On the Cartesian Product of Two Compact Spaces

By
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A. Tychonoff\(^1\) established that the Cartesian product of two biocompact spaces is a biocompact space. In this paper the solution of the following problem of Čech is given: Is the Cartesian product of two compact\(^2\) spaces always compact? Čech posed this problem in 1938 at the topological seminar in Brno. In the year 1949 at the Czechoslovak-Polish Congress of Mathematicians in Prague, I gave the solution\(^3\), namely that there exist two compact spaces whose Cartesian product fails to be compact. The present paper contains the complete proof of this assertion. The Čech biocompactification\(^4\) \(\beta(N)\) of the countable isolated set (the set of naturals \(N\)) was a useful tool for the solution of the problem. Using a theorem by Čech, Pospíšil proved\(^5\) that the cardinality of \(\beta(N)\) is \(2^{\aleph_0}\). In this paper I shall prove this statement directly without any reference to Čech’s theorem. At the end I shall add a remark concerning the Cartesian product of two spaces one of which is compact whereas the other is biocompact. M. Katětov proved that this type of Cartesian product is compact.

**Definition.** Let \(X\) be a given point-set. Let \(\mathcal{R}\) be a system of subsets of \(X\). The elements of \(\mathcal{R}\) are said to be biocompact if the product \(\bigcap_{i=1}^{k} M_{k+1}^{\mathcal{R}}\) is equal to 0 for every natural \(n\) and \(M_{k} \in \mathcal{R}\) where \(k=-1\) or \(-1\) and \(M_{k}^{-1} = X - M_{k}\).

**Lemma 1.** There is a system \(\mathcal{R}\) of power \(2^{\aleph_0}\) of independent sets whose elements are subsets of the set \(N\) of all naturals. Moreover, the product of any finite number \(\bigcap_{k=1}^{n} M_{k}^{\mathcal{R}}\) where \(N_{k} \in \mathcal{R}\) is infinite.


\(^{2}\) A topological space is called (compact) biocompact if it is countable and every covering contains a finite subcovering.


Now we prove the following theorem.

**Theorem.** Let \(U\) denote the space of members of which are infinite sequences \((x_{1}, x_{2}, ... , x_{n}, ... )\) of real numbers \(x\), such that \(\sum_{n=1}^{\infty} x_{k} < \infty\). A point \((r_{1}, r_{2}, ..., r_{n}, ... )\) will be called a rational point if every \(r_{n}\) is a rational number and if \(r_{n} = 0\) except for a finite number of values of \(n\). Evidently, the set of all rational points is countable. Hence, there is a one-to-one mapping \(\varphi(n)\) of \(N\) onto the set of all rational points. Let

\[ F_{\varphi}(x_{1}, x_{2}, ... , x_{n}, ...) = \sum_{n=1}^{\infty} \delta^{r_{n}} x_{n} = 0 \quad (n \in (0, 1)) \]

be the equation of a hyperplane \(A_{n}\) in the space \(U\). The elements of the system \(\mathcal{R}\) will be infinite sets \(N_{k} \in \mathcal{R}\) defined as follows: \(m \in N_{k}\) if and only if \(F_{\varphi}(\varphi(m)) > 0\).

Let \(0 < t < s < 1\) and let \(r_{1}, r_{2}\) be two rational numbers such that \(-w_{r_{1}} < r_{1} < -w_{r_{2}}\). Then \(m \in N_{r_{1}} - N_{r_{2}}\) where \(\varphi(m) = (r_{1}, r_{2}, 0, 0, ... )\). From this it follows that the power of the system \(\mathcal{R}\) is \(2^{\aleph_0}\).

Let \(N_{k} \in \mathcal{R}, k = 1, 2, ..., n\), where \(t_{k} \in (0, 1)\) and \(t_{k} = t_{k} / k\) for \(k \neq 1\). We shall now prove that the product \(\bigcap_{k=1}^{n} N_{k}^{\mathcal{R}}\) is infinite. Let

\[ F_{\varphi}(x_{1}, x_{2}, ... , x_{n}, ...) = \sum_{n=1}^{\infty} \delta^{r_{n}} x_{n} = 0 \quad (k = 1, 2, ..., n) \]

be \(n\) equations of the hyperplanes \(A_{k}\) in \(U\) and let

\[ G_{\varphi}(x_{1}, x_{2}, ... , x_{n}) = \sum_{n=1}^{\infty} \delta^{r_{n}} x_{n} = 0 \quad (k = 1, 2, ..., n) \]

be \(n\) corresponding equations of hyperplanes in \(n\)-dimensional Euclidean space \(E_{n}\) all containing the point \((0, 0, 0, 0, 0, ... )\). Since \(t_{k} = t_{k} / k\) for \(k + 1\) we get

\[ 1 \quad t_{1} \quad ... \quad t_{k-1} \]

\[ 1 \quad t_{2} \quad ... \quad t_{k-2} \]

\[ ... \]

\[ 1 \quad t_{k} \quad ... \quad t_{1} \]

