On the generation of a simple surface by means of a set of equicontinuous curves.

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If, in a three dimensional space S, l is a simple open [1])-curve and G is a self-compact [2]) set of simple open curves such that through each point of l there is just one curve of the set G and each curve of the set G contains just one point of l then, in order that the curves of the set G should form a simple open surface (i. e. a suface which is in one to one continuous correspondence with a plane) it is not sufficient that no two curves of the set G should have a point in common. Such a set of curves may form a surface containing a portion resembling a part of a coat with a pocket which, in addition to being attached as usual, is also sewed to the coat along its two „lateral edges". In this case the set of curves G is, however, not equicontinuous [3]) with respect to every

[1]) A *simple open curve* is a point-set which is in one to one continuous correspondence with a straight line. For a characterization of such a point-set see my paper *Concerning simple continuous curves*, Transactions of the American Mathematical Society, vol. 21, (1920), pp. 333—347.

[2]) A set G of curves is said to be self-compact if every infinite sequence of curves of the set G contains an infinite subsequence of curves which has a curve of the set G as its sequential limiting set. Cf. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22, (1906), pp. 1—72.

[3]) A set of curves G is said to be *equicontinuous with respect to a given point-set* M if, for every positive number ε, there exists a positive number $\delta_{M\varepsilon}$ such that if P_1 and P_2 are two points of M at a distance apart less than $\delta_{M\varepsilon}$ and lying on a curve g of the set G then that arc of g which has P_1 and P_2 as endpoints lies wholly within some circle of radius ε. Cf. R. L. Moore, *Concer-*

bounded point-set. A condition which is sufficient in oder that a set of curves may form a simple open surface is given below in the statement of Theorem 2. This theorem may be proved with the help of the following related theorem concerning certain sets of simple continuous arcs.

Theorem 1. *Suppose that, in a given three dimensional space S, $ABCD$ is a rectangle and G is a self-compact set of simple continuous arcs such that* (1) *through each point of $ABCD$ there is just one arc of G,* (2) *BC and AD are arcs of G,* (3) *no two arcs of G have a point in common* (4) *each arc of G has one endpoint on the interval AB and one endpoint on the interval CD but contains no other point in common with either of these intervals,* (5) *the set of arcs G is equicontinuous. Then the point-set R composed of all the arcs of the set G is in one to one continuous correspondence with the plane point-set formed by a rectangle together with its interior.*

The truth of this theorem will be established with the help of two lemmas. These lemmas will be proved first.

Definition. A connected point-set K is said to be *simply related* to a set of arcs G satisfying the conditions stated in the hypothesis of Theorem 1 if (1) every point of K belongs to some arc of the set G, (2) K contains the whole of every G-interval[1]) whose endpoints are in K, (3) if P is a point of K there exists a sphere with center at P such that every point of R within this sphere belongs to K, (4) there exist two G arcs g_1 and g_2 such that (a) g_1 lies above g_2, every point of K is between g_1 and g_2 and both g_1

ning certain equicontinuous systems of curves, Transactions of the American Mathematical Society, vol. 22. (1921), p. 42. As far as I know, the notion of a set of *equicontinuous functions* was first introduced by G. Ascoli in an article titled *Sulle curve limiti di una varieta data di curve*, Memoire della Reale Accademia dei Lincei, vol. 18, (1884), pp. 521 - 586.

[1]) If G is a set of arcs or curves, a *G-arc* or a *G-curve* is an arc or curve of the set G. A *G-interval* is an interval (and a *G-segment* is a segment) of such an arc or curve. If G is a set of arcs satisfying the conditions stated in the hypothesis of Theorem 1, the G-arc g_1 is said to be *above* the G-arc g_2 if that endpoint of g_1 which lies on AB is between B and that endpoint of g_2 which lies on AB. If P is a point of R, g_P denotes that arc of G which contains P. If P_1 and P_2 are points of R, P_1 will be said to lie *above* P_2 in case g_{P_1} is above g_{P_2}. If g_1 and g_2 are arcs of G and g_2 is above g_1 and below g_3 then g_2 is said to be *between* g_1 and g_3 and every point of g_2 is said to be *between* g_1 and g_3.

and g_2 have points in common with the boundary of K with respect to R, (b) the set of all those points that the boundary [1]) of K with respect to R has in common with g_2 is an interval t_2 of g_2 $(i = 1, 2)$ (c) no point of t_1 or of t_2, except their endpoints, is a limit point of any point-set which lies between g_1 and g_2 and contains no point of K. The interval t_1 minus its endpoints will be called the upper base and the interval t_2 minus its endpoints will be called the lower base of the set K.

