Let H_n denote the set of all $y \in Q$ such that, for an appropriate neighbourhood $V = Q$ of y, we have $P\times V\subset G_m$.

Clearly, the H_n are open, $H_n \subset H_{n+1}$. Let $b \in Q$ be arbitrary. We shall show that $b \in \bigcup_{n=1}^{\infty} H_n$. Putting $B = P\times(b)$, we evidently have, for some p, $B\subset G_p$. For any $x \in P$, there exist open (in P and, respectively, in Q) sets U_x, V_x such that $(x,b) \in U_x\times V_x \subset G_p$. Therefore, $B\subset \bigcup_{x\in P}(U_x\times V_x)\subset G_p$. Since B is bicompact, there exist $x_i \in P$ $(i=1,\dots,n)$ such that $B\subset \bigcup_{i=1}^{n}(U_{x_i}\times V_{x_i})$. Putting $V=\bigcap_{i=1}^{m} V_{x_i}$, we have $B\subset P\times V\subset G_p$. Therefore, V being a neighbourhood of b, $b \in H_p$. Hence $Q=\bigcup_{n=1}^{\infty} H_n$ which implies, Q being compact, that $Q=H_m$, for some m. Then clearly $P\times Q\subset G_m$

On Compact Measures *

By

E. Marczewski (Wrocław)

Let μ_j be a measure in abstract space X_j with $\mu_j(X_j)=1$ for $j=1,2,\dots$ Roughly speaking, a measure μ in the Cartesian product $X_1\times X_2\times\dots$ is called a *product* of $\{\mu_j\}$ (for the precise definition see below, Section 6), if always

$$\mu(X_1\times X_2\times\dots\times X_{n-1}\times E\times X_{n+1}\times\dots)=\mu_n(E),$$

and the *direct product* of $\{\mu_j\}$ if

$$\mu(E_1\times E_2\times\dots\times E_n\times X_{n+1}\times X_{n+2}\times\dots)=\mu_1(E_1)\cdot\mu_2(E_2)\cdot\ldots\cdot\mu_n(E_n).$$

Products of measures are especially important for Probability Theory, in which they correspond to joint distributions of random variables. Obviously, the direct product corresponds to the case of stochastic independence.

It is well known that for each family of σ-measures there is a uniquely determined direct σ-product [1]). The relations in the domain of non direct products are rather complicated. The important theorem formulated by Kolmogoroff [2]) concerns the case, in which each X_j is the real line [3]) and its abstract analogue is false, as was proved by Sparre-Andersen and Jessen [4]).

In Kolmogoroff's proof, the approximation of measurable sets by compact ones is important. By eliminating non-essential topological concepts from this proof, I arrived at the notion of compact measure. In this paper I shall establish the fundamental properties of this concept, especially some relations between compactness and independence in the sense of the General Theory of Sets [5]) (theorems 5 (iii)-(v)). Then I shall show that each product of compact measures is compact (6 (vii)),

*) Presented to the Polish Mathematical Society (Wrocław Section), on the 10th of November, 1950. Cf. preliminary reports [9] and [11].

[1]) See *e. g.* Halmos [5], p. 157, Theorem B.

[2]) See Kolmogoroff [6], p. 27, Halmos [5], p. 212, Theorem A.

[3]) — or bicompact topological space, cf. Halmos, l. c., p. 212.

[4]) Sparre-Andersen and Jessen [1]; cf. also Halmos [5], p. 211-213, and p. 214 (3).

[5]) Cf. *e. g.* Marczewski [7], [8] and [10].

which implies that it can always be extended to a σ-measure. Also the precise analogue of Kolmogoroff's theorem is fulfilled by compact measures [6]) (this follows from theorem 6 (viii)).

The concept of compact measure seems to be useful not only in problems of Cartesian multiplication, *e. g.* compactness is a sufficient condition for the countable additivity (theorem 4 (i)).

The converse theorem is false since the relative Lebesgue measure in a non-measurable set of Sierpiński is not compact (theorem 7 (iv)). A stronger result was recently obtained by Ryll-Nardzewski, who defined the notion of quasi-compact σ-measure and proved that the relative Lebesgue measure in any non-measurable set is never quasi-compact (of course every compact σ-measure is quasi-compact). Ryll-Nardzewski's results on quasi-compact measures will be published in the same Journal (cf. also preliminary report [12]). The notion of quasi-compact σ-measure is equivalent to that of perfect measure in the sense of Gnedenko and Kolmogoroff [7]).

1. Preliminaries. For each class $\boldsymbol{K}$ of sets we denote respectively by $(\boldsymbol{K})_s$, $(\boldsymbol{K})_d$, $(\boldsymbol{K})_\sigma$, and $(\boldsymbol{K})_\delta$ the class of all sets of the form

$$E_1+E_2+\ldots+E_n, \qquad E_1E_2\ldots E_n,$$
$$E_1+E_2+\ldots, \qquad E_1E_2\ldots$$

where $E_j \in \boldsymbol{K}$. Next, we denote by $(\boldsymbol{K})_0$ the smallest field (*i. e.* an additive and complementative class), and by $(\boldsymbol{K})_\beta$ the smallest σ-field (*i. e.* countably additive field) containing $\boldsymbol{K}$.

