On the Cartesian Product of Two Compact Spaces

By

J. Novák (Praha)

A. Tychonoff[1]) established that the Cartesian product of two bicompact spaces is a bicompact space. In this paper the solution of the following problem of Čech is given: Is the Cartesian product of two compact[2]) spaces always compact? Čech posed this problem in 1938 at the topological seminar in Brno. In the year 1949 at the Czechoslovak-Polish Congress of Mathematicians in Prague, I gave the solution[3]), namely that there exist two compact spaces whose Cartesian product fails to be compact. The present paper contains the complete proof of this assertion. The Čech bicompactification[4]) $\beta(N)$ of the countable isolated set (the set of naturals N) was a useful tool for the solution of the problem. Using a theorem by Čech, Pospíšil proved[5]) that the cardinality of $\beta(N)$ is $2^{2^{\aleph_0}}$. In this paper I shall prove this statement directly without any reference to Čech's theorem. At the end I shall add a remark concerning the Cartesian product of two spaces one of which is compact whereas the other is bicompact. M. Katětov proved that this type of Cartesian product is compact.

Definition. Let X be a given point-set. Let $\mathfrak{M}$ be a system of subsets of X. The elements of $\mathfrak{M}$ are said to be *independent* if the product $\bigcap_{k=1}^{n} M_k^{\lambda(k)} \neq 0$ for every natural n and $M_k \in \mathfrak{M}$ where $\lambda(k) = -1$ or $=1$ and $M_k^1 = M_k$ whereas $M_k^{-1} = X - M_k$.

Lemma 1. *There is a system $\mathfrak{N}$ of power $2^{\aleph_0}$ of independent sets whose elements are subsets of the set N of all naturals. Moreover, the product of any finite number $\bigcap_{k=1}^{n} N_k^{\lambda(k)}$ where $N_k \in \mathfrak{N}$ is infinite.*

[1]) A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Annalen **102** (1930), p. 544—561.

[2]) A topological space is called *(compact) bicompact* provided that every (countable) open covering contains a finite subcovering.

[3]) J. Novák, *On the Bicompact Space β (N)-N*, Časopis pro pěstování mat. a fys. **74** (1949), p. 238.

[4]) E. Čech, *On Bicompact Spaces*, Annals of Math. **38** (1937), p. 823.

[5]) B. Pospíšil, *Remark on Bicompact Spaces*, Annals of Math. **38** (1937), p. 845.

Proof. Let U denote the space the elements of which are infinite sequences $(x_1, x_2, \dots, x_n, \dots)$ of real numbers x_n such that $\sum_1^\infty x_n^2 < \infty$. A point $(r_1, r_2, \dots, r_n, \dots)$ will be called a *rational point* if every r_n is a rational number and if $r_n = 0$ except for a finite number of values of n. Evidently, the set of all rational points is countable. Hence, there is a one-to-one mapping $\varphi(n)$ of N onto the set of all rational points. Let

$$F_t(x_1, x_2, \dots, x_n, \dots) \equiv \sum_{i=1}^{\infty} t^{i-1} x_i = 0 \qquad t \in (0,1)$$

be the equation of a hyperplane A_t in the space U. The elements of the system $\mathfrak{N}$ will be infinite sets $N_t \subset N$ defined as follows: $m \in N_t$ if and only if $F_t(\varphi(m)) \geqslant 0$.

Let $0 < t < u < 1$ and let r_1, r_2 be two rational numbers such that $-ur_2 < r_1 < -tr_2$. Then $m \in N_u - N_t$ where $\varphi(m) = (r_1, r_2, 0, 0, \dots)$. From this it follows that the power of the system $\mathfrak{N}$ is $2^{\aleph_0}$.

