finite sequence of squares K_n with diameters $2h_n$ and with middle points (x_n, y_n). The sequence of subharmonic functions

$$u_n(x,y) = \frac{1}{\log h_n^2} \log [x - x_n + y - y_n]$$

converges asymptotically to 0 in K, but it diverges at every point in K.

4. Divide a fixed square K into four squares. Denote by $K_{1,1}$ and $K_{1,2}$ the squares lying on the principal diagonal of K, and by $K_{1,3}$ and $K_{1,4}$ the two remaining squares. Now divide each of the 4 squares $K_{1,j}$ $(j=1,\dots,4)$ into four squares and denote the squares thus obtained by $K_{2,i}$ $(i=1,\dots,4^2)$ so that the squares $K_{2,i}$ $(i=1,\dots,\frac{1}{2}\cdot 4^2)$ lie on principal diagonals of $K_{1,j}$. Continue this process to infinity so that $K_{n+1,i}$ $(i=1,\dots,\frac{1}{2}\cdot 4^{n+1})$ lie on principal diagonals of $K_{n,j}$ $(j=1,\dots,4^n)$. Let $2h_n$ be the diameter of $K_{n,j}$, and let $(x_{n,j}, y_{n,j})$, be the middle point of $K_{n,j}$.

The sequence of subharmonic functions

$$u_n(x,y) = \sum_{p=1}^{\frac{1}{2}\cdot 4^n} \frac{1}{\log h_n^2} \log (x - x_{n,p} + y - y_{n,p})$$

approximates pointwise the function $u(x,y)=1$ in K, but it does not converge asymptotically to $u(x,y)$.

On Partition of an Ordered Continuum.

By

Josef Novák (Praha).

The result of the consecutive division of an ordered continuum is a system of intervals which satisfy certain conditions (Theorem 3). By these conditions — as in axioms 1°-4° — a system $\mathfrak{P}$ of intervals of an ordered continuum C — called a *partition* of C — is defined. From a thorough study of these axioms a row of properties emerges concerning the partition $\mathfrak{P}$ and the ordered continuum C. For instance: For a given ordered continuum C all the partitions have the same cardinality equal to the least cardinal $\mathfrak{m}(C)$ of the set which is dense in C. Therefore the cardinality of the partition of C is topologically invariant.

In this paper the following theorem is proved: Let C be an ordered continuum. Then there exists an ordinal ϑ of power $\leqslant \mathfrak{m}(C)$ such that C is similar to a lexicographically ordered set whose elements are transfinite sequences of zeros and ones of order-types $\leqslant \vartheta$ (Theorem 4). It is interesting to compare this result with the following theorem of Sierpiński[1]): For a given ordinal ν every ordered set of power $\aleph_\nu$ is similar to a lexicographically ordered set whose elements are transfinite sequences of zeros and ones of type ω_ν.

We shall prove at the end of the paper that a necessary and sufficient condition for all ordered continua of power $\aleph_\sigma$ to contain a point with character c_{00} is the inequality: $\aleph_\sigma < 2^{\aleph_1}$ (Theorem 5). From this it follows that every ordered continuum of power $2^{\aleph_0}$ and of $\mathfrak{m}(C) = \aleph_1$ contains a subset which is dense in C and whose points have character c_{00} (Corollary).

[1]) See W. Sierpiński, *Sur une propriété des ensembles ordonnés*, Fund. Math. **36** (1949), p. 56.

Let C be an ordered continuum [2]). We say that the system $\mathfrak{P}$ of closed intervals [2]) of C is a *dyadic partition* or simply a *partition* of C if the following four conditions are satisfied:

1^0 For any two intervals $X \in \mathfrak{P}$ and $Y \in \mathfrak{P}$ the product $X \cap Y = X$ or $X \cap Y = Y$ or $X \cap Y$ is not an interval at all.

2^0 $C \in \mathfrak{P}$.

3^0 For any interval $X \in \mathfrak{P}$ there are two subintervals $X_1 \in \mathfrak{P}$ and $X_2 \in \mathfrak{P}$ such that $X = X_1 \cup X_2$ and such that the product $X_1 \cap X_2$ contains exactly one point [3]).

4^0 The product of every monotone system of intervals belonging to $\mathfrak{P}$ is either a one-point-set or an interval which belongs to the system $\mathfrak{P}$.

In the sequel, in Theorem 3, it will be proved that in every ordered continuum there exists at least one partition.