Lemma 1. *If G is a set of arcs satisfying the conditions stated in the hypothesis of Theorem 1, B_1C_1 and B_2C_2 are arcs of G, B_1 and B_2 are points of AB and C_1 and C_2 are points of CD. and B_1 is between B_2 and A, then C_1 is between C_2 and D.*

Proof. There exists a point E between A and B_2 such that if X is any point between A and E and XY is an arc of G then Y is between C_2 and D. For if there exists no such point E then there exists on AB a sequence of distinct points $X_1, X_2, X_3, \ldots$ such that as $n = \infty$, $X_n = A$, but such that, if Y_n denotes the other endpoint of that arc of G which has X_n as one endpoint, then Y_n is between C_2 and C. But since the set G is self-compact therefore the sequence of arcs $X_1Y_1, X_2Y_2, X_3Y_3, \ldots$ contains an infinite subsequence $X_{n_1}Y_{n_1}, X_{n_2}Y_{n_2}, X_{n_3}X_{n_3} \ldots$ such that as $m = \infty$ $X_{n_m}Y_{n_m}$ approaches, as a limiting set, an arc of G. Since as $m = \infty$ $X_{n_m} = A$, this limiting arc must pass through A and must therefore be the arc AD. But this is impossible since, for every m, Y_{n_m} is between C_2 and C. Thus the supposition that there exists no point E having the property stipulated above leads to a contradiction. Now let M denote the set of all points $[E]$ that have this property and let N denote the set of all the remaining points (if there are any) on the segment AB. Suppose that N is not vacuous. Then there exists a point O which is either the uppermost point in set M or the lowermost point in set N. That there is no uppermost point in set M can be proved by an argument entirely analogous to that employed above to prove the existence of a point belonging to the set M. Suppose that P is the lowermost point in the set N. Then every point between P and A belongs to the set M. Hence,

[1]) If the point-set K is a proper subset of the point-set R, the boundary of K with respect to R is the set of all points $[X]$ belonging to R such that every sphere which encloses X encloses at least one point of K and at least one point of $R - K$.

since G is self-compact, if PH denotes that arc of G which has P as an endpoint, G contains an infinite sequence of arcs P_1Z_1, P_2Z_2, $P_3Z_3, \ldots$ such that the points $P_1, P_2, P_3, \ldots$ all belong to M and such that PH is the limiting arc of this sequence. Hence H is a limiting point of the sequence of points $Z_1, Z_2, Z_3, \ldots$ But H is between C_2 and C and the points $Z_1, Z_2, Z_3, \ldots$ are all between D and C_2. Thus we again have a contradiction. The truth of Lemma 1 is therefore established.

Lemma 2. *Under the conditions stated in the hypothesis of Theorem* 1, *if P is a point (distinct from A_0 and from D_0) belonging to an arc A_0D_0 of the set G and S is a sphere with center at P and A_0D_0 is distinct from $\left\{\begin{matrix} BC \\ AD \end{matrix}\right\}$ then there exists, within S, a connected point-set $\left\{\begin{matrix} M \\ N \end{matrix}\right\}$ such that* (1) *every point of $\left\{\begin{matrix} M \\ N \end{matrix}\right\}$ lies on a G-arc $\left\{\begin{matrix} \text{above} \\ \text{below} \end{matrix}\right\}$ A_0D_0,* (2) *the point P is a limit point of $\left\{\begin{matrix} M \\ N \end{matrix}\right\}$,* (3) *there exists a sphere with center at P such that every point $\left\{\begin{matrix} \text{above} \\ \text{below} \end{matrix}\right\}$ A_0D_0 and within this sphere belongs to the point-set $\left\{\begin{matrix} M \\ N \end{matrix}\right\}$.*