If $\boldsymbol{M}$ is a field of subsets of a fixed set X, we call *measure in* $\boldsymbol{M}$ each non-negative and additive set function $\mu(E)$ defined for $E \in \boldsymbol{M}$ and such that $\mu(X)=1$. A measure is *countably additive* if for each sequence of disjoint sets $E_j \in \boldsymbol{M}$, with $E_1+E_2+\ldots \in \boldsymbol{M}$, we have

$$\mu(E_1+E_2+\ldots)=\mu(E_1)+\mu(E_2)+\ldots$$

It is well known that a measure μ is countably additive if and only if each sequence of sets $E_1 \supset E_2 \supset \ldots$, with $\mu(E_j)>a>0$, has a non-void product.

A countable additive measure μ in a σ-field is called a *σ-measure*. A measure is called *non-atomic* if for each set E with $\mu(E)>0$ there is a set D such that $\mu(E)>\mu(D)>0$.

If μ is a measure in $\boldsymbol{M}$, and $\boldsymbol{L}$ is a subfield of $\boldsymbol{M}$, then by $\mu|\boldsymbol{L}$ we understand the *partial measure* of μ in $\boldsymbol{L}$, *i. e.* the set function λ defined only in $\boldsymbol{L}$ and equal to μ in $\boldsymbol{L}$. Then we call μ an *extension* of λ. If μ is a σ-measure, then it is called a *σ-extension* of λ to $\boldsymbol{M}$. The well known theorem of Fréchet and Nikodym says that

[6]) In this connection cf. a theorem formulated by Doob and Jessen in the paper by Andersen and Jessen [3], p. 5.

[7]) Gnedenko and Kolmogoroff [4], p. 22-23.

(i) *Any countable additive measure μ in $\boldsymbol{M}$ has the unique σ-extension ν to $(\boldsymbol{M})_\beta$ and then we have*

$$\nu(E)=\inf\sum_{j=1}^{\infty}\mu(E_j),$$

where $\{E_j\}$ runs over all sequences of sets $E_j \in \boldsymbol{M}$ such that $E_1+E_2+\ldots \supset E$ [8]).

It easily follows from (i) that

(ii) *If μ is a measure in $\boldsymbol{M}$ and ν a σ-extension of μ to $(\boldsymbol{M})_\beta$, then for each $E \in (\boldsymbol{M})_\beta$ we have*

$$\nu(E)=\inf\nu(K)=\sup\nu(H),$$

where K runs over all sets containing E and belonging to $(\boldsymbol{M})_\sigma$ and H runs over all subsets of E belonging to $(\boldsymbol{M})_\delta$.

Finally I shall prove an elementary lemma:

(iii) *Let Z be a set of finite sequences of positive integers with the following properties:*

1° *There is an infinite sequence $(l_1,l_2,\ldots)$ such that, if $(k_1,k_2,\ldots,k_n) \in Z$, then $k_j \leqslant l_j$ for $j=1,2,\ldots,n$;*

2° *For each $n=1,2,\ldots$ there is a sequence $(k_1,k_2,\ldots,k_n) \in Z$;*

3° *If $(k_1,k_2,\ldots,k_n,k_{n+1}) \in Z$, then $(k_1,k_2,\ldots,k_n) \in Z$.*

Then there is an infinite sequence $(k_1,k_2,\ldots)$ such that $(k_1,k_2,\ldots,k_n) \in Z$ for $n=1,2,\ldots$

Applying 1° and 2° we define $(k_1,k_2,\ldots)$ by induction in such a way that for each positive integer n there exist arbitrarily long sequences $(k_1,k_2,\ldots,k_n,k'_{n+1},\ldots,k'_{n+m}) \in Z$. In view of 3°, $(k_1,k_2,\ldots)$ is the required infinite sequence.

2. Compact classes of sets. A class $\boldsymbol{F}$ of subsets of a set X is called *compact*, if for each sequence $P_n \in \boldsymbol{F}$ the relation $P_1P_2\ldots P_n \neq 0$ for $n=1,2,\ldots$ implies $P_1P_2\ldots \neq 0$. Obviously, a multiplicative class $\boldsymbol{F}$ of sets (*i. e.* a class $\boldsymbol{F}=\boldsymbol{F}_d$) is compact if and only if the product of any decreasing sequence of non-void sets belonging to it is non-void. A topological space is compact if and only if the class of all its closed subsets is compact. More generally, the class of all compact subsets of a topological space is compact. Evidently

(i) *Each subclass of a compact class is compact.*

Now we shall prove that

(ii) *If $\boldsymbol{F}$ is compact, then $(\boldsymbol{F})_\delta$ is compact.*

[8]) See *e. g.* Kolmogoroff [6], p. 15-16, Halmos [5], p. 54-56.

