Theorem VII permits us to prove that some measures are not quasi-compact. E.g. no proper extension of Lebesgue measure to a σ-measure is quasi-compact. In fact, by Theorem III it suffices to prove that no proper extension of Lebesgue measure to a separable σ-measure is compact. By a theorem of Rohlin [1] such extensions are not isomorphic with the Lebesgue measure and consequently are not compact.

5. Cartesian multiplication. Let \( \mu_i \) be a σ-measure in a σ-field \( \mathcal{M}_i \) of subsets of a space \( X_i \) (where \( i \) runs over any set \( T \) of indices). In addition to the terminology of C, we call the σ-product of \( \{ \mu_i \} \) the σ-extension of any product of \( \mu_i \).

**Theorem VIII.** Each σ-product \( \mu \) of quasi-compact σ-measures \( \{ \mu_i \} \) is quasi-compact.

By virtue of Theorem III it suffices to prove that \( \nu(D_0) \) is compact for each denumerable class \( D_0 \subset (\mathcal{M}_0) \). Obviously there is a family \( \{ D_i \} \) of denumerable classes such that

\[
D_i \subset \mathcal{M}_i, \quad \{ D_i \} \subset \bigcup \{ D_0 \}.
\]

We denote by \( L \) the last field.

It follows from O(ii), that \( \mu \mid L \) is compact, whence, by Theorems I and III, the measure \( \mu(D_0) \) is compact, q.e.d.

Notice that Theorem VIII can be generalized as follows: Each product of quasi-compact σ-measures has the quasi-compact σ-extension.

References


Undecidability of Some Simple Formalized Theories

By A. Janiczak (Warszawa)

The aim of this paper is to prove the undecidability of the theory of two equivalence-relations and of some related formalized theories.

With the exception of theorems 2 and 3 in section 2, I consider theories whose logical basis is the functional calculus of the first order with identity. Individual variables \( x_1, x_2, \ldots \) are the only variables which occur in those theories.

Negation, conjunction, alternation, implication, and equivalence will be denoted by the symbols \( ', \land, \lor, \rightarrow, \leftrightarrow \); the quantifiers by the symbols \( (\forall x), (\exists x) \). Multiple conjunctions and alternations will be denoted by Greek capitals \( \Pi \) and \( \Sigma \). The sign \( \equiv \) will be used as the symbol of identity within the theory, whereas \( \equiv \) denotes the relation of identity in the meta-theory.

When describing a formalized theory I shall enumerate its extralogical constants and axioms. It is known that these data determine the theory univocally.

**§ 1. The theory \( \mathcal{T}_1 \) of two equivalence relations.** The extralogical constants of the theory \( \mathcal{T}_1 \) are two functors \( R_1, R_2 \) each with two arguments. The axioms of \( \mathcal{T}_1 \) are as follows:

\[
\begin{align*}
(1) & \quad (x_1, x_2, R_1 x_1, R_2 x_2), \\
(2) & \quad (x_1, x_2, R_1 x_1, R_2 x_2), \\
(3) & \quad (x_1, x_2, R_1 x_1, R_2 x_2, R_1 x_3 \rightarrow x_3, R_2 x_3), \\
(4) & \quad (x_1, x_2, R_1 x_1, R_2 x_2), \\
(5) & \quad (x_1, x_2, R_1 x_1, R_2 x_2).
\end{align*}
\]

\( ^1 \) This paper is a modified version of a paper submitted by the author shortly before his unexpected death (July 1951) to the faculty of Mathematics of the University of Warsaw, to obtain a lower scientific grade in Mathematics. The paper was prepared for print by A. Mostowski with the assistance of A. Grzegorczyk.

\( ^2 \) For the notion of decidability see Tarski [9], p. 50. Numbers in brackets refer to the bibliography at the end of the paper.

\( ^3 \) In the terminology of Church [1] the theories are based on the applied functional calculus of the first order with identity.
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(iii) axioms securing the definability (within $T$) of the constants $I_1 \subseteq \mathcal{O}$, and $\prec$ in terms of $R_4$ and $R_1$; namely

$\langle x_1, x_2 \rangle \vdash (x_1, x_2) \in I_1(x_1, x_2)$.

The constant $\prec$ possesses in $T$ a definition resulting from the equivalence

$\langle x_1, x_2 \rangle \vdash (x_1, x_2) \in I_1(x_1, x_2)$.

by an elimination of the constant $\in$ and $\in$ by means of the axioms (iii).

It remains to show that $T$ is self-consistent. This will be done by exhibiting a model in which the axioms of $T$ are satisfied.

