Note on a paper of M. Banach.

By

Norbert Wiener (Cambridge, Mass., U. S. A.).

In a paper entitled „*Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*" [1]), M. Banach develops from the standpoint of the *calcul fonctionnel* the properties of certain entities which form a class E, for which he assings postulates. E is supposed to contain at least two members, and the elements of E are supposed subject to the three operations of addition, multiplication by a real number, and taking the norm (represented by $\| \|$). Concerning these operations, the following postulatet are assumed:

I_1] $X+Y$ is a determinate element of E,

I_2] $X+Y=Y+X$,

I_3] $X+(Y+Z)=(X+Y)+Z$,

I_4] If $X+Y=X+Z$, then $Y=Z$,

I_5] There is a determinate element ϑ of E such that for every X, $X+\vartheta=X$,

I_6] $a.X$ is a determinate element of E, if X belongs to E and is a real number,

I_7] If $a.X=\vartheta$, then either $a=0$ or $X=\vartheta$,

I_8] If $a \neq 0$ and $a.X=a.Y$, then $X=Y$,

I_9] If $X \neq \vartheta$ and $a.X=b.X$, then $a=b$,

I_{10}] $a.(X+Y)=a.X+a.Y$,

I_{11}] $(a+b).X=a.X+b.X$,

I_{12}] $1.X=X$,

I_{13}] $a.(b.X)=(a.b).X$,

[1]) Fundamenta Mathematicae, vol. III, p. 133.

$\mathrm{II}_1]$ $\|X\|$ is a non-negative real number,
$\mathrm{II}_2]$ If $\|X\| = 0$, $X = \vartheta$, and vice versa,
$\mathrm{II}_3]$ $\|a \,.\, X\| = |a| \,.\, \|X\|$,
$\mathrm{II}_4]$ $\|X + Y\| \leqslant \|X\| + \|Y\|$,
III] If (1) $\{X_n\}$ is a sequence of elements of E,
(2) $\lim\limits_{\substack{r=\infty \\ p=\infty}} \|X_r + (-1) \,.\, X_p\| = 0$,

then there is an element X of E such that

$$\lim_{n=\infty} \|X + (-1) \,.\, X_n\| = 0.$$

If E satisfies these postulates, we shall call E a *vector domain.* M. Banach defines the following operations:

(1) $$-X = (-1) \,.\, X,$$

(2) $$X - Y = X + (-Y),$$

(3) $$\overline{\overline{\lim_{n=\infty}}}\, X_n = X \text{ means}$$

$$\lim_{n=\infty} \|X - X_n\| = 0,$$

(4) $$\sum_{n=1}^{\infty} X_n = X \text{ means}$$

$$\lim_{\nu=\infty} \left\| \sum_{n=1}^{\nu} X_n - X \right\| = 0.$$

He then shows that many of the familiar theorems on limits and sums apply to the operators thus defined. He gives many examples of vector-domains, among them:

(C) E consists of all real functions continuous on the intervall (0, 1). Multiplication and addition receive their usual interpretations. $\|f(x)\| = \max |f(x)|$.

(S^2) E consists of all real functions summable on the interval (0, 1) with summable square, two functions being considered identical when they only differ over a point-set of zero measure. Multiplication and addition receive their usual interpretions. $\|f(x)\| = \left\{ \int\limits_0^1 |f(x)|^2 dx \right\}^{1/2}$.

Mr. Banach's set of postulates may readily be generalized to apply to what we may call complex vector domains. All that is necessary is to allow a and b in postulates $\mathrm{I}_{6,7,8,9,10,11,13}$, and II_3 to assume all complex values. Such a vectordomain, which will *a fortiori* be a system E, we shall denominate a system EC. Examples of systems EC are:

(C′) EC consists of all complex-valued functions continuous on the intervall (0, 1). Multiplicatton and addition receive their usual interpretation. $\|f(x)\| = \max|f(x)|$.

(S′) EC consists of all complex-valued functions summable on the intervall (0, 1), and such that the square of their modulus is summable. Multiplication and addition receive their usual interpretations. $\|f(x)\| = \left\{\int_0^1 |f(x)|^2 dx\right\}^{1/2}$.

Let us now consider functions $F(z)$ with arguments that are complex numbers and values that lie in EC. The derivative and curvilinear integral of such a function receive definitions in no essential wise different from the corresponding definitions in the case of numerically valued functions. For example, we can define

$$\int_C F(z)\,dz,$$

where C is a curvilinear path in the complex plane extending from z_0 to Z, as the vector S (if that vector exist) such that it is possible to make

$$\left\| S - \sum_{r=0}^{\nu} (z_{r+1} - z_r) F(z_r') \right\|$$

less than an arbitrary positive number ε by taking any ν points $z_1, z_2, \ldots, z_\nu$ in order on C (interpreting $z_{\nu+1}$ as Z) in such a way that

$$|z_{r+1} - z_r| < \delta(\varepsilon) \qquad (r = 0, 1, \ldots, \nu)$$

and putting z_r' on the curve between z_r and z_{r+1}. It is of course understood that $\delta(\varepsilon) > 0$ and that ν depends on δ.

