Let $H_y$ denote the set of all $y \in Q$ such that, for an appropriate neighbourhood $V = Q$ of $y$, we have $P \times V \subseteq G_y$.

Clearly, the $H_y$ are open, $H_y \subseteq H_{x_0}$. Let $b \in Q$ be arbitrary. We shall show that $b \in \bigcup H_y$. Putting $R = P \times \{b\}$, we evidently have, for some
$p \in B C G_y$. For any $x \in P$, there exist open (in $P$ and, respectively, in $Q$) sets $U_x, V_x$ such that $(x, b) \in U_x \times V_x \subseteq G_y$. Therefore, $B \bigcup_{i=1}^{n} (U_x \times V_x) \subseteq G_y$.

Since $B$ is bicoompact, there exist $n \in \mathbb{N}$ such that $B \bigcup_{i=1}^{n} (U_n \times V_n)$. Putting $V = \bigcap_{i=1}^{n} V_n$, we have $B \cap P \times V \subseteq G_y$. Therefore, $V$ being a neighbourhood of $b, b \in H_y$. Hence $Q = \bigcap_{i=1}^{n} H_i$, which implies, $Q$ being compact, that $Q = H_{x_0}$ for some $n$. Then clearly $P \times Q \subseteq G_y$.

On Compact Measures

By

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Let $\mu$ be a measure in abstract space $X_j$ with $\mu(X_j) = 1$ for $j = 1, 2, \ldots$. Roughly speaking, a measure $\mu$ in the Cartesian product $X_1 \times X_2 \times \ldots$ is called a product of $(\mu_j)$ (for the precise definition see below, Section 6), if always
$$\mu(X_1 \times X_2 \times \ldots \times X_{n-1} \times E \times X_{n+1} \times \ldots) = \mu(E),$$
and the direct product of $(\mu_j)$ if
$$\mu(E_1 \times E_2 \times \ldots \times E_{n-1} \times E_{n+1} \times \ldots \times E_k) = \mu(E_1) \cdot \mu(E_2) \cdots \mu(E_k).$$

Products of measures are especially important for Probability Theory, in which they correspond to joint distributions of random variables. Obviously, the direct product corresponds to the case of stochastic independence.

It is well known that for each family of $\sigma$-measures there is a uniquely determined direct $\sigma$-product. The relations in the domains of non direct products are rather complicated. The important theorem formulated by Kolmogoroff concerns the case, in which each $X_j$ is the real line and its abstract analogue is false, as was proved by Sparre-Andersen and Jessen.

In Kolmogoroff's proof, the approximation of measurable sets by compact ones is important. By eliminating non-essential topological concepts from this proof, I arrived at the notion of compact measure. In this paper I shall establish the fundamental properties of this concept, especially some relations between compactness and independence in the sense of the General Theory of Sets (theorems 5 (iii)-(v)). Then I shall show that each product of compact measures is compact (6 (vii)),


1) See e. g. Halmos [5], p. 157, Theorem B.

2) See Kolmogoroff [6], p. 27, Halmos [5], p. 212, Theorem A.

3) — or bicoompact topological space, cf. Halmos, loc. cit., p. 212.

4) Sparre-Andersen and Jessen [1]; cf. also Halmos [5], p. 211-212, and p. 214 (3).

5) Cf. e. g. Marczewski [7], [8] and [10].

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which implies that it can always be extended to a $\sigma$-measure. Also the precise analogue of Kolmogoroff's theorem is fulfilled by compact measures (this follows from theorem 6 (viii)).

The concept of compact measure seems to be useful not only in problems of Cartesian multiplication, e.g. compactness is a sufficient condition for the countable additivity (theorem 4 (ii)).

The converse theorem is false since the relative Lebesgue measure in a non-measurable set of Sierpiński is not compact (theorem 7 (iv)). A stronger result was recently obtained by Rybicki-Nardzewski, who defined the notion of quasi-compact $\sigma$-measure and proved that the relative Lebesgue measure in any non-measurable set is never quasi-compact (of course every compact $\sigma$-measure is quasi-compact). Rybicki-Nardzewski's results on quasi-compact measures will be published in the same Journal (cf. preliminary report (12)). The notion of quasi-compact $\sigma$-measure is equivalent to that of perfect measure in the sense of Gnedenko and Kolmogoroff (14).