Proof. There exists, within S, a sphere S_0, with center at P, such that if X and Y are two points lying within S_0 and on a G-arc g, then that interval of g whose endpoints are X and Y lies wholly within S. Let E and F denote two points on A_0D_0 in the order A_0EPFD_0 and such that the interval EPF of A_0D_0 lies wholly within S_0. Let S_P denote a sphere which lies within S_0, has its center at P, and neither contains nor encloses any point of the arc A_0D_0 which does not lie on the segment EF of A_0D_0. In view of the fact that the set of arcs G is equicontinuous, and satisfies the other conditions of the hypothesis of Theorem 1, it can be easily proved that there exist, within S_0, two spheres S_E and S_F, with centers at E and F respectively, such that if Z is a point within S_P and lying on a G-arc g which contains a point within S_E and a point within S_F, then Z lies on an interval of g whose endpoints lie within S_E and S_F respectively.

Suppose that A_0D_0 is distinct from BC. With the aid of the conditions (other than that of equicontinuity) stipulated in the hypothesis of Theorem 1, it can be shown that there exists a G-arc A_1D_1.

lying above A_0D_0 and such that every G-arc which lies between A_1D_1 and A_0D_0 contains a point within each of the spheres S_E, S_P and S_F. Let M denote the point-set composed of all points $[W]$ such that W is on a G-segment which lies between A_1D_1 and A_0D_0 and whose endpoints are within S_E and S_F respectively. The set M is connected. For suppose that M_1 and M_2 are two mutually exclusive point-sets such that $M_1 + M_2 = M$. If any one G-arc g contains both points of M_1 and points of M_2, then, since the set $g \times M$[1]) is clearly connected, it follows that one of the point-sets $g \times M_1$ and $g \times M_2$ contains a limit point of the other one, and therefore one of the sets M_1 and M_2 contains a limit point of the other one. Suppose that no arc of G contains both points of M_1 and points of M_2. Let $\overline{M}_1$ denote the set of all those points $[L]$ on the interval A_0A_1 of AB such that G-arc which has L as one of its endpoints contains points of M_1 and let $\overline{M}_2$ denote the set of all the remaining points of A_0A_1. Then every G-arc which has a point of $\overline{M}_2$ as an endpoint contains points of M_2. Since the interval A_0A_1 is a connected point-set, one of the sets $\overline{M}_1$ and $\overline{M}_2$ contains a limit point of the other one. Suppose, for instance, that $\overline{M}_1$ contains a point E which is a limit point of $\overline{M}_2$. Then E is the sequential limit point of some sequence of points $E_1, E_2, E_3, \ldots$ belonging to $\overline{M}_1$. The arc g_E is the limiting set of the sequence of arcs $g_{E_1}, g_{E_2}, g_{E_3}, \ldots$ The arc g_E contains a point T within the sphere S_P. The point T is the sequential limit point of a sequence of points $T_1, T_2, T_3, \ldots$ such that, for every n, T_n belongs to g_{E_n}. There exists an integer m such that if $n > m$ then T_n is within S_P. The point T is the sequential limit point of the sequence of points $T_{m+1}, T_{m+2}, T_{m+3}, \ldots$ But every point of this sequence belongs to M_2. Thus M_1 contains a point T which is a limit point of M_2.

So, no matter how M is divided into two mutually exclusive subsets, one of these subsets contains a limit point of the other one. It follows that M is connected. Clearly P is a limit point of M. It is clear that no point between A_1D_1 and A_0D_0 is a limit point of the point-set composed of all those points of R which are above A_1D_1 or below A_0D_0. It follows that there exists a sphere, with center at P, which lies within S_P and neither contains nor encloses any point of R which lies above A_0D_0 but does not lie

[1]) By $g \times M$ is meant the set of all points that are common to g and M.

between A_0D_0 and A_1D_1. Every point of R within this sphere belongs to the point-set M.

The remainder of Lemma 2 may be proved in an entirely analogous manner.

