $k_1 < k_2$ and $j_1 = 1$ or $k_1 > k_2$ and $j_2 = 1$.

Lemma 3. *An expression Φ of type AA is equivalent to an alternation ψ of conjunctions of mutually disjoint expressions of type A. The alternation ψ can be found explicitly once Φ is explicitly given.*

Lemma 4. *A conjunction of mutually disjoint expressions of type A is refutable in the theory T_3 if and only if it is identical with*

$$F_m = \prod_{r=1}^{m-1} A(1/0, 1/r) \cdot A(1/0, 0/m)$$

or differs from F_m by the order of terms.

From lemmas 1-4 we easily obtain

Theorem 4. *The theory T_3 is decidable* [12]).

§ 4. Undecidability of some other theories. Let T_4 be a theory whose extra-logical terms are R_0 and F and which is based on axioms (1)-(3) and the following 4 axioms:

$(x_1)(Ex_2)(x_1 F x_2)$,

$(x_2)(Ex_1)(x_1 F x_2)$,

$(x_1 x_2 x_3)[(x_1 F x_2) \cdot (x_1 F x_3) \to (x_2 \doteq x_3)]$,

$(x_1 x_2 x_3)[(x_2 F x_1) \cdot (x_3 F x_1) \to (x_2 \doteq x_3)]$.

T_4 can be termed a theory of one equivalence relation and one one-one relation.

Theorem 5. *The theory T_4 is undecidable.*

The proof is similar to that of Theorem 1. We have only to replace the R_1-classes used in the proof of Theorem 1 by pairs of points satisfying the propositional function $(x_1 F x_2) \cdot (x_2 F x_1)$.

[11]) Lemmata 1 and 2 are proved by means of a method known as the „elimination of quantifiers". Cf. Tarski [6], p. 15.

[12]) I was informed by Professor Tarski that the same result has been found independently by F. B. Thompson.

Theorem 6. *The theory* T_5 *of one one-one relation and of one function (one-many relation) is undecidable* [13]).

This results from the fact that the theory T_0 of non-densely ordered rings is consistently interpretable in T_5. Let us write the one-one relation in the form $x_1 \doteq F(x_2)$ and the one-many relation in the form $x_1 \doteq f(x_2)$. To define a theory T′ which is a common extension of T_5 and T_0 we add to T_5 the constants of T_0, and the axioms (ii), (iii) from p. 2, where the propositional functions $J(x_n)$, $S(x_k, x_l, x_m)$, and $P(x_k, x_l, x_m)$ are defined as follows:

$$J(x_n) \overline{\overline{\mathrm{df}}} (E\, x_{n+1} \ldots x_{n+7}) \prod_{1 \leqslant p < q \leqslant 7} \{(x_{n+p} \doteq x_{n+q})' \cdot [f(x_{n+p}) = x_n]\},$$

$$S(x_k, x_l, x_m) \overline{\overline{\mathrm{df}}} J(x_k) \cdot J(x_l) \cdot J(x_m) \cdot (E\, x_n \ldots x_{n+4})$$
$$\{\prod_{p=0}^{4} [f(x_{n+p}) = x_n] \cdot (x_{n+5})[(f(x_{n+5}) \doteq x_n) \to \sum_{p=0}^{4} (x_{n+5} \doteq x_{n+p})]$$
$$\cdot \prod_{p=0}^{2} [x_k \doteq f(F(x_{n+p}))] \cdot [x_l \doteq f(F(x_{n+3}))] \cdot [x_m \doteq f(F(x_{n+4}))]\},$$

$$P(x_k, x_l, x_m) \overline{\overline{\mathrm{df}}} J(x_k) \cdot J(x_l) \cdot (J(x_m) . (E x_n \ldots x_{n+5})\{ \prod_{p=0}^{5} [f(x_{n+p}) = x_n]$$
$$\cdot (x_{n+6})[(f(x_{n+6} \doteq x_n) \to \sum_{p=0}^{5} (x_{n+6} \doteq x_{n+p})] \cdot \prod_{p=0}^{3} [x_k \doteq f(F(x_{n+p})]$$
$$\cdot [x_l \doteq f(F(x_{n+4}))] \cdot [x_m \doteq f(F(x_{n+5}))]\}.$$

(In the last two formulae we put $n = k + l + m$).

As in section 1, it is sufficient to establish the consistency of T′. To this end we consider the same model as in the proof of Theorem 1, and define a one-one relation F_0 and a function f_0:

$$F_0((2k, n)) = (2k+1, n),$$
$$F_0((2k+1, n)) = (2k, n);$$

$f_0((k,n)) = (l,m)$ *if and only if* (l,m) *is in the same* R_0*-class as* (k,n) *and has the smallest possible ordinate (if there are many such points then* (l,m) *is defined as that one which has the smallest possible absciss).*

It is then easy to show that the following interpretation of the propositional functions $I(x_n)$, $x_k \doteq x_l \oplus x_m$, $x_k \doteq x_l \odot x_m$, $x_k \doteq f(x_l)$, $x_k \doteq F(x_l)$ yields a model of the theory T′:

$I(x_n)$: x_n is a point of the form $(2p, 0)$;

$x_k \doteq x_l \oplus x_m$: x_k, x_l, x_m are points of the forms $(2p,0)$, $(2q,0)$, $(2r,0)$ and $p = q + r$;

[13]) We omit the axioms of this theory. They are unambiguously determined by the name we nave given to the theory. A similar remark applies to theories mentioned in problems on p. 139.

$x_k \doteq x_l \odot x_m$: x_k, x_l, x_m are points of the forms $(2p,0)$, $(2q,0)$, $(2r,0)$ and $p = q \cdot r$;

$x_k \doteq f(x_l)$: x_k, x are points (p,q), (r,s) such that $(p,q) = f_0((r,s))$;

$x_k \doteq F(x_l)$: x_k, x are points (p,q), (r,s) such that $(p,q) = F_0((r,s))$.

Theorem 6 is thus demonstrated.

We conclude with some open problems:

1. Is the theory of one function (one-many relation) decidable?
2. Is the theory of one ordering relation decidable?
3. Is the theory of k distinct one-one relations decidable?

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