1. Preliminaries. For each class $K$ of sets we denote respectively by $(K)_i$, $(K)_i$, $(K)_i$, and $(K)_i$ the class of all sets of the form

$$E_1 + E_2 + \ldots + E_n, \quad E_1 = E_2 = \ldots = E_n,$$

where $E_i \in K$. Next, we denote by $(K)_i$, the smallest field (i.e. an additive and complementable class), and by $(K)_i$, the smallest $\sigma$-field (i.e. countably additive field) containing $K$. If $M$ is a field of subsets of a fixed set $X$, we call measure in $M$ each non-negative and additive set function $\mu(E)$ defined for $E \in M$ and such that $\mu(X) = 1$. A measure is countably additive if for each sequence of disjoint sets $E_1, E_2, \ldots \in M$, we have

$$\mu(E_1 + E_2 + \ldots) = \mu(E_1) + \mu(E_2) + \ldots$$

It is well known that a measurable $\mu$ is countably additive if and only if each sequence of sets $E_1, E_2, \ldots$, with $\mu(E_n) > 0$, has a non-empty product.

A countable additive measure $\mu$ in a $\sigma$-field is called a $\sigma$-measure. A measure is called non-atomic if for each set $E$ with $\mu(E) > 0$ there is a set $D$ such that $\mu(D) = \mu(E) > 0$. If $\mu$ is a measure in $M$, and $L$ is a subfield of $M$, then by $\mu|_L$ we understand the partial measure of $\mu$ in $L$, i.e. the set function $\mu$ defined only in $L$ and equal to $\mu$ in $L$. Then we call $\mu$ an extension of $\mu$. If $\mu$ is a $\sigma$-measure, then it is called a $\sigma$-extension of $\mu$ to $M$. The well-known theorem of Fréchet and Nikodym says that

(i) Any countable additive measure $\mu$ in $M$ has the unique $\sigma$-extension $\nu$ to $(M)_i$, and then we have

$$\nu(E) = \inf \sum_{i=1}^{\infty} \mu(E_i),$$

where $\{E_i\}$ runs over all sequences of sets $E_i \in M$ such that $E_1 + E_2 + \ldots \supseteq E^n$.

It easily follows from (i) that

(ii) If $\mu$ is a measure in $M$ and $\nu$ a $\sigma$-extension of $\mu$ to $(M)_i$, then for each $E \in (M)_i$ we have

$$\nu(E) = \inf \nu(E) = \sup \nu(H),$$

where $K$ runs over all sets containing $E$ and belonging to $(M)_i$, and $H$ runs over all subsets of $E$ belonging to $(M)_i$.

Finally I shall prove an elementary lemma:

(iii) Let $Z$ be a set of finite sequences of positive integers with the following properties:

1. There is an infinite sequence $(1, k_1, k_2, \ldots)$ such that, if $(k_1, k_2, \ldots, k_n) \in Z$, then $k_j \leq 2$ for $j = 1, 2, \ldots, n$;

2. For each $u = 1, 2, \ldots$ there is a sequence $(k_1, k_2, \ldots, k_u) \in Z$;

3. If $(k_1, k_2, \ldots, k_u+1) \in Z$, then $(k_1, k_2, \ldots, k_u) \in Z$.

Then there is an infinite sequence $(k_1, k_2, \ldots)$ such that $(k_1, k_2, \ldots, k_u) \in Z$ for $u = 1, 2, \ldots$

Applying 1 and 2 we define $(k_1, k_2, \ldots)$ by induction in such a way that for each positive integer $u$ there exist arbitrarily long sequences $(k_1, k_2, \ldots, k_n, \ldots, k_{u+1}, \ldots) \in Z$. In view of 3, $(k_1, k_2, \ldots)$ is the required infinite sequence.

2. Compact classes of sets. A class $F$ of subsets of a set $X$ is called compact, if for each sequence $F_1, F_2, \ldots$ the relation $P_1 P_2 \ldots P_n = 0$ for $n = 1, 2, \ldots$ implies $P_1 F_1 = 0$. Obviously, a multiplicative class $F$ of sets (i.e. a class $F = F_1 F_2$) is compact if and only if the product of any decreasing sequence of non-zero sets belonging to it is non-zero. A topological space is compact if and only if the class of all its closed subsets is compact. More generally, the class of all compact subsets of a topological space is compact. Evidently

(i) Each subclass of a compact class is compact.