Let $N_{t_k} \in \mathfrak{N}$, $k = 1, 2, \dots, n$, where $t_k \in (0,1)$ and $t_k \neq t_l$ for $k \neq l$. We shall now prove that the product $\bigcap_{k=1}^{n} N_{t_k}^{\lambda(t_k)}$ is infinite. Let

$$F_{t_k}(x_1, x_2, \dots, x_n, \dots) \equiv \sum_{i=1}^{\infty} t_k^{i-1} x_i = 0 \qquad (k = 1, 2, \dots, n)$$

be n equations of the hyperplanes A_{t_k} in U and let

$$G_{t_k}(x_1, x_2, \dots, x_n) \equiv \sum_{i=1}^{n} t_k^{i-1} x_i = 0 \qquad (k = 1, 2, \dots, n)$$

be n corresponding equations of hyperplanes in n-dimensional Euclidean space E_n all containing the point $(0, 0, \dots, 0) \in E_n$. Since $t_k \neq t_l$ for $k \neq l$ we get

$$\begin{vmatrix} 1 & t_1 & \dots & t_1^{n-1} \\ 1 & t_2 & \dots & t_2^{n-1} \\ \cdot & \cdot & \dots & \cdot \\ 1 & t_n & \dots & t_n^{n-1} \end{vmatrix} = \prod_{k<l<n} (t_k - t_l) \neq 0.$$

Therefore all these hyperplanes are linearly independent in E_n and the set C of all points $(r_1, r_2, \dots, r_n) \in E_n$ where r_k are rational numbers such that $\lambda(t_k) G_{t_k}(r_1, r_2, \dots, r_n) > 0$ for $k = 1, 2, \dots, n$, is infinite. Since

$$G_{t_k}(x_1, x_2, \dots, x_n) = F_{t_k}(x_1, x_2, \dots, x_n, 0, 0, \dots)$$

we have $(r_1, r_2, \dots, r_n, 0, 0, \dots) \in \bigcap_{k=1}^{n} N_{t_k}^{\lambda(t_k)}$ for every $(r_1, r_2, \dots, r_n) \in C$. This proves that the product $\bigcap_{k=1}^{n} N_{t_k}^{\lambda(t_k)}$ is an infinite set.

Let $f_t(x)$, $x \in N$, $t \in (0,1)$, be real valued continuous functions defined in the following manner:

$$(*) \qquad \begin{aligned} f_t(x) &= \frac{1}{x+2} \quad \text{for} \quad x \in N_t^{-1}, \\ f_t(x) &= \frac{x}{x+1} \quad \text{for} \quad x \in N_t^{1}, \end{aligned}$$

where $N_t^{\lambda(t)} \in \mathfrak{N}$. Let $0 < t < t' < 1$. Then, according to Lemma 1, there is an element $n \in N_t \cap N_{t'}^{-1}$. Hence, we have

$$f_t(n) = \frac{n}{n+1} \neq f_{t'}(n) = \frac{1}{n+2}$$

so that $f_t(x) \neq f_{t'}(x)$. Therefore the cardinal number of the system of all functions $f_t(x)$ is $2^{\aleph_0}$. Clearly $f_t(N_t^{-1}) \subset (0,1/3\rangle$, $f_t(N_t) \subset \langle 1/2,1)$ and $f_t(N) \subset (0,1)$.

Let $f_v(x)$, $x \in N$, $v \in (0,2)$, be real valued continuous and bounded functions whereby $f_v(x)$, $v \in (0,1)$, are functions defined by $(*)$. Let $T_v = \langle \inf f_v(x), \sup f_v(x) \rangle$; hence $T_v = \langle 0,1 \rangle$ for $v \in \langle 0,1 \rangle$. The Cartesian product $\mathfrak{P}T_v$, $v \in (0,2)$, is a bicompact Hausdorff space [6]. The transformation $\Phi(x) = \{\xi_v\}$ where $x \in N$, $\{\xi_v\} \in \mathfrak{P}T_v$ and $\xi_v = f_v(x)$ for $v \in (0,2)$ is a homeomorphism of N onto $\Phi(N) \subset \mathfrak{P}T_v$. After identifying $x \equiv \Phi(x)$ for $x \in N$ we have $\overline{N} = \overline{\Phi(N)} \subset \mathfrak{P}T_v$. The space $\overline{\Phi(N)}$ which is immerged in $\mathfrak{P}T_v$, $v \in (0,2)$, is the Čech bicompactification of the isolated set N of all naturals; it will be denoted by $\beta(N)$. In the rest of this paper the closure of a subset A in $\beta(N)$ will be denoted by βA.