Lemma 3. *If G is a set of arcs satisfying the conditions stated in the hypothesis of Theorem 1, and the point-set K is simply related to G, then any point on the upper base of K can be joined to any point on its lower base by a simple continuous arc that lies wholly in K and does not have more than one point in common with any arc of the set G.*

Proof. If P is a point of R and ε is a positive number, let $R_{P\varepsilon}$ denote the set of all points [X] belonging to R such that X lies on a G-interval ZW such that (1) Z and W are both within a sphere $S_{P\varepsilon}$ of radius ε and with center at P, and (2) some point of ZW can be joined to P by a closed and connected subset of R which lies wholly within $S_{P\varepsilon}$. That the set $R_{P\varepsilon}$ contains points above g_P (unless P is on BC) and points below g_P (unless P is on AD) can be easily proved with the aid of Lemma 2. It can also be easily proved that $R_{P\varepsilon}$ is connected. If, for a given point P and a given pair of positive numbers e and ε such that $e \leqq \varepsilon$, the point-set R_{Pe} has points between two distinct G-arcs g_1 and g_2 and also points on g_1 and points on g_2 then the set of all those points of R_{Pe} that lie between g_1 and g_2 will be called an *elemental set of rank* ε[1]. It follows from Lemma 2 that, if ε is a positive number, each point of K is in some elemental set of rank ε which lies, together with its boundary, wholly in the point-set K^* composed of K and its two bases. Such an elemental set will be called a K element of rank ε. If E and F are two points of K^* and E is above F, then by a *chain* of *K-elements from E to F*, or *joining E to F*, is meant a finite set of K-elements $K_1, K_2, K_3, \dots K_n$ such that (1) E belongs to the upper base of K_1 and F belongs to the lower base of K_n, (2) for each i ($l \leqq i \leqq n$) the lower base of K_i and the upper base of K_{i+1} lie on the same arc of the set G and have points in common, and the set of all their common points is a segment t_i. The point-set $K_1 + K_2 + K_3 + \dots + K_n + t_1 + t_2 + t_3 + \dots + t_{n-1}$ is simply related to G. It will be called the set associated with the chain

[1]) According to this definition, if $\varepsilon_1 < \varepsilon_2$ then every elemental set of rank ε_1 is also of rank ε_2.

$K_1 K_2 K_3 \ldots K_n$. Suppose that E is a point on the upper base of K, F is a point on its lower base and ε is a positive number. I will show that E can be joined to F by a chain of K-elements of rank ε. Let K denote the set of all those points of K that can be joined to E by chains of K-elements of rank ε. It can easily be seen with the aid of Lemma 2 that there exists a K-element of rank ε whose upper base contains the point E. It follows that the set $\overline{K}$ exists.