Let us consider the product

$$P=(P_1^1P_2^1\ldots)(P_1^2P_2^2\ldots)\ldots \qquad \text{where} \quad P_l^k\in \boldsymbol{F}$$

and let us suppose that

$$(*) \qquad (P_1^1P_2^1\ldots)(P_1^2P_2^2\ldots)\ldots(P_1^nP_2^n\ldots)\neq 0 \qquad \text{for} \quad n=1,2,\ldots$$

Obviously

$$(**) \qquad P=P_1^1P_2^1P_1^2P_3^1P_2^2P_1^3P_1^4\ldots$$

and each partial product of (**) is non-void because it contains a product of the form (*). The class $\boldsymbol{F}$ being compact, we obtain $P\neq 0$, q. e. d.

(iii) *If $\boldsymbol{F}$ is compact, then $(\boldsymbol{F})_s$ is compact.*

Let us consider the product

$$P=(P_1^1+P_2^1+\ldots+P_{l_1}^1)(P_1^2+P_2^2+\ldots+P_{l_2}^2)\ldots \qquad \text{where} \quad P_l^k\in \boldsymbol{F}$$

and let us suppose that

$$(P_1^1+P_2^1+\ldots+P_{l_1}^1)(P_1^2+P_2^2+\ldots+P_{l_2}^2)\ldots(P_1^n+P_2^n+\ldots+P_{l_n}^n)\neq 0 \qquad \text{for} \quad n=1,2,\ldots$$

Consequently for each natural n there is a sequence $k_1,k_2,\ldots,k_n$ of natural numbers such that

$$(\text{⁂}) \qquad P_{k_1}^1P_{k_2}^2\ldots P_{k_n}^n\neq 0.$$

Let us denote by Z the set of all finite sequences $\{k_j\}$ with the property (⁂). It follows from the lemma 1 (iii), that there exists an infinite sequence $\{k_j\}$ such that the inequality (⁂) holds for $n=1,2,\ldots$ The class $\boldsymbol{F}$ being compact, we obtain $P_{k_1}^1P_{k_2}^2\ldots\neq 0$, whence $P\neq 0$, q. e. d.

3. Approximation with respect to a measure. Let μ be a measure in a field $\boldsymbol{M}$ of subsets of X. Let $\boldsymbol{F}$ be a class of subsets of X. We say that *$\boldsymbol{F}$ approximates $\boldsymbol{M}$ with respect to μ* if for each $E\in\boldsymbol{M}$ and each $\eta>0$ there exists a set $P\in\boldsymbol{F}$ and a set $D\in\boldsymbol{M}$ such that

$$D\subset P\subset E \qquad \text{and} \qquad \mu(E-D)<\eta.$$

Let us consider, for instance, the field $\boldsymbol{E}$ of all finite sums of intervals of the form $a\leqslant x<\beta$, where $0\leqslant a<\beta\leqslant 1$. Evidently the class $\boldsymbol{F}$ of closed linear sets approximates $\boldsymbol{E}$ with respect to the ordinary measure.

Obviously a class $\boldsymbol{F}$ contained in $\boldsymbol{M}$ approximates $\boldsymbol{M}$, if and only if for each $E\in\boldsymbol{M}$ and $\eta>0$ there exists $P\in\boldsymbol{F}$ such that

$$P\subset E \qquad \text{and} \qquad \mu(E-P)<\eta.$$

(i) *Let μ be a measure in a field $\boldsymbol{M}$ of subsets of X, and ν be the σ-extension of μ to the σ-field $(\boldsymbol{M})_\beta$. If $\boldsymbol{F}$ approximates $\boldsymbol{M}$ with respect to μ then $(\boldsymbol{F})_\delta$ approximates $(\boldsymbol{M})_\beta$ with respect to ν* [9].

Let $E\in(\boldsymbol{M})_\beta$. It follows from 1 (ii) that for each $\eta>0$ there is a set $H\in(\boldsymbol{M})_\delta$ such that $H\subset E$ and

$$\nu(E-H)<\frac{\eta}{2}.$$

We have $H=E_1E_2\ldots$, where $E_j\in\boldsymbol{M}$. By hypothesis, for each $j=1,2,\ldots$ there is a set $P_j\in\boldsymbol{F}$ and a set $D_j\in\boldsymbol{M}$ such that

$$D_j\subset P_j\subset E_j \qquad \text{and} \qquad \mu(E_j-D_j)<\frac{\eta}{2^{j+1}}.$$

Setting $D=D_1D_2\ldots$ and $P=P_1P_2\ldots$, we obtain

$$D\subset P\subset H \qquad \text{and} \qquad \nu(H-D)<\frac{\eta}{2},$$

whence

$$D\subset P\subset E \qquad \text{and} \qquad \nu(E-D)<\eta.$$

Obviously $P\in(\boldsymbol{F})_\delta$ and $D\in\boldsymbol{M}_\beta$, and consequently $(\boldsymbol{F})_\delta$ approximates $\boldsymbol{M}_\beta$, q. e. d.

(ii) *Let μ be a measure in the field*

$$\boldsymbol{M}=(\sum_{t\in T}\boldsymbol{M}_t)_0,$$

where $\boldsymbol{M}_t$ are fields of subsets of X. Let $\boldsymbol{F}_t$ be classes of subsets of X. If $\boldsymbol{F}_t$ approximates $\boldsymbol{M}_t$ with respect to μ, then the class

$$\boldsymbol{G}=(\sum_{t\in T}\boldsymbol{F}_t)_{ds}$$

approximates $\boldsymbol{M}$ with respect to μ.