Therefore all these hyperplanes are linearly independent in \(E_{n}\) and the set \(C\) of all points \((x_{1}, x_{2}, ... , x_{n}) \in E_{n}\) where \(r_{k}\) are rational numbers such that \(\varphi(t_{k})G_{\varphi}(x_{1}, x_{2}, ... , x_{n}) > 0\) for \(k = 1, 2, ..., n\), is infinite. Since

\[ G_{\varphi}(x_{1}, x_{2}, ... , x_{n}) = F_{\varphi}(x_{1}, x_{2}, ... , x_{n}, 0, 0, ...) \]

we have \((t_{1}, t_{2}, ... , t_{n}, 0, 0, ...) \in \bigcap_{k=1}^{n} N_{k}^{\mathcal{R}}\) for every \((t_{1}, t_{2}, ... , t_{n}) \in C\). This proves that the product \(\bigcap_{k=1}^{n} N_{k}^{\mathcal{R}}\) is an infinite set.
Let \( f(x), x \in N, t \in (0,1) \), be real valued continuous functions defined in the following manner:

\[
\begin{align*}
  f(x) &= \frac{1}{x^2 + 2} \quad \text{for} \quad x \in N_0^{-1}, \\
  f(x) &= \frac{1}{x + 1} \quad \text{for} \quad x \in N_t
\end{align*}
\]

where \( N_0^{-1} \in \mathbb{R} \). Let \( 0 < t < \varepsilon < 1 \). Then, according to Lemma 1, there is an element \( u \in N_t \cap N_0^{-1} \). Hence, we have

\[
  f(u) = \frac{u}{u + 1} + f(\alpha) = \frac{1}{u^2 + 2}
\]

so that \( f(x) = f(x) \). Therefore the cardinal number of the set of all functions \( f(x) \) is \( 2^{\mathbb{R}} \). Clearly, \( f(N_0^{-1}) \subseteq (0,1/2], f(N_t) \subseteq (1/2,1) \) and \( f(N) \subseteq (0,1) \).

Let \( f(x), x \in N, t \in (0,2) \), be real valued continuous and bounded functions where \( f(x), x \in (0,1) \), are functions defined by \((*)\). Let \( T_k = \min f(x), x \in (0,1) \); hence \( T_k = (0,1) \) for \( x \in (0,1) \). The Cartesian product \( \mathbb{N} \times (0,2) \) is a bicompletely Hausdorff space. The transformation \( \Phi(x) = (\xi_x, \eta_x) \) where \( x \in \mathbb{N} \), \( \xi_x \in \mathbb{N} \), and \( \eta_x = f(x) \) for \( x \in (0,2) \) is a homeomorphism of \( \mathbb{N} \) onto \( \mathbb{N} \times (0,2) \). After identifying \( x = \Phi(x) \) for \( x \in \mathbb{N} \) we have \( N = \mathbb{N} \times (0,2) \). The space \( \mathbb{N} \) which is immersed in \( \mathbb{N} \times (0,2) \), is the Čech bicompletion of the isolated set \( N \) of all naturals; it will be denoted by \( \mathbb{B}(N) \). In the rest of this paper the closure of a subset \( A \) in \( \mathbb{B}(N) \) will be denoted by \( \beta A \).

**Theorem 1.** Let \( \xi_n \in \mathbb{B}(N), x \in (0,1) \), where \( \xi_n = 0 \) or \( -1 \) for \( x \in (0,1) \). Then there exists a point \( \xi_n \in \mathbb{B}(N) \) such that \( \xi_n = \xi_n \) for every \( x \in (0,1) \).

**Proof.** Let \( \xi_n \) be any point of the set \( \mathbb{B}(N), x \in (0,1) \). Let \( \mathbb{N} \) be the space of all closed subsets \( \mathbb{N} \times (0,1) \), where \( x \in (0,1) \) and \( 0 \) is the point according to whether \( \xi_n = 0 \) or \( -1 \).