Suppose that WZ is an arc of G that contains a point of $\overline{K}$. The set of points common to WZ and K is a segment $W^1 Z^1$. Every point of $W^1 Z^1$ must belong to $\overline{K}$. For suppose this is not the case. Then the segment $W^1 Z^1$ is the sum of two mutually exclusive point-sets S_1 and S_2 such that S_1 is a subset of $\overline{K}$ but no point of S_2 belongs to $\overline{K}$. There exists a point P which either belongs to S_1 and is a limit point of S_2 or belongs to S_2 and is a limit point of S_1. In the first case there is a chain α_2 of K-elements of rank ε from E to P. The lower base of the last element of this chain is a segment of $W^1 Z^1$ containing P. Since P is a limit point of S_2 this segment must contain at least one point P_2 of S_2. Thus α_2 is a chain of K elements of rank ε from E to P_2. Thus the supposition that S_1 contains a limit point of S_2 leads to a contradiction. Suppose now that S_2 contains a point P which is a limit point of S_1. There exists a K-element e of rank ε whose lower base $W_2 Z_2$ contains P and is a segment of $W^1 Z^1$. Since P is a limit point of S_1 there exists on the segment $W_2 Z_2$ a point P_1 belonging to S_1. There exists a chain $e_1, e_2, e_2, \ldots, e_n$ of K-elements of rank ε from E to P_1. The lower base of the last element e_n of this chain is a segment $W_1 Z_1$ containing P_1. There exists a G-arc $\bar{g}$ (lying above WZ) and two segments $\overline{W}_1 \overline{Z}_1$ and $\overline{W}_2 \overline{Z}_2$ such that (1) $\overline{W}_1 \overline{Z}_1$ is the set of all points common to e_n and $\bar{g}$, (2) $\overline{W}_2 \overline{Z}_2$ is the set of all points common to e and $\bar{g}$, (3) $\overline{W}_1 \overline{Z}_1$ and $\overline{W}_2 \overline{Z}_2$ have a segment in common. Let $\bar{e}_n$ denote that part ot e_n which lies between $\bar{g}$ and the arc of G that contains the upper base of e_n. Let $\bar{e}_{n+1}$ denote that part of e which lies between $\bar{g}$ and WZ. The set of elements $e_1, e_2, e_3, \ldots, e_{n-1}, \bar{e}_n, \bar{e}_{n+1}$ is a chain of K-elements of rank ε from E to P. It is thus established that if one point of $W^1 Z^1$ belongs to $\overline{K}$ then so does every other point of $W^1 Z^1$. It has been shown that if a G-arc above g_F contains a point of K then so must some lower arc of G. It follows that if F does not belong to K there exists an arc XY which is the uppermost arc of G that contains no point of $\overline{K}$. Let P denote a point of K on the arc XY. There exists a K-element e of rank ε whose lower base contains P. The set G contains an arc g that intersects e in a segment MN. Let $\overline{P}$ denote a point of MN. There exists a chain of K-elements of rank ε from E to $\overline{P}$. If to this chain of elements there is added that portion of the K-element e which lies between g and XY, there is obtained a chain of K-elements of rank ε from E to P. Thus the supposition that E can not be joined to F by a chain of K-elements of rank ε leads to a contradiction. It follows that there exists a simple chain, from E to F, whose links are all K-elements of rank 1. Let K_1 denote the set associated with this chain. There exists a simple chain of K_1-elements, of rank 1, from E to F. This process may be continued. It follows that there exists a sequence of simple chains $C_1, C_2, C_3,\ldots$ from E to F such that if, for each n, K_n denotes the set associated with C_n then (1) every link of C_{n+1} is a K_n-element of rank $1/n$, (2) K'_{n+1} is a subset of the point-set composed of K_n plus its bases. Let t denote the set of all points $[X]$ such that X belongs to every K_n. It can be proved[1]) that t is a simple continuous arc from E to F and that it does not have more than one point in common with any given arc of the set G. The truth of Lemma 3 is thus established.

Proof of Theorem 1. If X is a point of AB and XY is that arc of G which has X as one of its end-points, it may be easily proved, with the aid of the Heine-Borel Theorem, that there exists on XY a finite set of points $A_1, A_2, A_3,\ldots A_n$, in the order $XA_1A_2A_3A_4\ldots A_{n-1}A_nY$, such that each of the intervals XA_1, $A_1A_2,\ldots, A_{n-1}A_n$, A_nY of the arc XY, lies wholly within some sphere of radius 1. Let $C_1, C_2, C_3,\ldots C_n$ denote n points in the order $BC_1C_2C_3\ldots C_{n-1}C_nC$ on the arc BC and let $D_1, D_2, D_3,\ldots, D_n$ denote n points in the order $AD_1D_2\ldots D_{n-1}D_nD$ on the arc AD. With the use of Lemma 3 it is easily established that there exist two sets of arcs, $A_1C_1, A_2C_2,\ldots A_nC_n$ and $A_1D_1, A_2D_2,\ldots A_nD_n$, such that no arc of either set has a point in common with any other

[1]) Cf. the proof of Theorem 15 of my paper *On the foundations of plane analysis situs*, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 136–139.