Theorem 1. *Let $\{\zeta_t\} \in \mathfrak{P}T_t$, $t \in (0,1)$, where $\zeta = 0$ or $= 1$ for $t \in (0,1)$. Then there exists a point $\{\xi_v\} \in \beta(N) - N$ such that $\xi_v = \zeta_v$ for every $v \in (0,1)$.*

Proof. Let $\{\zeta_t\}$, where $\zeta_t = 0$ or $= 1$ for $t \in (0,1)$, be any point of the set $\mathfrak{P}T_t$, $t \in (0,1)$. Let $\mathfrak{S}$ be the system of all closed subsets $\beta N_t^{\lambda(t)} - N$ where $t \in (0,1)$ and $\lambda(t) = -1$ or $= 1$ according to whether $\zeta_t = 0$ or $= 1$.

Our next task will be to prove that the product of any finite number of sets which are elements of $\mathfrak{S}$ is non-void. As a matter of fact, let $\beta N_{t_k}^{\lambda(t_k)} - N \in \mathfrak{S}$, $k = 1,2,\ldots,n$. If $t_k = t_l$ then, evidently, $\lambda(t_k) = \lambda(t_l)$. Consequently, we can arrange the numbers t_k in ascending order rejecting repetitions $u_1 < u_2 < \ldots < u_m$, where $m \leqslant n$. According to Lemma 1 the set $\bigcap_{j=1}^{m} N_{u_j}^{\lambda(u_j)}$ is infinite so that $0 \neq \beta \bigcap_{j=1}^{m} N_{u_j}^{\lambda(u_j)} - N = \bigcap_{k=1}^{n} \beta N_{t_k}^{\lambda(t_k)} - N = \bigcap_{k=1}^{n} (\beta N_{t_k}^{\lambda(t_k)} - N)$, $\beta(N)$ being a bicompact space and N being an isolated set in it.

Thus we have proved that $0 \neq \bigcap \mathfrak{S} \subset \beta(N) - N$. Let $\{\xi_v\} \in \bigcap \mathfrak{S}$. If there were an index $u \in (0,1)$ such that $\xi_u \neq \zeta_u$, then the neighbourhood $V(\{\xi_v\}) = \mathfrak{P}V_v$, $v \in (0,2)$, of the point $\{\xi_v\} \in \mathfrak{P}T_v$, where $V_u = T_v$ for $v \in (0,2)$, $u \neq v$, and $V_u = \langle 0,1/3)$ in the case in which $\zeta_u = 1$ or $V_u = (1/2,1\rangle$ in the case when $\zeta = 0$, would not contain any point of the set $N_u^{\lambda(u)}$. Therefore the

[6]) See A. Tychonoff, l. c., p. 548.

point $\{\xi_v\}$ would not belong to the set $\beta N_u^{\lambda(u)} - N \in \mathfrak{S}$; but this is impossible. Thus the theorem is established.

From theorem 1 it follows immediately that

Corollary 1. *The cardinality of the space $\beta(N)$ is $2^{2^{\aleph_0}}$.*

Proof. According to Theorem 1 there correspond to two different points $\{\zeta_t\}$ and $\{\zeta'_t\}$ of the space $\mathfrak{P}T_t$, $t \in (0,1)$, where $\zeta_t = 0$ or $= 1$ and $\zeta'_t = 0$ or $= 1$, two different points $\{\xi_v\}$ and $\{\xi'_v\}$ of the space $\beta(N)$ such that $\xi_v = \zeta_v$ and $\xi'_v = \zeta'_v$ for $v \in (0,1)$. Consequently, the corollary given follows from the fact that the cardinal number of the set of all points $\{\zeta_t\} \in \mathfrak{P}T_t$, $t \in (0,1)$, such that $\zeta_t = 0$ or $= 1$ is $2^{2^{\aleph_0}}$.

Theorem 2 [7]). *The cardinal number of every infinite closed subset of the space $\beta(N)$ is $2^{2^{\aleph_0}}$.*

Proof. Let $A_0 \subset \beta(N)$ be an infinite closed subset. Since $\beta(N)$ is a Hausdorff space there is a point $a_1 \in A_0$ and a neighbourhood $V(a_1) \subset \beta(N)$ of a_1 such that $A_0 - V(a_1)$ is an infinite set. Therefore because of the regularity of the space $\beta(N)$ and by using the method of simple induction it is easy to choose points $a_n \in A_0$ and to construct their neighbourhoods $V(a_n) \subset \beta(N)$ such that $V(a_m) \cap V(a_n) = 0$ for $m \neq n$.