Our next task will be to prove that the product of any finite number of sets which elements of \( \mathbb{N} \) is non-empty. As a matter of fact, let \( \beta \mathbb{N} \times (0,1) \), where \( x \in (0,1) \) and \( \mathbb{N} \) is non-empty, \( \mathbb{N} \) is an ordinal. Consequently, we can arrange the numbers \( k \) in ascending order rejecting repetitions \( u_k = u_k = u \), where \( m \leq u \). According to Lemma 1 the set \( \mathbb{N} \) is infinite so that \( 0 = \mathbb{N} \times (0,1) \), where \( m \leq u \). Hence, the characteristic condition for the Čech bicompletion of an isolated countable set \( A \). Since \( \beta A \subseteq \mathbb{N} \), the assertion of the theorem follows from Corollary 1.

**Corollary 1.** The cardinality of the space \( \mathbb{B}(N) \) is \( 2^{\mathbb{R}} \).

**Proof.** According to Theorem 1 there correspond to two different points \( \xi_n, \eta_n \) of the space \( \mathbb{B}(N), x \in (0,1) \), where \( \xi_n = 0 \) or \( -1 \) and \( \eta_n = 0 \) or \( -1 \), two different points \( \xi_n \) and \( \eta_n \) of the space \( \mathbb{B}(N) \) such that \( \xi_n = \xi_n \) and \( \eta_n = \eta_n \) for \( x \in (0,1) \). Consequently, the corollary given follows from the fact that the cardinal number of the set of all points \( \xi_n, \eta_n \) of \( \mathbb{B}(N) \), such that \( \xi_n = 0 \) or \( -1 \) is \( 2^{\mathbb{R}} \).

**Theorem 2.** The cardinal number of every infinite closed subset of the space \( \mathbb{B}(N) \) is \( 2^{\mathbb{R}} \).

**Proof.** Let \( \mathbb{B}(N) \) be an infinite closed subset. Since \( \mathbb{B}(N) \) is a Hausdorff space there is a point \( a \in \mathbb{B}(N) \), and a neighbourhood \( V(a) \subseteq \mathbb{B}(N) \) of \( a \) such that \( V(a) \cap \mathbb{B}(N) \) is an infinite set. Therefore because of the regularity of the space \( \mathbb{B}(N) \) and by using the method of simple induction it is easy to choose points \( a \in \mathbb{B}(N) \) and to construct their neighbourhoods \( V(a) \subseteq \mathbb{B}(N) \) such that \( V(a) \cap \mathbb{B}(N) \) is an infinite set.

Now, we shall try to prove that the set \( \beta A \subseteq \mathbb{B}(N) \) is the Čech bicompletion of the isolated set \( A = \bigcup a_n \). As a matter of fact, \( \beta A \) is a bicomplete point-set and the isolated set \( A \) is dense in it. Now, let \( f(x), x \in A \), be any continuous and bounded real-valued function defined on the set \( A \). Then the function

\[
g(x) = \begin{cases} 
  f(a) & \text{for } x \in N \cap V(a), \\
  0 & \text{for } x \in N \cap V(a)
\end{cases}
\]

is a continuous and bounded real-valued function defined on the domain \( N \). Therefore, there exists a continuous function \( h(x), x \in \beta A \), of the function \( g(x) \) such that \( h(x) = g(x) \) for \( x \in N \).

Since \( a \in \beta A \) \( \cap \mathbb{B}(N) \), we have

\[
h(a) = h(N \cap V(a)) = g(N \cap V(a)) = f(a) = f(a)
\]

and thus \( f(a) = h(a) \) for \( n = 1,2, \ldots \). That is, the partial function \( h_{\alpha}(x) \) is continuous in the domain \( \beta A \) and the equality \( h_{\alpha}(x) = f(x) \) is valid. Therefore the function \( h_{\alpha}(x) \) is a continuous extension of the given function \( f(x) \) to the domain \( \beta A \). But this is the characteristic condition for the Čech bicompletion of an isolated countable set \( A \). Since \( \beta A \subseteq \mathbb{N} \), the assertion of the theorem follows from Corollary 1.

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1) This theorem was known to Professor E. Čech who communicated it at the topological seminar in Brno in the year 1939.
Theorem 3. There are two compact subsets $A_1, A_2$ such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \beta(N)$.

Proof. Let $\mathfrak{S}$ denote the system of all countable infinite subsets $S \subseteq \beta(N)$. The cardinal number of the system $\mathfrak{S}$ is -- according to Corollary 1 --

$$\vert \mathfrak{S} \vert = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$ 

The space $\beta(N) - N$ has the same cardinal number. Therefore the elements of the system $\mathfrak{S}$ and the elements of the set $\beta(N) - N$ can be arranged in transfinite sequences

$$S_1, S_2, \ldots, S_\xi, \ldots$$
$$x_1, x_2, \ldots, x_\xi, \ldots$$

where $\xi < \alpha_3$, where $\alpha_3$ is the least ordinal of power $2^{\aleph_0}$.