Now we shall prove that

(ii) If $F$ is compact, then $(F)_i$ is compact.

See e.g. Kolmogoroff [6], p. 51-56, Halmois [5], p. 54-56.

* See e.g. Kolmogoroff [6], p. 51-56, Halmois [5], p. 54-56.

* In this connection cf. a theorem formulated by Doob and Jensen in the paper by Anderson and Jensen [3], p. 5.

* Gnedenko and Kolmogoroff [4], p. 12-23.
Let us consider the product
\[ P = (P_1^1 P_1^2 ...)(P_2^1 P_2^2 ...)... \] where \( P_i^j \in F \)
and let us suppose that
\[ (P_1^1 P_1^2 ...)(P_2^1 P_2^2 ...)...(P_n^1 P_n^2 ...) \neq 0 \] for \( n = 1, 2, ... \)
Obviously
\[ P = P_1^1 P_1^2 P_1^3 \cdots \] and each partial product of \( (==) \) is non-void because it contains a product of the form \((*)\). The class \( F \) being compact, we obtain \( P \neq 0 \), q. e. d.

(iii) If \( F \) is compact, then \( (P_i) \) is compact.
Let us consider the product
\[ P = (P_1^1 + P_2^1 + \cdots + P_n^1)(P_1^2 + P_2^2 + \cdots + P_n^2) ... \] where \( P_i^j \in F \)
and let us suppose that
\[ (P_1^1 + P_2^1 + \cdots + P_n^1)(P_1^2 + P_2^2 + \cdots + P_n^2) \neq 0 \] for \( n = 1, 2, ... \)
Consequently for each natural \( n \) there is a sequence \( k_1, k_2, ..., k_n \) of natural numbers such that
\[ P_{k_1}^1 P_{k_2}^2 \cdots P_{k_n}^n \neq 0. \]

Let us denote by \( Z \) the set of all finite sequences \( (k) \) with the property \((*)\). It follows from the lemma 1 (iii), that there exists an infinite sequence \( (k) \) such that the inequality \((*)\) holds for \( n = 1, 2, ... \). The class \( F \) being compact, we obtain \( P_{k_1}^1 P_{k_2}^2 \cdots \neq 0 \), whence \( F \neq 0 \), q. e. d.

3. Approximation with respect to a measure. Let \( \mu \) be a measure in a field \( M \) of subsets of \( X \), and \( \nu \) be the \( \sigma \)-extension of \( \mu \) to the \( \sigma \)-field \( (M) \). If \( F \) approximates \( M \) with respect to \( \mu \), then \( (F) \) approximates \( (M) \) with respect to \( \nu \).

Let \( E \in (M) \). It follows from 1 (ii) that for each \( \eta > 0 \) there is a set \( H \in (M) \) such that
\[ \nu(E - H) < \eta. \]
We have \( H = E_1 E_2 \cdots \) where \( E_i \in (M) \). By hypothesis, for each \( j = 1, 2, ... \) there is a set \( F_j \in F \) and a set \( D_j \in M \) such that
\[ D_j \subseteq C_j \subseteq E_j \] and \[ \nu(E_j - D_j) < \eta. \]
Setting \( D = D_1 D_2 \cdots \) and \( P = F_1 F_2 \cdots \), we obtain
\[ D \subseteq C \subseteq E \] and \[ \nu(E - D) < \eta. \]

Obviously \( P \in (F) \) and \( D \in (M) \), and consequently \( (F) \) approximates \( (M) \), q. e. d.

(ii) Let \( \nu \) be a measure in the field
\[ M = (\sum_{i \in I} M_i)_I, \]
where \( M_i \) are fields of subsets of \( X \). Let \( F_i \) be classes of subsets of \( X \). If \( F_i \) approximates \( M_i \), with respect to \( \mu_i \), then the class
\[ G = (\sum_{i \in I} F_i)_I \]
approximates \( M \) with respect to \( \mu \).