Now, we shall try to prove that the set $\beta A \subset \beta(N)$ is the Čech bicompactification of the isolated set $A = \bigcup_{n=1}^{\infty} a_n$. As a matter of fact, βA is a bicompact point-set and the isolated set A is dense in it. Now, let $f(x)$, $x \in A$, be any continuous and bounded real-valued function defined on the set A. Then the function

$$g(x) = \begin{cases} f(a_n) & \text{for} \quad x \in N \cap V(a_n), \\ 0 & \text{for} \quad x \in N - \bigcup_{n=1}^{\infty} V(a_n) \end{cases} \qquad (n = 1,2,\ldots)$$

is a continuous and bounded real-valued function defined on the domain N. Therefore there exists a continuous extension $h(x)$, $x \in \beta(N)$, of the function $g(x)$ such that $h(x) = g(x)$ for $x \in N$.

Since $a_n \in \beta(N \cap V(a_n))$ we have

$$h(a_n) \in \overline{h(N \cap V(a_n))} = \overline{g(N \cap V(a_n))} = \overline{f(a_n)} = f(a_n)$$

and thus $f(a_n) = h(a_n)$ for $n = 1,2,\ldots$ That is, the partial function $h_{\beta A}(x)$ is continuous in the domain βA and the equality $(h_{\beta A}(x))_A = f(x)$ is valid. Therefore the function $h_{\beta A}(x)$ is a continuous extension of the given function $f(x)$ to the domain βA. But this is the characteristic condition for the Čech bicompactification of an isolated countable set A. Since $\beta A \subset A_0$, the assertion of the theorem follows from Corollary 1.

[7]) This theorem was known to Professor E. Čech who communicated it at the topological seminar in Brno in the year 1939.

Theorem 3. There are two compact subsets $A_1\subset\beta(N)$ and $A_2\subset\beta(N)$ such that $A_1\cap A_2=N$ and $A_1\cup A_2=\beta(N)$.

Proof. Let $\mathfrak{S}$ denote the system of all countable infinite subsets $S\subset\beta(N)$. The cardinal number of the system $\mathfrak{S}$ is — according to Corollary 1 —

$$(2^{2^{\aleph_0}})^{\aleph_0}=2^{\aleph_0 2^{\aleph_0}}=2^{2^{\aleph_0}}.$$

The space $\beta(N)-N$ has the same cardinal number. Therefore the elements of the system $\mathfrak{S}$ and the elements of the set $\beta(N)-N$ can be arranged in transfinite sequences

$$S_0, S_1, \ldots, S_\xi, \ldots \qquad \xi<\omega_\delta$$

$$x_0, x_1, \ldots, x_\xi, \ldots \qquad \xi<\omega_\delta$$

of the same order type ω_δ, where ω_δ is the least ordinal of power $2^{2^{\aleph_0}}$.

Now, using the following transfinite method, let us construct the subsets P_ξ and Q_ξ of the set $\beta(N)-N$: Suppose there are attached to every $S_\xi\in\mathfrak{S}$, where $\xi<\alpha$ $(\alpha<\omega_\delta)$, two subsets P_ξ and Q_ξ of $\beta(N)-N$ such that

$$P_0\subset P_1\subset\ldots\subset P_\xi\subset\ldots,$$

$$Q_0\subset Q_1\subset\ldots\subset Q_\xi\subset\ldots,$$

$$P_\xi\cap Q_\xi=0,$$

and such that the cardinal numbers of P_ξ and of Q_ξ are the same as the cardinal number of the set of all ordinals $\leqslant\xi$ (which is $<2^{2^{\aleph_0}}$). Then we shall attach to the element $S_\alpha\in\mathfrak{S}$ two subsets $P_\alpha\subset\beta(N)-N$ and $Q_\alpha\subset\beta(N)-N$ as follows: Since — according to Theorem 2 — the cardina number of the set $\bigcup_{\xi<\alpha}(P_\xi\cup Q_\xi)$ is $<2^{2^{\aleph_0}}$ the set

$$\beta S_\alpha-(N\cup S_\alpha)-\bigcup_{\xi<\alpha}(P_\xi\cup Q_\xi)\subset\beta(N)-N$$

is an infinite point-set. Consequently, we can choose in this set two different points x_μ and x_ν with the least possible indices μ and ν. Put

$$P_\alpha=x_\mu\cup\bigcup_{\xi<\alpha}P_\xi \qquad\text{and}\qquad Q_\alpha=x_\nu\cup\bigcup_{\xi<\alpha}Q_\xi.$$