Each set $E\in\boldsymbol{M}$ is obviously of the form

$$E=\sum_{i=1}^{n}\prod_{j=1}^{k_i}E_j^i \qquad \text{where} \quad E_j^i\in\boldsymbol{M}_{t_j^i}.$$

Let us put $k=\max(k_1,\ldots,k_n)$. By hypothesis for each $\eta>0$ and for each pair (i,j) (where $i\leqslant n$ and $j\leqslant k_i$), there is a set $P_j^i\in\boldsymbol{F}_{t_j^i}$ and a set $D_j^i\in\boldsymbol{M}_{t_j^i}$ such that

$$D_j^i\subset P_j^i\subset E_j^i \qquad \text{and} \qquad \mu(E_j^i-D_j^i)<\frac{\eta}{kn}.$$

Consequently we have

$$\prod_{j=1}^{k_i}D_j^i\subset\prod_{j=1}^{k_i}P_j^i\subset\prod_{j=1}^{k_i}E_j^i$$

[9]) Theorem proved in cooperation with R. Sikorski.

and

$$\mu\Big(\prod_{j=1}^{k_i} E_j^i - \prod_{j=1}^{k_i} D_j^i\Big) \leqslant \mu \sum_{j=1}^{k_i} (E_j^i - D_j^i) < \frac{\eta}{n}.$$

Hence, by putting

$$D = \sum_{i=1}^{n} \prod_{j=1}^{k_i} D_j^i \quad \text{and} \quad P = \sum_{i=1}^{n} \prod_{j=1}^{k_i} P_j^i$$

we obtain $D \subset P \subset E$ $(P \in \boldsymbol{G}, D \in \boldsymbol{M})$ and $\mu(E-D) < \eta$, q. e. d.

4. Compact measures. A measure μ defined in a field $\boldsymbol{M}$ is called *compact*, if there exists a compact class $\boldsymbol{F}$ which approximates $\boldsymbol{M}$ with respect to μ.

Examples of compact measures: the ordinary measure in the field $\boldsymbol{E}$ (cf. Section 3), the Lebesgue measure and more generally any σ-measure in the field of all Borel subsets of a separable and complete metric space [10]).

(i) *Every compact measure is countably additive.*

Let μ denote a measure in a field $\boldsymbol{M}$ and $\boldsymbol{F}$ a compact class which approximates $\boldsymbol{M}$. Let $E_j \in \boldsymbol{M}$ form a descending sequence of sets with $\mu(E_j) > a > 0$. By hypothesis, there are a sequence of sets $P_j \in \boldsymbol{F}$ and a sequence of sets $D_j \in \boldsymbol{M}$ such that:

$$D_j \subset P_j \subset E_j \quad \text{and} \quad \mu(E_j - D_j) < \frac{a}{2^j}.$$

Consequently,

$$\mu(E_n - D_1 D_2 \ldots D_n) = \mu(E_1 E_2 \ldots E_n - D_1 D_2 \ldots D_n) \leqslant \mu \sum_{j=1}^{n} (E_j - D_j) < a,$$

whence $D_1 D_2 \ldots D_n \neq 0$ for $n = 1, 2, \ldots$

Since

$$D_1 D_2 \ldots D_n \subset P_1 P_2 \ldots P_n$$

and since the class $\boldsymbol{F}$ is compact, we have $P_1 P_2 \ldots \neq 0$, and *a fortiori* $E_1 E_2 \ldots \neq 0$, which implies the countable additivity of μ.

It follows from the Fréchet-Nikodym theorem 1 (i), theorem (i), and propositions 2 (ii) and 3 (i), that

(ii) *Every compact measure has the compact σ-extension.*

Let us notice that this theorem gives the existence proof of the Lebesgue measure as the σ-extension of the ordinary measure in $\boldsymbol{E}$.

[10]) A non-negative charge in a compact space in the sense of A. D. Alexandroff (cf. [1], p. 314, definition 1, p. 327, definition 7, and [2], p. 567, definition 1) is a compact measure in the sense of this paper. In connection with (i), see [2], p. 590, Theorem 5.

Now we shall prove that

(iii) *If μ is a compact σ-measure in the σ-field $\boldsymbol{M}$, then there exists a compact class $\boldsymbol{G} \subset \boldsymbol{M}$ which approximates $\boldsymbol{M}$ with respect to μ* [11]).