Furthermore, using an obvious extension of the notion of lim, we write dF/dz for the unique value of

$$\overline{\lim_{\binom{x'=x}{y'=y}}} \frac{F(z')-F(z)}{z'-z}$$

where $z = x+iy$, $z' \equiv x'+iy'$, and z' approaches z by any route whatever.

It is hence possible to give an enunciation to Cauchy's theorem which will be significant when applied to functions with complex numerical arguments and values on *EC*. This enunciation will read: *if $F(z)$ possesses a derivative dF/dz at all points in the interior of a regular closed curve C with limited variations, and if F be continuous* [in the sense that $\lim z_n = z$ involves $\overline{\lim}\, F(z_n) = F(z)$] *throughout the closed region formed by C and its interior, then*

$$\int_C F(z)\, dz = 0.$$

This theorem is true, and its proof differs in no detail of importance from that of the ordinary Cauchy's theorem as expounded for example in the *Cambridge Tract on Complex Integration and Cauchy's Theorem*, by G. N. Watson. The entire argument there given can be translated directly so as to apply to non-numerically valued fonctions. We can consequently proceed without difficulty to the ordinary theorems on residues, such as those which concern Taylor's and Laurent's series. In particular, if $F(z)$ has a derivative at all points inside a circle of radius r with center at a, while ξ is any point inside this circle, then

$$F(\xi) = \overline{\sum_0^{\infty}} \frac{(\xi-a)^n}{n!}\left[\frac{d^n F(x)}{da^n}\right]_{x=a},$$

where the n-th derivative receives its obvious interpretation.

We are consequently led to consider power-series with vectorial coefficients. Among the most important theorems on power-series is Abel's theorem. If we follow out a line of argument such as that given by Bromwich [1]), we see that there is no step which demands that the coefficients be numerical. Here for the first time, however, we have employ postulate III. We arrive at the result that if $\overline{\Sigma} A_n$ exists,

$$\overline{\lim_{\xi=1}}\, \overline{\Sigma} A_n \xi^n = \overline{\Sigma} A_n.$$

[1]) *Theory of Infinite Series*, §§ 80 and 83.

where ξ approaches 1 along any regular curve which cuts the unit circle at a finite angle.

If EC is taken as (C′), we obtain what Bôcher[1]) calls semi-analytic functions. Bôcher proves that if $f(x, \lambda)$ is continuous in (x, λ) and analytic in λ throughout the region (X, Λ), X consisting of a real interval on the X-axis and Λ being a two-dimensional continuum (i. e., simply-connected region minus its boundary) in the λ-plane, then if $f(x, \lambda)$ is considered as a function of λ having as values functions of x, it admits a derivative with respect to λ, in the sense in which we have defined the derivative in this paper. He deduces the existence of a second derivative with respect to λ, which we could readily prove by a use of the generalised Cauchy theorem, an indefinite integral with respect to λ possessing a derivative in the generalized sense of this paper, and proves a generalization of Taylor's theorem. In addition he proves several theorems concerning $f(x, \lambda)$ *qua* function of x.

We shall consider more particularly the case where (S′) is taken as EC. Let $F(\lambda)$ be a function with complex numerical arguments over the two-dimensional continuum Λ of the λ-plane, and let the values of F be functions from (S′). We may regard $F(\lambda)$ as a function of two variables, namely λ and the real variable x which forms the argument of the functions from (S′) that constitute the values of $F(\lambda)$. To bring the matter out more clearly, we shall write

$$F(\lambda) = f(x, \lambda),$$

where $f(x, \lambda)$ has for its values complex numbers. We shall suppose $F(\lambda)$ to be differentiable over Λ in the sense indicated in this paper. Under this hypotheses our previous discussion will enable us to prove the differentiabiality over Λ of $dF/d\lambda$ and $\int_{\lambda_0}^{\lambda} F(\mu)\,d\mu$. It will of course follow that if $\partial f/\partial \lambda$ and $\int_{\lambda_0}^{\lambda} f(x, \mu)\,d\mu$ are defined except for a set of values of x of zero mesure, they are diffrentiable over Λ in our sense, for $\partial f/\partial \lambda$ is the limit of a sequence of functions

[1]) *On Semi-Analytic Functions of two Variables*, Annals of Mathematics, Second Series, Vol. 12, p. 18.

of which $dF/d\lambda$ is the limit in the mean, and a similar result holds of the integrals.