arc of that set and such that, for every n, (1) $A_n C_n$ lies, except for its endpoints, entirely within[1] $XBCY$, (2) $A_n D_n$ lies, except for its endpoints, entirely within $AXYD$, (3) neither $A_n C_n$ nor $A_n D_n$ has more than one point in common with any one arc of the set G. With the help of the fact that the set G is equicontinuous it is easy to show that there exist two points X^1 and X in the order $A X^1 X \overline{X} B$, and two arcs $X^1 Y^1$ and $\overline{X}\,\overline{Y}$ belonging to G, such that if for every $i (1 \leqq i \leqq n) A_i^1$ is the point in which $X^1 Y^1$ intersects $A_i D_i$ and $\overline{A}_i$ is the point in which $\overline{X}\,\overline{Y}$ intersects $A_i C_i$, then the subset of R which forms the interior, with respect to R, of the closed curve formed by the intervals $A_i^1 A_i$, $A_i \overline{A}_i$, $\overline{A}_i \overline{A}_{i+1}$, $\overline{A}_{i+1} A_{i+1}$, $A_{i+1} A_{i+1}^1$ and $A_{i+1}^1 A_i^1$ of the arcs $A_i D_i$, $A_i C_i$, $\overline{X}\,\overline{Y}$, $A_{i+1} C_{i+1}$, $A_{i+1} D_{i+1}$ and $X^1 Y^1$ respectively, lies entirely within some sphere of radius 1. For each point X of AB make a similar construction and apply the Heine-Borel Theorem to the set of segments $[X^1 \overline{X}]$. If certain arcs are properly extended there will result a double ruling[2]) T_1 of $ABCD$ such that (1) each arc of one of its rulings is an arc of G and each arc of its other single ruling has its endpoints on BC and AD respectively and has just one point in common with each arc of the set G, (2) each of the subdivision[3]) into which T_1 divides R lies within some sphere of radius 1. In a similar manner each subdivision α of this set can itself be subdivided by a double ruling $T_{1\alpha}$ such that (1) each arc of one of its single rulings is an inter-

[1]) If $\overline{A}\,\overline{D}$ and $\overline{C}\,\overline{B}$ are two G-intervals not belonging to the same G-arc and $\overline{A}\,\overline{B}$ and $\overline{D}\,\overline{C}$ are two simple continuous arcs each of which lies wholly in R and has not more than one point in common with any one G-arc and $\overline{A}\,\overline{B}$ has not point in common with $\overline{D}\,\overline{C}$, then by the *interior, with respect to* R, of the simple closed curve $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ is meant the set of all points $[X]$ such that X is on a G-segment which has one endpoint on $\overline{A}\,\overline{B}$ and the other endpoint on $\overline{C}\,\overline{D}$. A point will be said to be *within* $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ if it belongs to the interior, with respect to R of $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$.

[2]) Cf. my paper *Concerning a set of postulates for plane analysis situs*, Transactions of the American Mathematical Society, vol. 20 (1919), p. 172 (footnote) and pp. 172—175.

[3]) If $\overline{A}\,\overline{D}$ and $\overline{C}\,\overline{B}$ are intervals of two arcs g_1 and g_2 of G belonging to one of the single rulings of T_1 and such that there is no arc of T_1 lying between g_1 and g_2 and belonging to the same single ruling and $\overline{A}\,\overline{B}$ and $\overline{D}\,\overline{C}$ are intervals of arcs of the other single ruling of T_1 such that there is no arc of that single ruling between them then the interior with respect to R of the simple closed curve $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ is called a *subdivision* of the ruling T_1, or one of the *subdivisions into which* T_1 *divides* R.