Each set \( E \in M \) is obviously of the form
\[ E = \sum_{i \in I} E_i \] where \( E_i \in M_i \).
Let us put \( k = \max(k_1, k_2, ... k_n) \). By hypothesis for each \( \eta > 0 \) and for each pair \( (i, j) \) (where \( i < n \) and \( j \leq k_i \)), there is a set \( F_i^j \in F_i \) and a set \( D_i^j \in M_i \) such that
\[ D_i^j \subseteq C_i^j \subseteq E_i \] and \[ \nu(E_i - D_i^j) < \frac{\eta}{K_i}. \]
Consequently we have
\[ \sum_{i \in I} D_i^j \subseteq \sum_{i \in I} F_i^j \subseteq \sum_{i \in I} E_i \] and \[ \nu(E - D) < \eta. \]

*) Theorem proved in cooperation with R. Sikorski.
4. Compact measures. A measure \( \mu \) defined in a field \( F \) is called compact, if there exists a compact class \( F \) which approximates \( M \) with respect to \( \mu \).

Examples of compact measures: the ordinary measure in the field \( E \) (cf. Section 3), the Lebesgue measure and more generally any \( \sigma \)-measure in the field of all Borel subsets of a separable and complete metric space.\(^1\)

(i) Every compact measure is countably additive.

Let \( \mu \) denote a measure in a field \( M \) and \( F \) a compact class which approximates \( M \). Let \( E, F \) form a descending sequence of sets with \( \mu(E) > a > 0 \). By hypothesis, there are a sequence of sets \( P_1 \in F \) and a sequence of sets \( D_1 \in M \) such that:

\[
\mu(E) = \sum_{j=1}^{\infty} \mu(D_j) = \sum_{j=1}^{\infty} \mu(E_j - D_j) < a.
\]

Consequently,
\[
\mu(E - D_{j+1} - D_j) = \mu(E_{j+1} - D_{j+1} - D_j) < \mu(E_j - D_j) < a,
\]
whence \( D_1 D_2 \ldots D_n = 0 \) for \( n = 1, 2, \ldots \).

Since
\[
D_1 D_2 \ldots D_n \subseteq P_1 P_2 \ldots P_n,
\]
and since the class \( F \) is compact, we have \( P_1 P_2 \ldots = 0 \), and a fortiori \( E_0 = D_0 = 0 \), which implies the countable additivity of \( \mu \).

It follows from the Fréchet-Nikodym theorem (1), theorem (i), and propositions (2) and (3), that

(ii) Every compact measure has the compact \( \sigma \)-extension.

Let us notice that this theorem gives the existence proof of the Lebesgue measure as the \( \sigma \)-extension of the ordinary measure in \( E \).

---

\(^1\) A non-negative charge in a compact space in the sense of J. D. Alexandroff (ed. (1)), p. 314, definition 1, p. 327, definition 7, and (2), p. 567, definition 1) is a compact measure in the sense of this paper. In connection with (i), see (2), p. 390, Theorem 8.

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Now we shall prove that

(iii) If \( \mu \) is a compact \( \sigma \)-measure in the \( \sigma \)-field \( M \), then there exists a compact class \( G \subseteq M \) which approximates \( M \) with respect to \( \mu \).\(^2\)

More precisely we shall prove that

(iv) If \( \mu \) is a \( \sigma \)-measure in a \( \sigma \)-field \( M \) and a class \( F \) with respect to \( \mu \), then the class \( G = \{ F \} \subseteq M \) approximates \( M \) too.

Let \( E_0 \subseteq M \) and \( \eta > 0 \). Then there exist two sequences of sets \( \{ P_n \} \) and \( \{ E_n \} \) such that

\[
E_0 \supseteq P_1 \supseteq E_1 \supseteq P_2 \supseteq E_2 \ldots, \quad E_0 \subseteq G, \quad P_n \subseteq F,
\]
and
\[
\mu(E_{j+1} - E_j) < \frac{\eta}{2^j} \quad \text{for} \quad j = 1, 2, \ldots
\]

Let us put

\[
P_0 = P_1 P_2 \ldots = E_0 E_1 \ldots
\]

Hence
\[
\mu(E_0 - P_0) < \eta \quad \text{and} \quad P_0 \subseteq G, \quad \text{q. e. d.}
\]

Theorem (iii) follows from (iv), 2(i) and 2(ii).