From a discussion of the Taylor expansion of $F(\lambda)$, it may be shown that if Λ' is the interior of a circle with center λ_0, lying with its circumference entirely within Λ, then the series

$$\sum_0^\infty \frac{(\lambda-\lambda_0)^n}{n!}\left\| \frac{d^n F(\lambda)}{d\lambda^n} \right\|_{\lambda=\lambda_0}$$

converges uniformly over Λ', whence it follows that

$$\lim_{m=\infty}\left\| \sum_0^m \frac{(\lambda-\lambda_0)^n}{n!}\left[\frac{d^n F(\lambda)}{d\lambda^n}\right]_{\lambda=\lambda_0} - F(\lambda)\right\| = 0$$

uniformly over Λ'. From this and the Schwarz inequality it follows that if $g(x)$ is a summable function of summable square,

$$\lim_{m=\infty}\left[\sum_0^m \frac{(\lambda-\lambda_0)^n}{n!}\int_0^1\left[\frac{d^n F(\lambda)}{d\lambda^n}\right]_{\lambda=\lambda_0} g(x)\,dx - \int_0^1 f(x,\lambda)g(x)dx\right| = 0$$

uniformly over Λ', from which it results that

$$\int_0^1 f(x,\lambda)g(x)\,dx$$

is analytic ovec every Λ', and hence over Λ.

To conclude, let us consider a little more in detail the behavior of a series such as

$$\sum_0^\infty \lambda^n f_n(x)$$

for $|\lambda| < 1$, given that

$$\lim_{\substack{n=\infty\\ p=\infty}}\int_0^1 |f_n(x)+f_{n+1}(x)+\ldots+f_p(x)|^2 dx = 0.$$

It clearly follows from this that

$$\lim_{n=\infty}\int_0^1 |f_n(x)|^2 dx = 0.$$

Let $M_n(A)$ be the measure of the set of points between 0 and 1 for which $|f_n(x)| \geqslant \vartheta^{-\frac{n}{2}} A$, given that ϑ is a positive number less than 1. Clearly if for all n

$$\int_0^1 |f_n(x)|^2 < B,$$

then

$$M_n(A) < B\vartheta^n/A^2$$

Hence the total measure of all the points between 0 and 1 for which some $|f_n(x)| \geqslant \vartheta^{-\frac{n}{2}} A$ is less than

$$\frac{B}{A^2}\sum_0^\infty \vartheta^n = \frac{B}{A^2(1-\vartheta)}.$$

In other words, except for a set of measure at most $B/[A^2(1-\vartheta)]$, we have for all n

$$|f_n(x)| < \vartheta^{-\frac{n}{2}} A,$$

or if $|\lambda| \leqslant \vartheta$,

$$\left|\sum_0^m \lambda^n f_n(x)\right| \leqslant \sum_0^m |\lambda|^n |f_n(x)| < \sum_0^m \vartheta^{\frac{n}{2}} A = A/(1-\sqrt{\vartheta}).$$

If we choose A large enough, we can make $B/[A^2(1-\vartheta)]$ as small as we wish. Hence the series $\sum_0^\infty \lambda^n f_n(x)$ converges uniformly and absolutely except for a set of values of x, the measure of which we can make as small as we like.

It follows from this theorem and our generalization of Taylor's series that if $F(\lambda)$ is a function with a numerical argument and a value on (S′), admitting of a derivative over a region Λ such as we have already discussed, and if we write it in the form $f(x, \lambda)$, as we have already indicated then over any region Λ' interior to a circle entirely within a circle within Λ, $f(x, \lambda)$ is analytic in Λ except for a set of points of zero measure in x. It follows at once from this and our previons discussion that $\partial f/\partial\lambda$ and $\int_{\lambda_0}^{\lambda} f(x, \mu)d\mu$, considered *qua* functions of λ, admit a derivative in the sense of this paper [1]).

Massachusetts Institute of Technology, Cambridge, U. S. A.

[1]) As a final comment on this paper, I wish to indicate the fact that postulates not unlike those of M. Banach have been given by me on several occasions (*Comptes rendus* of the Strasboug mathematical conference of 1920; *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 20, Part 5, pp. 332, 333, a forthcoming paper in the Bulletin de la société mathématique de France). However, as my work dates only back to August and September, 1920, and M. Banach's work was already presented for the degree of doctor of philosophy in June, 1920, he has the priority of original composition. I have here employed M. Banach's postulates rather than my own because they are in a form more immediately adopted to the treatment of the problem in hand.