More precisely we shall prove that

(iv) *If μ is a σ-measure in a σ-field $\boldsymbol{M}$ and a class $\boldsymbol{F}$ approximates $\boldsymbol{M}$ with respect to μ, then the class $\boldsymbol{G} = (\boldsymbol{F})_\delta \cdot \boldsymbol{M}$ approximates $\boldsymbol{M}$ too.*

Let $E_0 \in \boldsymbol{M}$ and $\eta > 0$. Then there exist two sequences of sets: $\{P_n\}$ and $\{E_n\}$ such that

$$E_0 \supset P_1 \supset E_1 \supset P_2 \ldots, \quad E_j \in \boldsymbol{M}, \quad P_j \in \boldsymbol{F}$$

and

$$\mu(E_{j-1} - E_j) < \frac{\eta}{2^j} \quad \text{for} \quad j = 1, 2, \ldots$$

Let us put

$$P_0 = P_1 P_2 \ldots = E_1 E_2 \ldots$$

Hence

$$\mu(E_0 - P_0) < \eta \quad \text{and} \quad P_0 \in \boldsymbol{G}, \quad \text{q. e. d.}$$

Theorem (iii) follows from (iv), 2 (i) and 2 (ii).

5. Compactness and independence. We say that the classes $\boldsymbol{F}_t$ (where $t \in T$) of subsets of a fixed set X are *countably independent*, if for each sequence of different indices $t_n \in T$ and each sequence P_n of non-void sets such that for each n

$$(*) \qquad \text{either} \quad P_n \in \boldsymbol{F}_{t_n} \quad \text{or} \quad X - P_n \in \boldsymbol{F}_{t_n}$$

we have $P_1 P_2 \ldots \neq 0$.

Replacing the condition $(*)$ by the condition $P_n \in \boldsymbol{F}_{t_n}$ we obtain the definition of *countably pseudo-independent* classes $\boldsymbol{F}_t$.

In the case of complementative classes of sets, in particular in that of fields, independence and pseudo-independence obviously coincide.

The most important examples of independent classes are the classes of cylinders in the theory of Cartesian multiplication (see Section 6 below).

(i) *If the classes $\boldsymbol{F}_t$ (where $t \in T$) are countably multiplicative (i. e. $(\boldsymbol{F}_t)_\delta = \boldsymbol{F}_t$), countably pseudo-independent and compact, then the class $\boldsymbol{F} = \sum_{t \in T} \boldsymbol{F}_t$ is compact.*

Let $P = P_1 P_2 \ldots$, where $P_j \in \boldsymbol{F}$ and

$$(**) \qquad P_1 P_2 \ldots P_n \neq 0 \quad \text{for} \quad n = 1, 2, \ldots$$

Obviously P can be represented in the form of a finite or denumerable product $P = Q_1 Q_2 \ldots$, where

$$Q_m = P_{j_1^m} \cdot P_{j_2^m} \ldots, \qquad P_{j_k^m} \in \boldsymbol{F}_{t_m} \quad \text{and} \quad t_{m'} \neq t_{m''} \quad \text{for} \quad m' \neq m''.$$

[11]) This proposition is due to C. Ryll-Nardzewski. We do not know, whether an analogous proposition holds without the assumption of countable additivity.

It follows from (**) and the compactness of F_t that $Q_m \neq 0$. The families F_t being countably multiplicative and countably quasi-independent, we have $Q_m \in F_{t_m}$ and $Q_1 Q_2 \ldots \neq 0$, q. e. d.

The preceding proposition and 2 (i)-(iii) imply directly that

(ii) *Under the hypotheses of* (i), *the class* $(\boldsymbol{F})_{ds}$ *is compact.*

Applying this proposition and 3 (ii) we obtain directly the following general theorem:

(iii) *Let* μ *be a measure in the field*

$$\boldsymbol{M} = (\sum_{t \in T} \boldsymbol{M}_t)_0,$$

where the $\boldsymbol{M}_t$ *are fields of subsets of* X. *Let us suppose that* $\mu|\boldsymbol{M}_t$ *is compact and, what is more, that there exist compact, countably multiplicative and countably pseudo-independent classes* $\boldsymbol{F}_t$ *which approximate* $\boldsymbol{M}_t$ *with respect to* μ. *Then* μ *is compact, namely the class*

$$(\sum_{t \in T} \boldsymbol{F}_t)_{ds}$$

approximates $\boldsymbol{M}$ *with respect to* μ.

In the most important case of the countable addivity theorem (iii) implies the following theorem:

(iv) *Let* μ *be a measure in the field* $\boldsymbol{M} = (\sum_{t \in T} \boldsymbol{M}_t)_0$ *where the* $\boldsymbol{M}_t$ *are countably independent* σ*-fields of subsets of* X. *If all the partial measures* $\mu|\boldsymbol{M}_t$ *are compact then* μ *is compact.*

In fact, the measures $\mu|\boldsymbol{M}_t$ being compact, they are σ-additive in virtue of 4 (i) and since the $\boldsymbol{M}_t$ are σ-fields, the $\mu|\boldsymbol{M}_t$ are σ-measures. By 4 (iii) there exist compact and countably multiplicative classes $\boldsymbol{F}_t \subset \boldsymbol{M}_t$ which approximate the $\boldsymbol{M}_t$ with respect to μ. Since the fields $\boldsymbol{M}_t$ are countably independent, the classes $\boldsymbol{F}_t$ are also countably independent and we can apply theorem (iii).