5. Compactness and independence. We say that the classes \( F_t \) (where \( t \in T \)) of subsets of a fixed set \( X \) are \( \sigma \)-countably independent, if for each sequence of different indices \( t \in T \) and each sequence \( P_n \) of non-vanishing sets such that for each \( n \)

(*)

either \( P_n \subseteq F_{t_n} \) or \( X - P_n \subseteq F_{t_n} \)

we have \( P_1 P_2 \ldots = 0 \).

Replacing the condition (*) by the condition \( P_n \subseteq F_{t_n} \) we obtain the definition of \( \sigma \)-countably pseudo-independent classes \( F_t \).

In the case of complementary classes of sets, in particular in that of fields, independence and pseudo-independence obviously coincide.

The most important examples of independent classes are the classes of cylinders in the theory of Carleson multiplication (see Section 6 below).

(i) If the classes \( F_t \) (where \( t \in T \)) are \( \sigma \)-countably multiplicative (i.e. \( F_{t_1} F_{t_2} = F_{t_1} \)), \( \sigma \)-countably pseudo-independent and compact, then the class \( F = \sum_{t \in T} F_t \) is compact.

Let \( F = P_1 P_2 \ldots \), where \( P_j \subseteq F \) and

(**)

\[
P_j P_{j+1} \ldots P_n \neq 0 \quad \text{for} \quad n = 1, 2, \ldots
\]

Obviously \( P \) can be represented in the form of a finite or denumerable product \( P = Q_1 Q_2 \ldots \), where

\[
Q_n = P_{j_n} P_{j_n+1} \ldots, \quad P_n \subseteq F_{t_n} \quad \text{and} \quad t_n \neq t_{n'} \quad \text{for} \quad m \neq m'.
\]

\(^2\) This proposition is due to C. Ryll-Nardzewski. We do not know, whether an analogous proposition holds without the assumption of countable additivity.
It follows from \((\ast\ast)\) and the compactness of \(F_i\), that \(Q_\alpha \neq 0\). The families \(F_i\) being countably multiplicative and countably quasi-independent, we have \(Q_\alpha \in F_{m_\alpha}\), and \(Q_\alpha Q_m \neq 0\), \(\alpha, \ldots, m \neq 0\). The preceding proposition and 2 (i)–(iii) imply directly that

(ii) Under the hypotheses of (i), the class \((F_{m_\alpha})\) is compact.

Applying this proposition and 3 (ii) we obtain directly the following general theorem:

(iii) Let \(\mu\) be a measure in the field

\[
M = \left( \sum_{i \in T} M_i \right)_\alpha,
\]

where the \(M_i\) are fields of subsets of \(X\). Let us suppose that \(\mu|M\) is compact and, what is more, that there exist compact, countably multiplicative and countably pseudo-independent classes \(F_i\) which approximate \(M\), with respect to \(\mu\). Then \(\mu\) is compact, namely the class

\[
\left( \sum_{i \in T} F_i \right)_\alpha
\]

approximates \(M\) with respect to \(\mu\).

In the most important case of the countable additivity theorem (iii) implies the following theorem:

(iv) Let \(\mu\) be a measure in the field \(M = \left( \sum_{i \in T} M_i \right)_\alpha\), where the \(M_i\) are countably independent \(\sigma\)-fields of subsets of \(X\). If all the partial measures \(\mu|M_i\) are compact then \(\mu\) is compact.

In fact, the measures \(\mu|M_i\), being compact, are \(\sigma\)-additive in virtue of 4 (i) and since the \(M_i\) are \(\sigma\)-fields, the \(\mu|M_i\) are \(\sigma\)-measures. By 4 (iii) there exist compact, countably multiplicative classes \(F_i \subseteq M_i\), which approximate the \(M_i\), with respect to \(\mu\). Since the \(M_i\) are countably independent, the classes \(F_i\) are also countably independent and we can apply theorem (iii).

Obviously, it follows from (iv) that there is a compact \(\sigma\)-extension of \(\mu\). We shall prove that some partial measure of this \(\sigma\)-extension is also compact:

(v) Let \(M_0 = (\bigcup_{i \in T} M_i)\), and \(M_{0,1} = \sum_{i \in T} I_{i,1}\), where \((t_1, t_2, \ldots, t_k)\) runs over all finite systems of indices belonging to \(T\). Let \(\lambda\) be a measure in \(L\) such that \(\lambda|I_{i,1,\ldots,k}\) is a \(\sigma\)-measure for each \((t_1, t_2, \ldots, t_k)\). If all partial measures \(\lambda|M_i\) are compact, then \(\lambda\) is compact \(^{20}\).