The elements of the model will be points $(m, n)$ of the Cartesian plane with integral coordinates such that $n \geq 0$. In the sequel letters $k, l, m, n$ with or without indices denote arbitrary integers. Abstraction-classes of a relation $R_1$ will be called briefly $R_1$-classes.

We define the $R_1$-classes of our model as pairs $(\{m, n\}, \{(2m, n), (2m + 1, n)\})$. Each $R_1$-class contains just two points: one point with an even abscissa and its right-hand neighbour. The $R_1$-classes will either be infinite or will contain one, five, or six points.

The infinite $R_1$-classes are sets of the form

$\{(2m, 0), (2m + 1), (2m + 2), \ldots\}$.

Every point $(2m, 0)$ constitutes a one-element $R_1$-class.

It follows from the above definitions that points $(2m, 0)$ satisfy the propositional function $P(x_1)$ and according to (iii) we shall consider them as the elements of a ring. Subsequent definitions will be arranged so that no other point will satisfy the propositional function $P(x_1)$.

We now define the $R_1$-classes containing 5 and 6 elements.

First we remark that every pair of the form $\langle(2m + 1, n), (2m, 0)\rangle$ satisfies the propositional function $P(x_1)$.

We shall require that for arbitrary $k, l, m$ the equation $k = l + m$ be true if and only if there exists an $R_1$-class containing just five points three of which have the form $(2k - 1, n_2)$ with $n_1 > 0$, and the fourth and fifth the forms $(2l - 1, n_3)$ and $(2m - 1, n_4)$ where $n_3, n_4 > 0$. If this equivalence is true, then the triple $\langle(2k, 0), (2l, 0), (2m, 0)\rangle$ satisfies the propositional function $P(x_1, x_2)$ and hence (according to axiom (iii)) the propositional function $x_1 \in x_2 = x_1 \in x_2$ if, and only if, $k = l + m$.

* See [7] and [3].
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From this we obtain a result concerning the monadic functional calculus of the $2$-nd order ($i.e.$, one in which only variables for individuals, classes of individuals, and classes of such classes are allowed).

It is well known that the monadic functional calculus of the $1$-st order is decidable. Skolem's result has shown that the calculus remains decidable even if quantification of class-variables is introduced into the system.

We now enlarge this system and introduce monadic variables of the next higher type (no quantification of these variables being allowed).

Let us call $S$ the resulting system.

Theorem 3. The system $S$ is undecidable.

Proof. Let $\Phi_0$ and $\Phi_1$ be two variables of the type of a class of classes and $X$ a variable of the type of a class. We let an expression $\alpha$ of $S$ correspond to every expression $\phi$ of the theory $T_0$, replacing in $\phi$ every atomic expression $\alpha_0 R_0 \alpha_1$ by

$$(EX)[\theta(X), X(\alpha_0), X(\alpha_1)].$$

Let $C$ be the conjunction of the axioms of $T_0$ and $K$ the class of all theorems of $S$ having the form $C \rightarrow \alpha$ where $\alpha$ is an expression of $T_0$.

We shall show that the class $K$ is undecidable. Indeed, if $\alpha$ is provable in $T_0$, then $C \rightarrow \alpha$ is provable within the functional calculus, and hence $\beta \rightarrow \beta$ is provable in $S$. Conversely, if $\alpha$ is not provable in $T_0$, then $C \rightarrow \alpha$ is not provable in the functional calculus, and hence by Gödel's completeness theorem $^1$ there is a model for the conjunction $C \rightarrow \alpha$. Since $C$ is true of the model, $R_0$ and $R_1$ are interpreted in the model as two equivalence relations. These relations define two decompositions of the class of elements of the model into mutually disjoint sets. We now let the class of sets of the first decomposition correspond to $\Phi_0$ and the class of sets of the second correspond to $\Phi_1$, and obtain a model for the conjunction $C \rightarrow \alpha$. Hence $C \rightarrow \alpha$ is not provable in $S$.

We have thus proved that $\alpha$ is provable in $T_0$ if and only if $C \rightarrow \alpha$ is in $K$. It follows that if $S$ were decidable, we should have a method allowing us to decide whether an arbitrary expression of $T_0$ is provable or unprovable in $T_0$. Since this contradicts Theorem 2, the proof of Theorem 3 is complete.

§ 3. The Theory of One Equivalence-Relation. The following problem is suggested by Theorems 1 and 2. Is the theory of one equivalence relation decidable?

$^1$) Cf. e. g. Church [1], p. 94.

$^2$) [5].

$^3$) [2].
We denote this theory by \( T_P \). It is based on the functional calculus of the first order with identity and has just one extra-logical term \( R_p \).

The axioms of \( T_P \) are (1)-(3).

It can be shown that the theory \( T_P \) is decidable. To save space we restrict ourselves to formulating the pertinent definitions and lemmata. The proofs of those lemmata is essentially a routine matter.