val of an arc of G, (2) each arc of its other single ruling has its endpoints on the arcs which form respectively the upper and the lower base of α and no arc of this ruling has more than one point in common with any arc of G, (3) each of the subdivisions into which $T_{1\alpha}$ divides α is within a sphere of radius 1/2. It follows that there exists a double ruling T_2 satisfying the conditions (1) and (2) stated above as being satisfied by T_1 and also satisfying the additional condition that each of its subdivisions is within some sphere of radius 1/2, for every α each arc of $T_{1\alpha}$ being an interval of an arc of one or the other of the rulings of T_2. This process may be continued. It follows that there exists an infinite sequence of double rulings $T_1, T_2, T_3, \dots$ such that for every n, (1) T_n satisfies the conditions (1) and (2) stated above for T_1, (2) each arc of T_n is an arc of T_{n+1}, (3) each subdivision of T_n is within a sphere of radius $1/n$. Let β be the set of all arcs $[t]$ such that, for some n, t belongs to one of the rulings of T_n and has its endpoints on AD and CB respectively. If P is a point on BC which is not an endpoint of an arc of the set β then there exists just one arc t_P that has one endpoint at P and the other one on AD, lies except for its endpoints entirely in $R - ABCD$ and has no point in common with any arc of the set β. Let γ be the set of all such arcs t_P for all such points P. Let G^1 denote the set of arcs composed of all the arcs of β together with all the arcs of γ and the straight line intervals AB and CD. The points of each of the intervals AD and AB can be brought into one to one continuous correspondence with the numbers of the set $(0 \leqq x \leqq 1)$ in such a way that B and D correspond to 1 and in each case A correspond to 0. If P is a point of R let h_P denote the number which, in the above mentioned correspondence, corresponds to the point of intersection of AD with that arc of G^1 which passes trough P. Let k_P denote the number corresponding to the point in which AB intersects that arc of G which passes through P. Let OX and OY be the axes of Y in a rectangular system of coordinates in some plane M and let $\overline{R}$ denote the set of all points (x, y) in M such that $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$. If P is a point of R let $\overline{P}$ denote the point in M whose coordinates referred to OX and OY are (h_P, k_P). Let $\overline{T}$ denote the transformation of R into $\overline{R}$ such that if P is any point of R then $\overline{T}(P) = P^1$. It is easy to see that the transformation $\overline{T}$ is continuous and that it satisfies all the other requirements of Theorem 1

Theorem 2. *Suppose that, in a space S of three dimensions, l is a simple open curve and G is a self-compact set of simple open curves such that* (1) *through each point of l there is just one curve of the set G,* (2) *each curve of the set G contains a point of l,* (3) *no two curves of the set G have a point in common, and* (4) *the set of curves G is equicontinuous with respect to every bounded point-set. Then the point-set obtained by adding together all the points of all the curves of the set G is a simple open surface.*

Indication of proof of Theorem 2. Let R denote the set of all points $[X]$ such that X is on some curve of the set G. The open curve l may be brought into a definite one to one correspondence with the set of all real numbers. If, in this correspondence, the points X and Y are paired with the numbers x and y respectively then X will be said to be *above* Y if, and only il, $x > y$. If α and β are two curves of the set G, then α will be said to be *above* β if, and only if, the point in which α intersects l is above the point in which β intersects l; and if A and B are two points of R which do not both lie on l then A is said to be *above* B if that G-curve which passes through A is above the one which passes through B. A point of R is said to be *above* a curve of the set G if it is above every point of that curve. By an argument closely related to that used in a portion of the above proof of Lemma 1, it may be shown that if α and β are two curves of G such that α is above β then no point which lies between α and β (i. e. which is above β and below α) is a limit point of the set of all points which lie above α or below β.

With the aid of these notions and a theorem identical with Theorem 1, except that the *rectangle $ABCD$* is replaced by a *simple closed curve $ABCD$*, it is possible to prove, by a line of reasoning analagous to that beginning with the second line from the bottom of page 50 and ending on page 52 of my paper *Concerning certain systems of equicontinuous curves* [1]), that there exists a set of open curves H such that (1) through each point of R there is just one curve of the set H, (2) each curve of the set G lies wholly in R and has just one point in common with each curve of the set G. The truth of Theorem 2 easily follows.

The following theorem may also be established.

[1]) Loc. cit.

Theorem 3. *Suppose that O is a point and J is a simple closed curve and G is an equicontinuous and compact set of simple continuous arcs such that* (1) *each arc of the set G has one endpoint at O and its other endpoint on J,* (2) *for each point P of J there is an arc of the set G which has P and O as its endpoints, and* (3) *no two arcs of G have, in common, any point except O. Then the point-set composed of all the arcs of the set G is in one to one continuous correspondence with the plane point-set composed of a circle plus its interior.*