Now, using the transfinite method, let us construct the subsets $P_1$ and $Q_1$ of the set $\beta(N) - N$: Suppose there are associated with every $S_\xi \in \mathfrak{S}$, where $\xi < \alpha_3$ (where $\alpha_3$ is the least ordinal of power $2^{\aleph_0}$), two subsets $P_\xi$ and $Q_\xi$ of $\beta(N) - N$ such that

$$P_\xi \subseteq P_{\xi + 1} \subseteq \ldots \subseteq P_1 \subseteq \ldots$$
$$Q_\xi \subseteq Q_{\xi + 1} \subseteq \ldots \subseteq Q_1 \subseteq \ldots$$

and such that the cardinal numbers of $P_1$ and of $Q_1$ are the same as the cardinal number of the set $\beta(N) - N$ as follows: Since -- according to Theorem 2 -- the cardinal number of the set $\beta(N) - N$ is $2^{\aleph_0}$, the set

$$\beta(N) - (N \cup \mathfrak{S}) = \bigcup_{\xi < \alpha_3} (P_\xi \cup Q_\xi)$$

is an infinite point-set. Consequently, we can construct the set $P_\xi$ in the same way. Then $\beta(N) - (N \cup \mathfrak{S}) = \bigcup_{\xi < \alpha_3} (P_\xi \cup Q_\xi)$ and $\beta(N) - (N \cup \mathfrak{S}) = \bigcup_{\xi < \alpha_3} (P_\xi \cup Q_\xi)$.

Thus we have constructed two point-sets $P = \bigcup_{\xi < \alpha_3} (P_\xi \cup Q_\xi)$ and $Q = \bigcup_{\xi < \alpha_3} (P_\xi \cup Q_\xi)$. Since $P_\xi \subseteq P_{\xi + 1} \subseteq \ldots \subseteq P_1 \subseteq \ldots$ and $Q_\xi \subseteq Q_{\xi + 1} \subseteq \ldots \subseteq Q_1 \subseteq \ldots$ for $\xi < \alpha_3$ we have $P \cap Q = \emptyset$. On the other hand $P \cup Q = \beta(N) - N$. As a matter of fact, let $a$ be any point of the set $\beta(N) - N$. Then $a = x_\xi$ for some $\xi < \alpha_3$. Consider infinite countable sets $M \cup \mathfrak{M}$ of $\beta(N) - N$ that do not belong either to $P$ or to $Q$.

It remains to prove that both sets $A_1 = P \cup N$ and $A_2 = Q \cup N$ are compact. Suppose $M$ is an infinite subset of the set $A_1$ and let $M_\alpha$ be an infinite countable subset of $M$. Then $M_\alpha = B_\xi \subseteq \mathfrak{S}$ for some ordinal $\xi$. There is a point $x_\xi \in A_1$, $x_\xi \in B_\xi \subseteq \mathfrak{S}$ which is attached to the set $S_\xi$; this point is an accumulation point of the set $M$. The statement about the set $A_2$ can be proved analogously. Thus the proof of the theorem is complete.

Theorem 4. The Cartesian product of two compact regular spaces need not be compact.

Proof. Consider the Cartesian product $A_1 \times A_2$ of the spaces $A_1$ and $A_2$, which were constructed above. Both spaces $A_1$ and $A_2$ are compact and regular, both being subsets of the Cartesian product $\beta(N)$. Since $A_1 \cap A_2 = \emptyset$, the diagonal set $D$ of points $(x, y)$ for which $x = y$ and $x \neq N$. Hence the set $D$ is infinite and isolated in $A_1 \times A_2$. But there is no accumulation point of the set $D$ in the space $A_1 \times A_2$. Suppose the contrary: that there is a point $(a, b) \in \beta(N)$. Then $a \in \beta(N)$ and $b \in \beta(N)$ and $a \neq b$. Since $\beta(N)$ is a Hausdorff space there are two neighbourhoods: $V(a) \subseteq \beta(N)$ of the point $a$ and $V(b) \subseteq \beta(N)$ of the point $b$. Therefore no point $(x, y)$ of the set $D$ can belong to the neighbourhood $V(a) \times V(b)$ of the point $(a, b)$ in $A_1 \times A_2$. This contradicts our hypothesis. Thus we have established that the space $A_1 \times A_2$ fails to be compact.