Put \(\mu_0 = \lambda|M_0\). In view of 4 (iii), there is a compact class \(F_0 \subseteq M_0\), which approximates \(M_0\) with respect to \(\mu_0\). It follows from (iii) and 3 (i)\(^{20}\) at first I proved only that \(\lambda\) is countably additive. The stronger formulation was suggested to me by C. Pytli-Nardzewski.

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that the class \((\bigcup_{i \in T} F_i, \alpha)\) approximates \(L_{i_1,\ldots,i_n}\) with respect to \(\lambda|L_{i_1,\ldots,i_n}\). Consequently the class

\[
F = \left( \sum_{i \in T} F_i \right)_{\alpha_0}
\]

approximates \(L\) with respect to \(\mu\), and since the class \(F\) is compact by (ii) and 2 (ii), the measure \(\lambda\) is compact.

6. Compactness and Cartesian Multiplication. We shall apply the theorems of the preceding Section to Cartesian multiplication. (\(X\)) is a family of sets, of which \(T\) runs over a set \(T\), we denote by \(X^T\) the Cartesian product of \((X_i)_i\), i.e., the set of all functions \(x\) which attach to every \(t \in T\) a point \(x_t \in X_t\). Any set \(Z \subseteq X^T\) is called a countably reduced Cartesian product of \(X_i\) if for each sequence of indices \(i \in \tau\) and each sequence of elements \(x_i \in X_i\) there is an \(x \in \tau\) such that \(x_i = x\).

We fix a countably reduced Cartesian product \(Z \subseteq X^T\) and for each \(E \subseteq X_i\) we put

\[
C_i(E) = \bigcup_{x \in Z} \left( \left( x_i \right) \subseteq \bigcup_{E \subseteq X_i} \right).
\]

We call \(C_i(E)\) a cylinder in \(Z\) with the index \(i\) and with the base \(E\).

It follows easily from the fact that \(Z\) is a countably reduced Cartesian product of \((X_i)_i\) that

(i) The classes \(C_i\) of all cylinders with index \(i\) are countably independent.

(ii) If for any \(t \in T\), \(F_t\) denotes a class of subsets of \(X_t\), and \(\varphi\) is a set function in \(X_0\), then we denote by

\[
F' = \bigcup_{t \in T} F_t\]

the class of all sets \(C_i(E)_0\), where \(E \subseteq F_t\), \(\varphi\) is the set function:

\[
\varphi[C_i(E)_0] = \varphi_0[C_i(E)].
\]

(iii) If \(M_0\) is a multiplicative class [a field, a \(\sigma\)-field] of subsets of \(X\), then \(M_0^*\) is a multiplicative class [a field, a \(\sigma\)-field] of subsets of \(Z\).

(iii) If \(\mu_0\) is a measure [countably additive measure] in \(M_0\), then \(\mu_0^*\) is a measure [countably additive measure] in \(M_0^*\).

(iv) If \(F_t\) is a compact class of subsets of \(X_t\), then \(F_t^*\) is a compact class of subsets of \(Z\).

(v) If \(F_t\) approximates \(M_t\) with respect to \(\mu_t\), then \(F_t^*\) approximates \(M_t^*\) with respect to \(\mu_t^*\).

(vi) If \(\mu_t\) is a compact measure then \(\mu_t^*\) is a compact measure.

Let us suppose that \(\mu_t\) is for each \(t \in T\) a measure in a field \(M_t\) of subsets of \(X_t\). By the product of \((\mu_t)\) we understand any common extension of \((\mu_t)\) to the field

\[
M = \left( \sum_{i \in T} M_i \right)\alpha_0.
\]

The problem arises whether each product of \(\sigma\)-measures is countably additive. The negative answer follows from a result by Sparre-Andersen.
sen and Jensen [3]). Nevertheless the answer is positive in the case of compact measures:

(vii) Each product of compact measures is compact.