We introduce the following abbreviations:

\[
B(0, n, x_i) \triangleq (E(x_1, x_2, \ldots, x_n) \quad \forall x_1, x_2, \ldots, x_n \quad (x_i = x_0))
\]

(there are at least \( n \) elements bearing the relation \( R_p \) to \( x_0 \)),

\[
B(1, n, x_i) \triangleq B(0, n, x_i) \cdot B(0, 0 + 1, x_i)
\]

(there are exactly \( n \) elements bearing the relation \( R_p \) to \( x_0 \)),

\[
A(0, m, i, n) \triangleq (E(x_1, x_2, \ldots, x_m) \quad \forall x_1, x_2, \ldots, x_m \quad (x_i = x_n))
\]

(there exist at least \( n \) elements),

\[
A(1, m, i, n) \triangleq A(0, m, i, n) \cdot A(0, 0 + 1, i, n)
\]

\[
A(1, 0, i, n) \triangleq A(0, 0, i, n),
\]

The intuitive content of expressions \( A(i, m, j, n) \) is as follows: there exist at least \( n \) [if \( i = 0 \) exactly \( m \) [if \( i = 1 \)] abstraction classes of \( R_p \) each of which contains at least \( n \) elements [if \( j = 0 \) exactly \( m \) [if \( j = 1 \)].

\[
B(0, n) \triangleq (E(x_1, \ldots, x_n) \quad \forall x_1, \ldots, x_n \quad (x_i = x_0))
\]

(there exist at least \( n \) elements),

\[
E(1, n) \triangleq E(0, n) \cdot E(0, 0 + 1)
\]

(there exist exactly \( n \) elements).

Finally we set

\[
\forall \not A(0, 0, 0, 0)
\]

\( \Gamma \) is evidently provable in \( T_p \).

Expressions (14)-(15) are said to be of type \( B \), expressions (16)-(18) of type \( A \), and expressions (19)-(30) of type \( E \).

Alternations of conjunctions whose terms are of type \( A \), are called expressions of type \( AA \). Alternations of conjunctions whose terms are of type \( A \) or \( E \), are called expressions of types \( AE \).

3) If \( n = 0 \), we assume that \( B(0, n, x_i) \) is a void expression \( 1, i.e. such that \( \phi \vdash A = \phi \vdash 1 \cdot \phi = \phi \) for any \( \phi \).

\[\text{Undecidability of Some Simple Formalized Theories:} \]

**Lemma 1.** Let \( \Phi \) be an expression of the theory \( T_A \) built up from atomic propositional functions \( x_i \cdot R_x \) and from expressions of type \( B \) by means of propositional connectives \( \rightarrow, \cdot \), and \( ' \). Then the expression \( (E_x \Phi) \) is equivalent to an expression \( \psi \) which is an alternation of conjunction of atomic expressions and of expressions of types \( B, A, E \). The variables free in \( \psi \) are the same as in \( (E_x \Phi) \). The expression \( \psi \) can be found explicitly if \( \Phi \) is explicitly given.

**Lemma 2.** Every sentence \( \Phi \) of \( T_A \) involving no free variables is equivalent to an expression \( \psi \) of type \( AA \); the expression \( \psi \) can be found explicitly once \( \Phi \) is explicitly given.

**Definition.** Expressions \( \Lambda(i, \eta, \xi, j, k) \) and \( \Lambda(i, \xi, \eta, j, k) \) are disjoint if either \( k_i < k \) and \( j_i = 1 \) or \( k_i > k \) and \( j_i = 1 \).

**Lemma 3.** An expression \( \Phi \) of type \( AA \) is equivalent to an alternation \( \alpha \) of conjunctions of mutually disjoint expressions of type \( A \). The alternation \( \alpha \) can be found explicitly once \( \Phi \) is explicitly given.

**Lemma 4.** A conjunction of mutually disjoint expressions of type \( A \) is refutable in the theory \( T_A \) if and only if it is identical with

\[
\prod_{i} \Lambda(i, 0, 1, \eta) \cdot A(0, 0, 0, 0)
\]

or differs from \( \Lambda \) by the order of terms.

From lemmas 1-4 we easily obtain

**Theorem 4.** The theory \( T_A \) is decidable.

\[\text{§ 4. Undecidability of Some Other Theories.} \]

Let \( T_d \) be a theory whose extra-logical terms are \( R_x \) and \( F \) and which is based on axioms (1)-(3) and the following 4 axioms:

\[
x_i \cdot R_x \cdot x_i \cdot (x_i \cdot F_x \cdot R_x) \rightarrow \neg x_i \cdot R_x \cdot x_i
\]

\[
x_i \cdot F_x \cdot x_i \cdot (x_i \cdot F_x \cdot R_x) \rightarrow \neg x_i \cdot R_x \cdot x_i
\]

\( T_d \) can be termed a theory of one equivalence relation and one one-one relation.