Remar. A. Tychonoff proved that the Cartesian product of two bicomplete spaces is bicomplete. In the present paper it is established that the Cartesian product of two compact regular spaces need not be compact. As to the Cartesian product of two compact spaces one of which is bicomplete the following statement holds:

Theorem 5. If $P$ is a bicomplete space, $Q$ is a compact one, then the Cartesian product $P \times Q$ is compact.

Proof. We have to show: if $G_\xi \subseteq P \times Q$ are open, $G_\xi \subseteq G_{\xi + 1}$ ($\xi = 1, 2, \ldots, \alpha_3$), and $G_\alpha = P \times Q$ then $G_\xi \subseteq P \times Q$ for some $\xi$.

\footnote{Given by M. Katér.}
Let \( H_* \) denote the set of all \( y \in Q \) such that, for an appropriate neighbourhood \( V = Q \) of \( y \), we have \( P \times V \subset C_{u_*} \).

Clearly, the \( H_* \) are open, \( H_* \subset H_{u_*} \). Let \( b \in Q \) be arbitrary. We shall show that \( b \in \bigcup_{i=1}^{\infty} H_* \). Putting \( R = P \times \{ b \} \), we evidently have, for some \( p \), \( B \subset C_{u_*} \). For any \( z \in P \), there exist open (in \( P \) and, respectively, in \( Q \)) sets \( U_z, V_z \) such that \( (z, b) \in U_z \times V_z \subset C_{u_*} \). Therefore, \( B \subset \bigcup_{i=1}^{\infty} (U_z \times V_z) \subset C_{u_*} \).

Since \( B \) is bicomplete, there exist \( z_i \in P \) (\( i = 1, \ldots, \infty \)) such that \( B \subset \bigcup_{i=1}^{\infty} (U_{z_i} \times V_{z_i}) \). Putting \( V = \bigcap_{i=1}^{\infty} V_{z_i} \), we have \( B \subset \bigcap_{i=1}^{\infty} (U_{z_i} \times V_{z_i}) \subset C_{u_*} \). Therefore, \( V \) being a neighbourhood of \( b \), \( b \in H_* \). Hence \( Q = \bigcup_{i=1}^{\infty} H_* \) which implies, \( Q \) being compact, that \( Q = H_{u_*} \) for some \( u_* \). Then clearly \( P \times Q \subset C_{u_*} \).

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On Compact Measures *

By

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Let \( \mu \) be a measure in abstract space \( X \) with \( \mu(X) = 1 \) for \( j = 1, 2, \ldots \). Roughly speaking, a measure \( \mu \) in the Cartesian product \( X_1 \times X_2 \times \ldots \) is called a product of \( \{ \mu_j \} \) (for the precise definition see below, Section 6), if always

\[
\mu(X_1 \times X_2 \times \ldots \times X_{n-1} \times E \times X_{n+1} \times \ldots) = \mu(E),
\]

and the direct product of \( \{ \mu_j \} \)

\[
\mu(E_1 \times E_2 \times \ldots \times E_i \times X_{i+1} \times X_{i+2} \times \ldots) = \mu_1(E_1) \cdot \mu_2(E_2) \cdot \ldots \cdot \mu_i(E_i) \cdot \ldots.
\]

Products of measures are especially important for Probability Theory, in which they correspond to joint distributions of random variables. Obviously, the direct product corresponds to the case of stochastic independence.

It is well known that for each family of \( \sigma \)-measures there is a uniquely determined direct \( \sigma \)-product \(^1\). The relations in the domain of non direct products are rather complicated. The important theorem formulated by Kolmogoroff \(^2\) concerns the case, in which each \( X_j \) is the real line \(^2\) and its abstract analogue is false, as was proved by Sparre-Andersen and Jessen \(^4\).

In Kolmogoroff's proof, the approximation of measurable sets by compact ones is important. By eliminating non-essential topological concepts from this proof, I arrived at the notion of compact measure. In this paper I shall establish the fundamental properties of this concept, especially some relations between compactness and independence in the sense of the General Theory of Sets \(^5\) (theorems 5 (iii)-(v)). Then I shall show that each product of compact measures is compact (6 (vii)),

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1) See e. g. Halmos [5], p. 157, Theorem 5.

2) See Kolmogoroff [6], p. 27, Halmos [5], p. 212, Theorem A.

3) or bicomplete topological space, cf. Halmos, I. e., p. 212.

4) Sparre-Andersen and Jessen [1]; cf. also Halmos [5], p. 211-212, and p. 211 (3).

5) Cf. e. g. Marczewski [7], [8] and [10].

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