Obviously, it follows from (iv) that there is a compact σ-extension of μ. We shall prove that some partial measure of this σ-extension is also compact:

(v) *Let* $\boldsymbol{M}_t$ $(t \in T)$ *be countably independent* σ*-fields of subsets of* X, *let* $\boldsymbol{L}_{t_1 t_2 \ldots t_n} = (\boldsymbol{M}_{t_1} + \ldots + \boldsymbol{M}_{t_n})_\delta$ *and* $\boldsymbol{L} = \sum \boldsymbol{L}_{t_1 t_2 \ldots t_n}$, *where* $(t_1, t_2, \ldots, t_n)$ *runs over all finite systems of indices belonging to* T. *Let* λ *be a measure in* $\boldsymbol{L}$ *such that* $\lambda|\boldsymbol{L}_{t_1 t_2 \ldots t_n}$ *is a* σ*-measure for each* $t_1, t_2, \ldots, t_n$. *If all partial measures* $\lambda|\boldsymbol{M}_t$ *are compact, then* λ *is compact* [12]).

Put $\mu_t = \lambda|\boldsymbol{M}_t$. In view of 4 (iii), there is a compact class $\boldsymbol{F}_t \subset \boldsymbol{M}_t$ which approximates $\boldsymbol{M}_t$ with respect to μ_t. It follows from (iii) and 3 (i) that the class $(\boldsymbol{F}_{t_1} + \ldots + \boldsymbol{F}_{t_n})_{ds}$ approximates $\boldsymbol{L}_{t_1 t_2 \ldots t_n}$ with respect to $\lambda|\boldsymbol{L}_{t_1 t_2 \ldots t_n}$. Consequently the class

$$\boldsymbol{F} = (\sum_{t \in T} \boldsymbol{F}_t)_{ds\delta}$$

approximates $\boldsymbol{L}$ with respect to μ, and since the class $\boldsymbol{F}$ is compact by (ii) and 2 (ii), the measure λ is compact.

[12]) At first I proved only that λ is countably additive. The stronger formulation was suggested to me by C. Ryll-Nardzewski.

6. Compactness and Cartesian multiplication. We shall apply theorems of the preceding Section to Cartesian multiplication.

$\{X_t\}$ being a family of sets, where t runs over a set T, we denote by X^T the *Cartesian product* of $\{X_t\}$, *i. e.* the set of all functions x which attach to every $t \in T$ a point $x_t \in X_t$. Any set $Z \subset X^T$ is called a *countably reduced Cartesian product* of X_t if for each sequence of indices $t_j \in T$ and each sequence of elements $\xi_j \in X_{t_j}$ there is an $x \in Z$ such that $x_{t_j} = \xi_j$.

We fix a countably reduced Cartesian product $Z \subset X^T$ and for each $E \subset X_t$ we put

$$C_t(E) = \underset{x}{E}[x \in Z;\ x_t \in E].$$

We call $C_t(E)$ a *cylinder* (in Z) with the index t and with the base E.

It follows easily from the fact that Z is a countably reduced Cartesian product of $\{X_t\}$ that

(i) *The classes* $\boldsymbol{C}_t$ *of all cylinders with index* t *are countably independent.*

If for any $t \in T$ $\boldsymbol{F}_t$ denotes a class of subsets of X_t, and φ_t a set function in X_t, then we denote by

$\boldsymbol{F}_t^*$ the class of all sets $C_t(E)$, where $E \in \boldsymbol{F}_t$,

φ_t^* the set function: $\varphi_t^*[C_t(E)] = \varphi_t(E)$.

(ii) *If* $\boldsymbol{M}_t$ *is a multiplicative class* [*a field, a* σ*-field*] *of subsets of* X_t *then* $\boldsymbol{M}_t^*$ *is a multiplicative class* [*a field, a* σ*-field*] *of subsets of* Z.

(iii) *If* μ_t *is a measure* [*countably additive measure*] *in* $\boldsymbol{M}_t$ *then* μ^* *is a measure* [*countably additive measure*] *in* $\boldsymbol{M}_t^*$.

(iv) *If* $\boldsymbol{F}_t$ *is a compact class of subsets of* X_t, *then* $\boldsymbol{F}_t^*$ *is a compact class of subsets of* Z.

(v) *If* $\boldsymbol{F}_t$ *approximates* $\boldsymbol{M}_t$ *with respect to* μ_t, *then* $\boldsymbol{F}_t^*$ *approximates* $\boldsymbol{M}_t^*$ *with respect to* μ_t^*.

(vi) *If* μ_t *is a compact measure then* μ_t^* *is a compact measure.*

Let us suppose that μ_t is for each $t \in T$ a measure in a field $\boldsymbol{M}_t$ of subsets of X_t. By the *product* of $\{\mu_t\}$ we understand any common extension of $\{\mu_t^*\}$ to the field

$$\boldsymbol{M} = (\sum_{t \in T} \boldsymbol{M}_t^*)_0.$$

The problem arises whether each product of σ-measures is countably additive. The negative answer follows from a result by Sparre-Andersen and Jessen [13]). Nevertheless the answer is positive in the case of compact measures:

(vii) *Each product of compact measures is compact.*

In fact, if the μ_t are compact measures in the fields $\boldsymbol{M}_t$ of subsets of X_t, then there exists for each $t \epsilon T$ a compact and countably multiplicative class $\boldsymbol{F}_t$ of subsets of X_t which approximates $\boldsymbol{M}_t$ with respect to μ_t. By (ii)-(vi) for each $t \epsilon T$, $\boldsymbol{M}_t^*$ is a field of subsets of Z, μ_t^* is a measure in $\boldsymbol{M}_t^*$ and $\boldsymbol{F}_t^*$ a compact and countably multiplicative class approximates $\boldsymbol{M}_t^*$ with respect to μ_t^*. Since the classes $\boldsymbol{F}_t^*$ are countably independent in virtue of (i), then by applying of the fundamental theorem 5 (iii), μ is compact.

