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On Quasi-Compact Measures

By

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This paper *) is a continuation of paper *On Compact Measures* by Marczewski [6] (quoted in the sequel as C). Here I consider only σ-measures, *i. e.* countably additive measures in a countably additive field and I define the notion of *quasi-compact σ-measure*. This notion is equivalent to that of *perfect* measure introduced by Gnedenko and Kolmogoroff [1]).

It is known that the distribution function of a measurable real function $f(x)$, *i. e.* the set function defined by the formula $\mu_f(E)=\mu[f^{-1}(E)]$ can be considered either for Borel sets E, or for all sets E possessing measurable inverse images $f^{-1}(E)$. In the case of Lebesgue measure these two variants are not essentially different, as was proved by Hartman [2]). Theorem VI proves that this property is characteristic of quasi-compact measures.

In connection with the abstract characterization of the Lebesgue measure, formulated by Halmos, von Neumann [3] and Rohlin [7] I shall prove that in the domain of separable measures the compactness, the quasi-compactness and the point-isomorphism with the Lebesgue measure are equivalent (Theorem VII).

Other relations between the compactness and quasi-compactness are stated in Theorems II and III.

Applying Marczewski's theorem on the invariance of compactness under Cartesian multiplication (C 6 (vii)), I shall prove that quasi-compactness has the same property (Theorem VIII).

In this paper I shall preserve the terminology and notation of C, in particular the letter X will always denote a set, on subsets of which the considered measure is defined.

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*) Presented in part to the Polish Mathematical Society, Wrocław Section, on November 17, 1950. Cf. the preliminary report [8].

[1]) Gnedenko and Kolmogoroff [1], § 3, p. 22-23. This equivalence follows from Theorem VI.

[2]) Hartman [4], p. 21, III.

In many proofs I shall use Marczewski's notion of characteristic function of a sequence of sets [3]) $E_n \subset X$, *i. e.* the function h defined as follows:

$$h(x) = \frac{i_1}{3} + \frac{i_2}{9} + \frac{i_3}{27} + \ldots, \quad \text{where} \quad i_n = \begin{cases} 0 \text{ if } x \in X - E_n, \\ 2 \text{ if } x \in E_n. \end{cases}$$

1. Definitions. Homomorphisms. Measurable functions. We say that a σ-measure μ in a σ-field $\boldsymbol{M}$ is *quasi-compact*, if each sequence of sets $Q_n \in \boldsymbol{M}$ satisfies the following condition:

(q) for each $\eta > 0$ there is a $Q_0 \in \boldsymbol{M}$ such that $\mu(Q_0) > 1 - \eta$ and that the sequence $\{Q_0 Q_n\}$ forms a compact class.

Let μ be a σ-measure in a σ-field $\boldsymbol{M}$ of subsets of X and ν a σ-measure in a σ-field $\boldsymbol{N}$ of subsets of Y. A mapping h of X onto Y is called a *homomorphism of μ to ν* if $\boldsymbol{M}$ is the class of all $h^{-1}(E)$, where $E \in \boldsymbol{N}$ and if $\mu[h^{-1}(E)] = \nu(E)$ [4]). If h is a homomorphism after the removing of a subset of X of measure μ zero and a subset of Y of measure ν zero then it is called an *almost homomorphism* [4]).

If h is one-one then it is called an *isomorphism* or an *almost-isomorphism* respectively.

It is easy to prove the following

Lemma. *If ν is an almost-homomorphic image of μ and ν is compact [quasi-compact], then μ is also compact [quasi-compact].*

A σ-measure μ in a σ-field $\boldsymbol{M}$ being fixed, we call *measurable* a real function f such that $f^{-1}(G) \in \boldsymbol{M}$ for each open set G of real numbers.

Theorem I. *A σ-measure μ in a σ-field $\boldsymbol{M}$ is quasi-compact if and only if, for each real valued measurable function there exists a set $Q \in \boldsymbol{M}$ such that $\mu(Q) = 1$ and that $f(Q)$ is a Borel set.*

Sufficiency. Let $Q_n \in \boldsymbol{M}$ for $n = 1, 2, \ldots$ and h be the characteristic function of the sequence of sets $\{Q_n\}$. By hypothesis there is a Borel set $Y \subset h(X)$ such that $\mu[h^{-1}(Y)] = 1$. For each Borel subset E of Y we put

$$\nu(E) = \mu[h^{-1}(E)].$$

Let $\boldsymbol{M}_0$ denote the smallest σ-field containing the sets Q_n and $h^{-1}(Y)$. It is easy to see that the σ-measure ν is an almost homomorphic image of $\mu|\boldsymbol{M}_0$. By Lemma, the measure $\mu|\boldsymbol{M}_0$ is quasi-compact and consequently the sequence $\{Q_n\}$ satisfies condition (q). Therefore the measure μ is quasi-compact.