In fact, if the \( \mu_i \) are compact measures in the fields \( M_i \) of subsets of \( X_i \), then there exists for each \( t \in T \) a compact and countably multiplicative class \( F_t \) of subsets of \( X_t \) which approximates \( M_t \) with respect to \( \mu_t \). By (ii)-(vii) for each \( t \in T \), \( M^*_t \) is a field of subsets of \( X_t \). \( \mu^*_t \) is a measure in \( \mathcal{F}^\ast \) and \( \mathcal{F}^\ast \) is a compact and countably multiplicative class approximates \( M^*_t \) with respect to \( \mu^*_t \). Since the classes \( \mathcal{F}^\ast \) are countably independent in virtue of (i), then by applying of the fundamental theorem 5(iii), \( \mu \) is compact.

Analogously, theorem 5(v) implies the following abstract generalization of Kolmogoroff’s theorem:

(viii) For each \( t \in T \) let \( \mu_t \) be a compact measure in a field \( M_t \) of subsets of \( X_t \). Let us put

\[
L_{L_1,\ldots, L_t} = (M^*_1 + M^*_2 + \cdots + M^*_t)
\]

and

\[
L_t = \sum_{(i_1,\ldots,i_t) \in L_{L_1,\ldots, L_t}} L_{i_1,\ldots,i_t} \cdot \mu_{i_1,\ldots, i_t},
\]

If \( \lambda \) is a measure in \( L \) such that \( \lambda | L_{i_1,\ldots, i_t} \) is a \( \sigma \)-measure and \( \lambda | M^*_t \) is \( \lambda^*_t \), then \( \lambda \) is compact.

7. Sets of measure zero. In this section we shall prove the compactness of a measure excludes the possibility of certain singularities.

At first we shall prove two lemmas:

(i) Let \( \mu \) be a compact non-atomic \( \sigma \)-measure in a \( \sigma \)-field \( M \). Let \( F \) be a compact subclass of \( M \) which approximates \( M \) with respect to \( \mu \). Then, for each \( E \in M \) with \( \mu(E) > 0 \) there is a subset \( D \in M \) belonging to \( F \) and such that

\[
0 < \mu(D) \leq \mu(E)/2.
\]

(ii) Under the assumption of (i) there are two disjoint subsets \( E_1 \) and \( E_2 \) of \( E \) belonging to \( F \) and such that

\[
\mu(E_1) > 0, \quad \mu(E_2) > 0, \quad \mu(E_1 + E_2) < \mu(E)/4 \cdot \mu(E) \quad (\text{iii}).
\]

Applying (i) we obtain a subset \( D \in F \) of \( E \) such that

\[
0 < \mu(D) \leq \mu(E)/2.
\]

7. Sets of measure zero. In this section we shall prove the compactness of a measure excludes the possibility of certain singularities.

At first we shall prove two lemmas:

(i) Let \( \mu \) be a compact non-atomic \( \sigma \)-measure in a \( \sigma \)-field \( M \). Let \( F \) be a compact subclass of \( M \) which approximates \( M \) with respect to \( \mu \). Then, for each \( E \in M \) with \( \mu(E) > 0 \) there is a subset \( D \in M \) belonging to \( F \) and such that

\[
0 < \mu(D) \leq \mu(E)/2.
\]

(ii) Under the assumption of (i) there are two disjoint subsets \( E_1 \) and \( E_2 \) of \( E \) belonging to \( F \) and such that

\[
\mu(E_1) > 0, \quad \mu(E_2) > 0, \quad \mu(E_1 + E_2) < \mu(E)/4 \cdot \mu(E) \quad (\text{iii}).
\]

Applying (i) we obtain a subset \( D \in F \) of \( E \) such that

\[
0 < \mu(D) \leq \mu(E)/2.
\]

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Applying (i) twice we obtain \( E_1 \in F \) and \( E_2 \in F \) such that

\[
E_1 \subset D, \quad 0 < \mu(E_1) \leq \mu(D) \leq \mu(E)/2.
\]

and

\[
E_2 \subset D, \quad 0 < \mu(E_2) \leq \mu(D) \leq \mu(E)/2.
\]

which implies (ii).

We shall prove the following theorem:

(iii) If \( \mu \) is a compact and non-atomic \( \sigma \)-measure, then each set \( E \) with \( \mu(E) > 0 \) contains a set \( N \) with \( \mu(N) = 0 \) of the power of the continuum.