Then $P_\xi\subset P_\alpha\subset\beta(N)-N$ and $Q_\xi\subset Q_\alpha\subset\beta(N)-N$ for all $\xi<\alpha$ and $P_\alpha\cap Q_\alpha=0$. Evidently, the cardinal number of the set P_α is the same as the cardinal number of the set Q_α and both are equal to the cardinal number of the set of all ordinals $\leqslant\alpha$.

Thus we have constructed two point-sets $P=\bigcup_{\xi<\omega_\delta}P_\xi\subset\beta(N)-N$ and $Q=\bigcup_{\xi<\omega_\delta}Q_\xi\subset\beta(N)-N$. Since $P_0\subset P_1\subset\ldots$ and $Q_0\subset Q_1\subset\ldots$ and $P_\xi\cap Q_\xi=0$ for $\xi<\omega_\delta$ we have $P\cap Q=0$. On the other hand $P\cup Q=\beta(N)-N$. As a matter of fact, let a be any point of the set $\beta(N)-N$. Then $a=x_\varrho$ for a suitable index $\varrho<\omega_\delta$. Consider infinite countable sets $N\cup(x)$ where $x\neq a$ denotes any point of the set $\beta(N)-N-(a)$. Every set like this belongs to the system $\mathfrak{S}$ and may be denoted by S_{ξ_λ}. Now, let $(x_{\mu_\lambda},x_{\nu_\lambda})$ be a pair of points which are attached to the element S_{ξ_λ}. If the point x_ϱ did not belong either to P or to Q we should have — with respect to the minimality of indices μ mentioned above — $\mu_\lambda<\varrho$ and $\nu_\lambda<\varrho$ for all ordinals $\lambda<\omega_\delta$. This contradicts the fact that the subsystem of all sets $S_{\xi_\lambda}=N\cup(x)$ has the cardinality $2^{2^{\aleph_0}}$.

It remains to prove that both sets $A_1=P\cup N$ and $A_2=Q\cup N$ are compact. Suppose M is an infinite subset of the set A_1 and let M_0 be an infinite countable subset of M. Then $M_0=S_\xi\in\mathfrak{S}$ for a suitable ordinal ξ. There is a point $x_\mu\in A_1$, $x_\mu\in\beta S_\xi-S_\xi$ which is attached to the set S_ξ; this point is an accumulation point of the set M. The statement about the set A_2 can be proved analogously. Thus the proof of the theorem is complete.

Theorem 4. The Cartesian product of two compact regular spaces need not be compact.

Proof. Consider the Cartesian product $A_1\times A_2$ of the two spaces A_1 and A_2 which were constructed above. Both spaces A_1 and A_2 are compact and regular, both being subsets of the regular space $\beta(N)$. Since $A_1\cap A_2=N$ the diagonal set D of all points (x,x) consists of those points $(x,y)\in A_1\times A_2$ for which $x=y$ and $x\in N$. Hence the set D is infinite and isolated in $A_1\times A_2$. But there is no accumulation point of the set D in the space $A_1\times A_2$. Suppose the contrary: that there is a point $(a,b)\in\overline{D}-D$. Then $a\in\beta(N)$, $b\in\beta(N)$ and $a\neq b$. Since $\beta(N)$ is a Hausdorff space there are two neighbourhoods: $V(a)\subset\beta(N)$ of the point a and $V(b)\subset\beta(N)$ of the point b such that $V(a)\cap V(b)=0$. Therefore no point (x,x) of the set D can belong to the neighbourhood $(A_1\cap V(a))\times\times(A_2\cap V(b))$ of the point (a,b) in $A_1\times A_2$. This contradicts our hypotheses. Thus we have established that the space $A_1\times A_2$ fails to be compact.