Necessity. Let μ be a quasi-compact σ-measure, f — a real measurable function and $\{I_n\}$ the sequence of all rational intervals. Then

[3]) Marczewski [5].
[4]) Cf. Rohlin [7], p. 109 and 112.

the sequence $f^{-1}(I_n)$ satisfies the condition (q) and consequently there is a set $Q_0 \in \boldsymbol{M}$ such that $\mu(Q_0) > 1 - \eta$ and that

(*) the sequence $\{Q_0 f^{-1}(I_n)\}$ forms a compact class.

The number $\eta > 0$ being arbitrarily small, it suffices to prove that $f(Q_0)$ is closed. Let y denote a point of accumulation of $f(Q_0)$ and $\{I_{n_k}\}$ a decreasing subsequence of $\{I_n\}$ such that $(y) = I_{n_1} \cdot I_{n_2} \cdot \ldots$ Since $Q_0 f^{-1}(I_{n_k}) \neq 0$ for $k = 1, 2, \ldots$, then by (*) $Q_0 f^{-1}(y) \neq 0$, *i. e.* $y \in f(Q_0)$.

2. Compactness and quasi-compactness.

Theorem II. *Each compact σ-measure is quasi-compact.*

Proof. Let μ denote a compact σ-measure in a σ-field $\boldsymbol{M}$ of subsets of X. By virtue of C4(iii), there exists a compact class $\boldsymbol{F} \subset \boldsymbol{M}$ which approximates $\boldsymbol{M}$ with respect to μ. Let $Q_n \in \boldsymbol{M}$ for $n = 1, 2, \ldots$ Then there exist two sequences of sets: $P_n, R_n \in \boldsymbol{F}$ such that

$$P_n \subset Q_n, \qquad R_n \subset X - Q_n,$$

$$\mu(Q_n - P_n) < \eta/2^{n+1}, \qquad \mu[(X - Q_n) - R_n] < \eta/2^{n+1}.$$

Let us put

$$Q_0 = \prod_{n=1}^{\infty} (P_n + R_n).$$

Obviously $\mu(Q_0) > 1 - \eta$ and $Q_n Q_0 = P_n Q_0 \in \boldsymbol{F}_{s\delta}$. Consequently, it follows from C2(ii) and (iii), that the sequence $\{Q_n Q_0\}$ forms a compact class.

Theorem III. *A σ-measure μ in a σ-field $\boldsymbol{M}$ of subsets of X is quasi-compact if and only if the σ-measure $\mu|(\boldsymbol{D})_\beta$ is compact for each denumerable class $\boldsymbol{D} \subset \boldsymbol{M}$.*

Sufficiency. Since $\mu|(\boldsymbol{D})_\beta$ is compact by hypothesis and consequently quasi-compact by Theorem II, there is a set $Q \in (\boldsymbol{D})_\beta$ such that $\mu(Q) > 1 - \eta$ and that the class of sets of the form EQ, where $E \in \boldsymbol{D}$ is compact.

Necessity. Let $\boldsymbol{M} \supset \boldsymbol{D} = \{E_n\}$. The characteristic function h of $\{E_n\}$ transform $(\boldsymbol{D})_\beta$ onto the σ-field $\boldsymbol{B}$ of sets $\boldsymbol{B} h(X)$, where $\boldsymbol{B}$ runs over the class of Borel sets. The function h determines a homomorphism of the measure $\mu|(\boldsymbol{D})_\beta$ to $\mu_h|\boldsymbol{B}$. In view of Theorem I, there is in $h(X)$ a Borel set A such that $\mu_h(A) = 1$. Consequently the measure $\mu_h|\boldsymbol{B}$ is compact [5]), whence, by the Lemma the measure $\mu(\boldsymbol{D})_\beta$ is also compact.

Theorem III implies directly

Theorem IV. *If μ is a quasi-compact σ-measure in a σ-field M, and if L is a σ-subfield of $\boldsymbol{M}$, then $\mu|L$ is also quasi-compact:*

[5]) This follows from the compactness of each σ-measure defined in the field of all Borel subsets of a Borel linear set, cf C4.

3. Completion. Distribution function. A σ-measure μ is called *complete* if any subset of a set of measure zero is measurable. By the *completion* of a σ-measure μ we mean the smallest extension of μ to a complete σ-measure [6]).