Let \( M \) be the \( \sigma \)-field in which \( \mu \) is defined and \( F \) a compact subclass of \( M \) which approximates \( M \). Then, thanks to (ii), we can build a dyadic set contained in \( E \). Thus it follows from (iii) that there exists a system of sets \( E_{i_1,\ldots, i_n} \in F \), where \( i_1,\ldots, i_n \) such that

\[
\mu(E_{i_1,\ldots, i_n}) > \mu(E)/2.
\]

We shall prove the following theorem:

(iii) If \( \mu \) is a compact and non-atomic \( \sigma \)-measure, then each set \( E \) with \( \mu(E) > 0 \) contains a set \( N \) with \( \mu(N) = 0 \) of the power of the continuum.

Let \( M \) be the \( \sigma \)-field in which \( \mu \) is defined and \( F \) a compact subclass of \( M \) which approximates \( M \). Then, thanks to (ii), we can build a dyadic set contained in \( E \). Thus it follows from (iii) that there exists a system of sets \( E_{i_1,\ldots, i_n} \in F \), where \( i_1,\ldots, i_n \) such that

\[
\mu(E_{i_1,\ldots, i_n}) > \mu(E)/2.
\]

Put

\[
N = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\ldots, i_n} E_{i_1,\ldots, i_n},
\]

where \( (i_1,\ldots, i_n) \) runs over the set of all systems consisting of \( n \) numbers 1 or 2.

Obviously \( N \notin M \) and in virtue of (iv) \( \mu(N) = 0 \). It follows from (iv) and from the compactness of \( F \) that for each sequence \( t \) of numbers 1 or 2, we have

\[
E_{t_1} E_{t_2} E_{t_3}, \ldots = \emptyset
\]

and in virtue of (v) these products are disjoint. Thus the set \( N \) is of the power of the continuum, e. g.

Sierpinski proved with the aid of the continuum hypothesis that there is a non-denumberable subset \( S \) of the unit interval such that each of its subsets of Lebesgue measure zero is at most denumberable [1]. Let \( M \) be the \( \sigma \)-field of subsets of \( S \) which are Borel sets with respect to \( S \). Let \( \mu \) be the exterior Lebesgue measure in \( M \). Then \( \mu \) is a \( \sigma \)-measure with the following property: each set of measure \( \mu \) zero is at most denumberable. We call Sierpinski measure any measure having this property. It follows from (iii) that

(iv) No non-atomic Sierpinski \( \sigma \)-measure is compact.

\[1) \text{ Cf. e. g. Sierpinski [13], p. 81.}\]
References


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On Quasi-Compact Measures

By C. Ryll-Nardzewski (Wrocław)

This paper *) is a continuation of paper *On Compact Measures* by Marczewski [6] (quoted in the sequel as C). Here I consider only -measures, i.e. countably additive measures in a countably additive field and I define the notion of quasi-compact -measure. This notion is equivalent to that of perfect measure introduced by Gnedenko and Kolmogoroff .

It is known that the distribution function of a measurable real function \( f(x) \), i.e. the set function defined by the formula \( \mu\{E\} = \int f^{-1}(E) \) can be considered either for Borel sets \( E \), or for all sets \( E \) possessing measurable inverse images \( f^{-1}(E) \). In the case of Lebesgue measure these two variants are not essentially different, as was proved by Hartman .

Theorem VI proves that this property is characteristic of quasi-compact measures.

In connection with the abstract characterization of the Lebesgue measure, formulated by Halmos, von Neumann [3] and Rohlin [7] I shall prove that in the domain of separable measures the compactness, the quasi-compactness and the point-isomorphism with the Lebesgue measure are equivalent (Theorem VII).

Other relations between the compactness and quasi-compactness are stated in Theorems II and III.

Applying Marczewski’s theorem on the invariance of compactness under Cartesian multiplication (C 6 (vii)), I shall prove that quasi-compactness has the same property (Theorem VIII).

In this paper I shall preserve the terminology and notation of C, in particular the letter \( X \) will always denote a set, on subsets of which the considered measure is defined.

*) Presented in part to the Polish Mathematical Society, Wroclaw Section, on November 17, 1950. Cf. the preliminary report [8].

1) Gnedenko and Kolmogoroff [1], § 3, p. 22-23. This equivalence follows from Theorem VI.

2) Hartman [4], p. 21, III.