Analogously, theorem 5 (v) implies the following abstract generalization of Kolmogoroff's theorem:

(viii) *For each $t \epsilon T$ let μ_t be a compact measure in a field $\boldsymbol{M}_t$ of subsets of X_t. Let us put*

$$\boldsymbol{L}_{t_1 t_2 \dots t_n} = (\boldsymbol{M}_{t_1}^* + \boldsymbol{M}_{t_2}^* + \dots + \boldsymbol{M}_{t_n}^*)_\beta \qquad \text{and} \qquad \boldsymbol{L} = \sum_{(t_1 t_2 \dots t_n)} \boldsymbol{L}_{t_1 t_2 \dots t_n}.$$

If λ is a measure in $\boldsymbol{L}$ such that $\lambda | \boldsymbol{L}_{t_1 t_2 \dots t_n}$ is a σ-measure and $\lambda | \boldsymbol{M}_t^* = \mu_t^*$, then λ is compact.

7. Sets of measure zero. In this section we shall prove that the compactness of a measure excludes the possibility of certain singularities.

At first we shall prove two lemmas:

(i) *Let μ be a compact non-atomic σ-measure in a σ-field $\boldsymbol{M}$. Let $\boldsymbol{F}$ be a compact subclass of $\boldsymbol{M}$ which approximates $\boldsymbol{M}$ with respect to μ. Then, for each $E \epsilon \boldsymbol{M}$ with $\mu(E) > 0$ there is a subset D of E belonging to $\boldsymbol{F}$ and such that $0 < \mu(D) \leqslant \mu(E)/2$.*

The measure μ being non-atomic there is a subset $E^* \epsilon \boldsymbol{M}$ of E such that $0 < \mu(E^*) \leqslant \mu(E)/2$ and by hypothesis there is a subset $D \epsilon \boldsymbol{F}$ of E^* such that $0 < \mu(D) \leqslant \mu(E)/2$.

(ii) *Under the assumption of* (i) *there are two disjoint subsets E_1 and E_2 of E belonging to $\boldsymbol{F}$ and such that*

$$\mu(E_1) > 0,\ \mu(E_2) > 0 \qquad \text{and} \qquad \mu(E_1 + E_2) < {}^3/_4\,\mu(E) \text{ [14]}.$$

Applying (i) we obtain a subset $D \epsilon \boldsymbol{F}$ of E such that $0 < \mu(D) \leqslant \mu(E)/2$.

[13]) More precisely, it follows from Andersen and Jessen's result [3] that there is a not countably additive product of σ-measures $\mu_1, \mu_2, \dots$ such that for each n the partial product of $\mu_1, \mu_2, \dots, \mu_n$ is countably additive. Ryll-Nardzewski remarked that there is a product of two σ-measures which is no countably additive.

[14]) By using stronger means, one can replace the coefficient 3/4 by any positive number.

Applying (i) twice we obtain $E_1 \epsilon \boldsymbol{F}$ and $E_2 \epsilon \boldsymbol{F}$ such that

$$E_1 \subset D, \qquad 0 < \mu(E_1) \leqslant \tfrac{1}{2}\mu(D) \leqslant \tfrac{1}{4}\mu(E)$$

and

$$E_2 \subset E - D, \qquad 0 < \mu(E_2) \leqslant \tfrac{1}{2}\mu(E - D) \leqslant \tfrac{1}{2}\mu(E)$$

which implies (ii).

We shall prove the following theorem:

(iii) *If μ is a compact and non-atomic σ-measure, then each set E with $\mu(E) > 0$ contains a set N with $\mu(N) = 0$ of the power of the continuum.*

Let $\boldsymbol{M}$ be the σ-field in which μ is defined and $\boldsymbol{F}$ a compact subclass of $\boldsymbol{M}$ which approximates $\boldsymbol{M}$. Then, thanks to (ii), we can build a dyadic set contained in E. Thus it follows from (ii) that there exists a system of sets $E_{i_1 i_2 \dots i_n} \epsilon \boldsymbol{F}$, where $i_j = 1, 2, \dots$ such that

$$(*) \qquad \mu(E_{i_1 i_2 \dots i_n 1} + E_{i_1 i_2 \dots i_n 2}) < {}^3/_4\,\mu(E_{i_1 i_2 \dots i_n}),$$

$$(**) \qquad E_i \subset E; \quad E_{i_1 i_2 \dots i_n i_{n+1}} \subset E_{i_1 i_2 \dots i_n}; \quad \mu(E_{i_1 i_2 \dots i_n}) > 0,$$

$$(\overset{**}{*}) \qquad E_{i_1 i_2 \dots i_n 1} \cdot E_{i_1 i_2 \dots i_n 2} = 0.$$

Put

$$N = \prod_{n=1}^{\infty} \sum_{(i_1 i_2 \dots i_n)} E_{i_1 i_2 \dots i_n},$$

where $(i_1, i_2, \dots, i_n)$ runs over the set of all systems consisting of n numbers 1 or 2.