Theorem V. *Let μ and $\bar{\mu}$ be σ-measures in σ-fields $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$ respectively, and let us suppose that $\bar{\mu}$ is the completion of μ. The measure μ is compact [quasi-compact] if and only if $\bar{\mu}$ is compact [quasi-compact].*

Obviously the compactness of μ implies the compactness of $\bar{\mu}$. Conversely, let $\bar{\mu}$ be compact and let $\overline{\boldsymbol{F}}$ be a compact class which approximates $\overline{\boldsymbol{M}}$ with respect to $\bar{\mu}$ and for which $\overline{\boldsymbol{F}}=\overline{\boldsymbol{F}}_\delta\subset\overline{\boldsymbol{M}}$ (cf. C4 (iii)). It suffices to prove that the class $\overline{\boldsymbol{F}}\boldsymbol{M}$ approximates $\boldsymbol{M}$. In fact, it is easy to see that for each $E_0\in\boldsymbol{M}$ and $\eta>0$ there exist the sets $P_n\in\overline{\boldsymbol{F}}$ and $E_n\in\boldsymbol{M}$ such that

$$P_{n+1}\subset E_n\subset P_n\subset E_0 \quad \text{for} \quad n=1,2,\ldots$$

and

$$\bar{\mu}(P_n-E_n)=0, \qquad \mu(E_n-E_{n+1})<\eta/2^{n+1} \quad \text{for} \quad n=0,1,\ldots,$$

whence $P_1P_2\ldots=E_0E_1\ldots=P\in\overline{\boldsymbol{F}}\boldsymbol{M}$ and $\mu(E_0-P)<\eta$.

For quasi-compactness the proof reduces to the case of compact measures in virtue of Theorem III.

For each measurable function f we denote by $\boldsymbol{M}_f$ the σ-field of all linear sets E with $f^{-1}(E)\in\boldsymbol{M}$. The set function

$$\mu_f(E)=\mu[f^{-1}(E)] \quad \text{for} \quad E\in\boldsymbol{M}$$

is called the *distribution function* of f.

Obviously μ is a σ-measure and if μ is complete, then μ_f is also complete.

In what follows we shall also consider the partial measure $\mu_f|\boldsymbol{B}$, where $\boldsymbol{B}$ is the field of all Borel linear sets.

Theorem VI. *A complete σ-measure μ is quasi-compact if and only if for each real measurable function f the σ-measure μ_f is the completion of $\mu_f|\boldsymbol{B}$.*

Sufficiency. If f is a measurable function then $f(X)\in\boldsymbol{M}_f$. Since μ_f is the completion of $\mu_f|\boldsymbol{B}$, there exists a Borel set $A\subset f(X)$ such that $\mu[f^{-1}(A)]=\mu_f(A)=1$. It follows from Theorem I that μ is quasi-compact.

Necessity. Let $f^{-1}(Y)\in\boldsymbol{M}$, where f is a measurable function. Put

$$g(x)=\begin{cases} f(x) & \text{if } f(x)\in Y,\\ y_0 & \text{if } f(x)\,\bar{\in}\, Y,\end{cases}$$

[6]) Cf. *e. g.* Halmos [2], p. 55.

where y_0 is any point with $y_0\,\bar{\in}\,Y$. Since g is measurable and μ is quasi-compact, there is by Theorem I a Borel set $A\subset g(X)$ such that $\mu[g^{-1}(A)]=1$.

It is easy to see that

$$A-(y_0)\subset Y \qquad \text{and} \qquad \mu_f[A-(y_0)]=\mu_f(Y).$$

Consequently, μ_f is the completion of $\mu_f|\boldsymbol{B}$, q. e. d.

Let us remark that Theorem VI can be formulated without the assumption on the completness of μ, as follows: A σ-measure μ is quasi-compact if and only if for each real measurable function f the completions of μ_f and $\mu_f|\boldsymbol{B}$ are identical.

4. Separable measures. A σ-measure μ is *separable* [7]) if there exists a basis $\{E_n\}$ of measurable sets, *i. e.* such a sequence of sets that

1° if $x\neq y$ then $x\in E_{n_0}$ and $y\,\bar{\in}\,E_{n_0}$ for a number n_0;

2° the measure μ coincides with the completion of $\mu|\boldsymbol{M}_0$, where $\boldsymbol{M}_0$ denotes the σ-field spaned on $\{E_n\}$.

Obviously

(**) A separable σ-measure remains separable after the removing of any set of measure zero.

A σ-measure is called *almost-separable* if it is separable after the removing of a set of measure μ zero.

Theorem VII. *The compactness, the quasi-compactness, and the point almost-isomorphism with the Lebesgue measure are equivalent for almost-separable complete σ-measures.*

Each σ-measure almost-isomorphic with the Lebesgue measure being obviously compact and each compact σ-measure being quasi-compact in view of Theorem II, it remains to prove that each almost-separable and quasi-compact σ-measure μ is almost-isomorphic with the Lebesgue measure. Obviously, we can suppose that μ is separable. Let $\{E_n\}$ be a basis of measurable sets and h the characteristic function of the sequence $\{E_n\}$.