Remark. A. Tychonoff proved that the Cartesian product of two bicompact spaces is bicompact. In the present paper it is established that the Cartesian product of two compact spaces need not be compact. As to the Cartesian product of two compact spaces one of which is bicompact the following statement holds:

Theorem 5. If P is a bicompact space, Q is a compact one, then the Cartesian product $P\times Q$ is compact.

Proof [8]. We have to show: if $G_n\in P\times Q$ are open, $G_n\subset G_{n+1}$ $(n=1,2,\ldots)$, $\bigcup_{n=1}^{\infty}G_n=P\times Q$ then $G_m=P\times Q$ for some m.

[8]) Given by M. Katětov.

Let H_n denote the set of all $y \in Q$ such that, for an appropriate neighbourhood $V = Q$ of y, we have $P \times V \subset G_m$.

Clearly, the H_n are open, $H_n \subset H_{n+1}$. Let $b \in Q$ be arbitrary. We shall show that $b \in \bigcup_{n=1}^{\infty} H_n$. Putting $B = P \times (b)$, we evidently have, for some p, $B \subset G_p$. For any $x \in P$, there exist open (in P and, respectively, in Q) sets U_x, V_x such that $(x,b) \in U_x \times V_x \subset G_p$. Therefore, $B \subset \bigcup_{x \in P} (U_x \times V_x) \subset G_p$. Since B is bicompact, there exist $x_i \in P$ $(i = 1, \dots, n)$ such that $B \subset \bigcup_{i=1}^{n} (U_{x_i} \times V_{x_i})$. Putting $V = \bigcap_{i=1}^{m} V_{x_i}$, we have $B \subset P \times V \subset G_p$. Therefore, V being a neighbourhood of b, $b \in H_p$. Hence $Q = \bigcup_{n=1}^{\infty} H_n$ which implies, Q being compact, that $Q = H_m$, for some m. Then clearly $P \times Q \subset G_m$

On Compact Measures *

By

E. Marczewski (Wrocław)

Let μ_j be a measure in abstract space X_j with $\mu_j(X_j) = 1$ for $j = 1, 2, \dots$ Roughly speaking, a measure μ in the Cartesian product $X_1 \times X_2 \times \dots$ is called a *product* of $\{\mu_j\}$ (for the precise definition see below, Section 6), if always

$$\mu(X_1 \times X_2 \times \dots \times X_{n-1} \times E \times X_{n+1} \times \dots) = \mu_n(E),$$

and the *direct product* of $\{\mu_j\}$ if

$$\mu(E_1 \times E_2 \times \dots \times E_n \times X_{n+1} \times X_{n+2} \times \dots) = \mu_1(E_1) \cdot \mu_2(E_2) \cdot \dots \cdot \mu_n(E_n).$$

Products of measures are especially important for Probability Theory, in which they correspond to joint distributions of random variables. Obviously, the direct product corresponds to the case of stochastic independence.

It is well known that for each family of σ-measures there is a uniquely determined direct σ-product [1]). The relations in the domain of non direct products are rather complicated. The important theorem formulated by Kolmogoroff [2]) concerns the case, in which each X_j is the real line [3]) and its abstract analogue is false, as was proved by Sparre-Andersen and Jessen [4]).

In Kolmogoroff's proof, the approximation of measurable sets by compact ones is important. By eliminating non-essential topological concepts from this proof, I arrived at the notion of compact measure. In this paper I shall establish the fundamental properties of this concept, especially some relations between compactness and independence in the sense of the General Theory of Sets [5]) (theorems 5 (iii)-(v)). Then I shall show that each product of compact measures is compact (6 (vii)),

---

*) Presented to the Polish Mathematical Society (Wrocław Section), on the 10^{th} of November, 1950. Cf. preliminary reports [9] and [11].

[1]) See *e. g.* Halmos [5], p. 157, Theorem B.

[2]) See Kolmogoroff [6], p. 27, Halmos [5], p. 212, Theorem A.

[3]) — or bicompact topological space, cf. Halmos, l. c., p. 212.

[4]) Sparre-Andersen and Jessen [1]; cf. also Halmos [5], p. 211-213, and p. 214 (3).

[5]) Cf. *e. g.* Marczewski [7], [8] and [10].