Obviously $N \epsilon \boldsymbol{M}$ and in virtue of $(*)$ $\mu(N) = 0$. It follows from $(**)$ and from the compactness of $\boldsymbol{F}$ that for each sequence $\{i_n\}$ of numbers 1 or 2, we have

$$E_{i_1} E_{i_1 i_2} E_{i_1 i_2 i_3} \dots \neq 0$$

and in virtue of $(\overset{**}{*})$ these products are disjoint. Thus the set N is of the power of the continuum, q. e. d.

Sierpiński proved with the aid of the continuum hypothesis that there is a non-denumerable subset S of the unit interval such that each of its subsets of Lebesgue measure zero is at most denumerable [15]). Let $\boldsymbol{M}$ be the σ-field of subsets of S which are Borel sets with respect to S. Let μ be the exterior Lebesgue measure in $\boldsymbol{M}$. Then μ is a σ-measure with the following property: each set of measure μ zero is at most denumerable. We call *Sierpiński measure* any measure having this property. It follows from (iii) that

(iv) *No non-atomic Sierpiński σ-measure is compact.*

[15]) Cf. *e. g.* Sierpiński [13], p. 81.

References

[1] A. D. Alexandroff, *Additive Set Functions in Abstract Spaces*, Introduction, Chapter 1, Recueil Mathématique **8** (1940), p. 307-348.

[2] — *Additive Set Functions in Abstract Spaces*, Chapter 2 and 3, Recueil Mathématique **9** (1941), p. 563-628.

[3] E. Sparre-Andersen and B. Jessen, *On the Introduction of Measures in Infinite Product Sets*, Danske Vid. Selbskab. Mat. Fys. Medd. **25** (1948), no. 4, p. 1-7.

[4] B. W. Gnedenko and A. N. Kolmogoroff, Предельные распределения для сумм независимых случайных величин, Москва-Ленинград 1949.

[5] P. R. Halmos, *Measure Theory*, New York 1950.

[6] A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin 1933.

[7] E. Marczewski, *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, Coll. Math. **1** (1948), p. 122-132.

[8] — *Ensembles indépendants et leurs applications à la théorie de la mesure* Fund. Math. **35** (1948), p. 13-28.

[9] — *On a Test of the σ-Additivity of Measure*, Čas. pro pest. mat. a fys. **75** (1949), p. 140.

[10] — *Measures in Almost Independent Fields*, Fund. Math. **38** (1951), p. 217-229.

[11] — *On Compact Measures*, Coll. Math. **2** (1951), p. 321.

[12] C. Ryll-Nardzewski, *On Quasi-Compact Measures*, Coll. Math. **2** (1951), p. 321-322.

[13] W. Sierpiński, *Hypothèse du continu*, Warszawa-Lwów 1934.

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On Quasi-Compact Measures

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This paper *) is a continuation of paper *On Compact Measures* by Marczewski [6] (quoted in the sequel as C). Here I consider only σ-measures, *i. e.* countably additive measures in a countably additive field and I define the notion of *quasi-compact σ-measure*. This notion is equivalent to that of *perfect* measure introduced by Gnedenko and Kolmogoroff [1]).

It is known that the distribution function of a measurable real function $f(x)$, *i. e.* the set function defined by the formula $\mu_f(E)=\mu[f^{-1}(E)]$ can be considered either for Borel sets E, or for all sets E possessing measurable inverse images $f^{-1}(E)$. In the case of Lebesgue measure these two variants are not essentially different, as was proved by Hartman [2]). Theorem VI proves that this property is characteristic of quasi-compact measures.

In connection with the abstract characterization of the Lebesgue measure, formulated by Halmos, von Neumann [3] and Rohlin [7] I shall prove that in the domain of separable measures the compactness, the quasi-compactness and the point-isomorphism with the Lebesgue measure are equivalent (Theorem VII).

Other relations between the compactness and quasi-compactness are stated in Theorems II and III.

Applying Marczewski's theorem on the invariance of compactness under Cartesian multiplication (C 6 (vii)), I shall prove that quasi-compactness has the same property (Theorem VIII).

In this paper I shall preserve the terminology and notation of C, in particular the letter X will always denote a set, on subsets of which the considered measure is defined.

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*) Presented in part to the Polish Mathematical Society, Wrocław Section, on November 17, 1950. Cf. the preliminary report [8].

[1]) Gnedenko and Kolmogoroff [1], § 3, p. 22-23. This equivalence follows from Theorem VI.

[2]) Hartman [4], p. 21, III.