By Theorem I there is a Borel set $Y\subset h(X)$ such that $\mu h^{-1}(Y)=1$. In view of (**) we may assume $h^{-1}(Y)=X$. Since $\{E_n\}$ have the property 1°, h is a one-one mapping of X on the Borel set Y and the Borel subsets $h(E_n)$ of Y have the analogous property in Y. Consequently, h transforms the σ-field $\boldsymbol{M}_0$ on the field of all Borel subsets of Y [8]). In virtue of 2°, h is an isomorphism of μ and a σ-measure in Y, isomorphic with the Lebesgue measure.

[7]) Halmos and von Neumann [3], p. 333.

[8]) Because for each Borel linear set Y, the σ-field spanned by a sequence of Borel subsets of Y with the property 1° is that of all Borel subsets of Y. Cf. Halmos and von Neumann [3], p. 335 and 337.

Theorem VII permits us to prove that some measures are not quasi-compact. *E. g.* no proper extension of Lebesgue measure to a σ-measure is quasi-compact. In fact, by Theorem III it suffices to prove that no proper extension of Lebesgue measure to a separable σ-measure is compact. By a theorem of Rohlin[9]) such extensions are not isomorphic with the Lebesgue measure and consequently are not compact.

5. Cartesian multiplication. Let μ_t be a σ-measure in a σ-field $\boldsymbol{M}_t$ of subsets of a space X_t (where t runs over any set T of indices). In addition to the terminology of C, we call the *σ-product* of $\{\mu_t\}$ the σ-extension of any product of μ_t.

Théorem VIII. *Each σ-product μ of quasi-compact σ-measures $\{\mu_t\}$ is quasi-compact.*

By virtue of Theorem III it suffices to prove that $\mu|(\boldsymbol{D})_\beta$ is compact for each denumerable class $\boldsymbol{D}\subset(\boldsymbol{M})_\beta$. Obviously there is a family $\{\boldsymbol{D}_t\}$ of denumerable classes such that

$$\boldsymbol{D}_t\subset \boldsymbol{M}_t, \qquad (\boldsymbol{D})_\beta\subset[\sum_t (\boldsymbol{D}_t)^*_\beta]_\beta .$$

We denote by $\boldsymbol{L}$ the last field.

It follows from C6(vii), that $\mu|\boldsymbol{L}$ is compact, whence, by Theorems I and III, the measure $\mu|(\boldsymbol{D})_\beta$ is compact, q. e. d.

Notice that Theorem VIII can be generalized as follows: Each product of quasi-compact σ-measures has the quasi-compact σ-extension.

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[9]) Rohlin [7], p. 123.

Undecidability of Some Simple Formalized Theories

By

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The aim of this paper[1]) is to prove the undecidability of the theory of two equivalence-relations and of some related formalized theories[2]).

With the exception of theorems 2 and 3 in section 2, I consider theories whose logical basis is the functional calculus of the first order with identity. Individual variables $x_1, x_2, \ldots$ are the only variables which occur in those theories[3]).

Negation, conjunction, alternation, implication, and equivalence will be denoted by the symbols $'$, $\cdot$, $+$, $\rightarrow$, $\leftrightarrow$; the quantifiers by the symbols (Ex_j), (x_j). Multiple conjunctions and alternations will be denoted by Greek capitals Π and Σ. The sign $\doteq$ will be used as the symbol of identity within the theory, whereas $=$ denotes the relation of identity in the meta-theory.

When describing a formalized theory I shall enumerate its extralogical constants and axioms. It is known that those data determine the theory univoquely.

§ 1. The theory T_1 of two equivalence relations. The extralogical constants of the theory T_1 are two functors R_0, R_1 each with two arguments. The axioms of T_1 are as follows:

(1) $(x_1)x_1R_0x_1$,

(2) $(x_1x_2)(x_1R_0x_2\rightarrow x_2R_0x_1)$,

(3) $(x_1x_2x_3)(x_1R_0x_2\cdot x_2R_0x_3\rightarrow x_1R_0x_3)$,

(4) $(x_1)x_1R_1x_1$,

(5) $(x_1x_2)(x_1R_1x_2\rightarrow x_2R_1x_1)$,

[1]) This paper is a modified version of a paper submitted by the author shortly before his unexpected death (July 1951) to the faculty of Mathematics of the University of Warsaw, to obtain a lower scientific grade in Mathematics. The paper was prepared for print by A. Mostowski with the assistance of A. Grzegorczyk.

[2]) For the notion of decidability see Tarski [6], p. 50. Numbers in brackets refer to the bibliography at the end of the paper.

[3]) In the terminology of Church [1] the theories are based on the applied functional calculus of the first order with identity.