**Theorem 5.** The theory \( T_d \) is undecidable.

The proof is similar to that of Theorem 1. We have only to replace the \( R_x \)-classes used in the proof of Theorem 1 by pairs of points satisfying the propositional function \( (x_i \cdot F_x \cdot R_x) \cdot (x_i \cdot F_x) \).

\[\text{\footnote{3} Lemmas 1 and 2 are proved by means of a method known as the "elimination of quantifiers."} \]

\[\text{\footnote{4} I was informed by Professor Taraki that the same result has been found independently by P. B. Thompson.} \]
Theorem 6. The theory \( T_2 \) of one one-one relation and of one function (one-many relation) is undecidable \(^{11}\).

This results from the fact that the theory \( T_2 \) of non-densely ordered rings is consistently interpretable in \( T_3 \). Let us write the one-one relation in the form \( x_1 \vdash F(x_2) \) and the one-many relation in the form \( x_1 \vdash f(x_2) \). To define a theory \( T' \) which is a common extension of \( T_2 \) and \( T_3 \), we add to \( T_2 \) the constants of \( T_3 \) and the axioms (ii), (iii) from p. 2, where the propositional functions \( F(x_2) \), \( S(x_2, x_2, x_2) \), and \( P(x_2, x_2, x_2) \) are defined as follows:

\[
J(x_2) \equiv (E x_{a+1} \cdots x_{a+n}) \cdot \bigwedge_{1 \leq a < b < c} [(x_{a+b} \equiv x_{a+c}) \cdot [(f(x_{a+b}) \equiv x_2) \cdot \left( \prod_{p=0}^{b} [f(x_{p+b}) = x_2] \cdot (x_{a+b} \equiv x_{a+p+b}) \right)]
\]

\[S(x_2, x_2, x_2) \equiv J(x_2) \cdot J(x_2) \cdot J(x_2) \cdot (E x_{a+1} \cdots x_{a+n}) \]

\[
\prod_{p=0}^{b} [f(x_{p+b}) = x_2] \cdot (x_{b+1} \equiv x_{a+p+b})
\]

\[P(x_2, x_2, x_2) \equiv J(x_2) \cdot J(x_2) \cdot J(x_2) \cdot (E x_{a+1} \cdots x_{a+n}) \]

\[
\prod_{p=0}^{b} [f(x_{p+b}) = x_2] \cdot (x_{b+1} \equiv x_{a+p+b}) \cdot \left( \prod_{p=0}^{b} [x_0 \equiv f(x_{p+b})] \cdot \left( x_0 = f(F(x_{b+1})) \right) \right)
\]

(In the last two formulae we put \( n = b = k + m \).

As in section 1, it is sufficient to establish the consistency of \( T' \). To this end we consider the same model as in the proof of Theorem 1, and define a one-one relation \( F_0 \) and a function \( f_0 \):

\[
F_0(2k+1, n) = (2k + 1, n),
\]

\[
F_0(2k+2, n) = (2k, n);
\]

\[
f_0(k, n) = (l, m) \text{ if and only if } (l, m) \text{ is in the same } R_v \text{ class as } (k, n) \text{ and has the smallest possible } \alpha \text{ if there are many such points then } (l, m) \text{ is defined as that one which has the smallest possible } \alpha.
\]

It is then easy to show that the following interpretation of the propositional functions \( I(x_2), x_0 \Rightarrow x_0 \cdot x_1, x_0 \Rightarrow x_0 \cdot x_2, x_0 \Rightarrow f(x_0), x_0 \cdot f(x_0) \) yields a model of the theory \( T' \):

\[
I(x_2) \equiv x_2 \text{ is a point of the form } (2p, 0);
\]

\[
x_0 \cdot x_1 \odot x_2 : x_0, x_1, x_2 \text{ are points of the forms } (2p, 0), (2q, 0), (2r, 0) \text{ and } p = q + r;
\]

\[x_0 \vdash f(x_0) : x_0, x_0 \text{ points } (p, q), (r, s) \text{ such that } (p, q) = f_0((r, s));
\]

\[x_1 \vdash f(x_1) : x_0, x_0 \text{ points } (p, q), (r, s) \text{ such that } (p, q) = f_0((r, s)).
\]

Thus theorem 6 is thus demonstrated.

We conclude with some open problems:

1. Is the theory of one function (one-many relation) decidable?
2. Is the theory of one ordering relation decidable?
3. Is the theory of \( k \) distinct one-one relations decidable?

Bibliography