Let $X \prec Y$ or $Y \succ X$ mean the same as $X \neq Y$ and $X \supset Y$. Then every partition of an ordered continuum is a partially ordered system with respect to the relation $\prec$.

Lemma 1. *Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C. Then $\mathfrak{P}$ does not contain any infinite (strictly) decreasing sequence $Y_1 \succ Y_2 \succ \ldots \succ Y_n \succ \ldots$ of intervals $Y_n \in \mathfrak{P}$.*

Proof. Suppose, on the contrary, the existence of a partition $\mathfrak{P}$ of an ordered continuum C and the existence of a sequence $Y_1 \succ Y_2 \succ \ldots \succ Y_n \succ \ldots Y_n \in \mathfrak{P}$, $n = 1, 2, \ldots$ in it. Then $\bigcup_1^\infty Y_n$ is an interval in C. Denote by $\mathfrak{P}'$ the subsystem of all intervals $X' \in \mathfrak{P}$, $X' \supset \bigcup Y_n$. Since $C \in \mathfrak{P}'$ we have $\mathfrak{P}' \neq 0$. Now, we shall prove the following statement:

If $X' \in \mathfrak{P}'$ and if $X' = X_1 \cup X_2$ where $X_1 \in \mathfrak{P}$ and $X_2 \in \mathfrak{P}$ then either $X_1 \in \mathfrak{P}'$ or $X_2 \in \mathfrak{P}'$.

Otherwise there would be $\bigcup Y_n \not\subset X_1$ and $\bigcup Y_n \not\subset X_2$; consequently there would exist an integer m such that neither $Y_m \subset X_1$ nor $Y_m \subset X_2$ so that $Y_m \cap X_1$ and $Y_m \cap X_2$ are intervals both different from Y_m. On the other hand $Y_m \neq \bigcup Y_n$ and $Y_m \subset \bigcup Y_n \subset X' = X_1 \cup X_2$ so that either $X_1 \not\subset Y_m$ or $X_2 \not\subset Y_m$. Therefore $Y_m \cap X_1 \neq X_1$ or $Y_m \cap X_2 \neq X_2$. But this contradicts the property 1^0.

If $X \in \mathfrak{P}'$, $Y \in \mathfrak{P}'$ then $\bigcup Y_n \subset X \cap Y$ and according to 1^0 we have $X \cap Y = X$ or $X \cap Y = Y$, in other words $X \subset Y$ or $Y \subset X$. Hence $\mathfrak{P}'$ is a monotone system and from the property 4^0 it follows that $\bigcap \mathfrak{P}' \in \mathfrak{P}'$. According to 3^0 there are two intervals $P_1 \in \mathfrak{P}$ and $P_2 \in \mathfrak{P}$ such that $\bigcap \mathfrak{P}' = P_1 \cup P_2$ and that $P_1 \cap P_2$ contains only one point. As we have just shown either $P_1 \in \mathfrak{P}'$ or $P_2 \in \mathfrak{P}'$ so that $\bigcap \mathfrak{P}' \subset P_1$ or $\bigcap \mathfrak{P}' \subset P_2$; consequently $\bigcap \mathfrak{P}' \neq P_1 \cup P_2$. This is a contradiction.

Lemma 2. *Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C and let $X \in \mathfrak{P}$. Then there exists exactly one couple of intervals $X_1 \in \mathfrak{P}$ and $X_2 \in \mathfrak{P}$ satisfying the condition 3^0.*

Proof. Let $X \in \mathfrak{P}$. Let $X_i \in \mathfrak{P}$, $i = 1, 2, 3, 4$, be intervals such that $X = X_1 \cup X_2 = X_3 \cup X_4$ and that the products $X_1 \cap X_2$ and $X_3 \cap X_4$ contain no more than one point. From 1^0 it follows that either $X_1 = X_3$ and $X_2 = X_4$ or $X_1 = X_4$ and $X_2 = X_3$.

Definitions. Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C. According to the condition 3^0, inside every interval $X \in \mathfrak{P}$ there is a common end-point of two intervals $X_1 \in \mathfrak{P}$ and $X_2 \in \mathfrak{P}$ where $X = X_1 \cup X_2$. According to lemma 2 there is only one point in X like this. It will be called a ***d-point*** of the interval X. The set of all d-points in C will be denoted by $D(\mathfrak{P})$ or simply by D.

Let A, $A \neq C$, be an interval or a point of an ordered continuum C. Let $\mathfrak{P}$ be a dyadic partition of C and let A be no d-point. Denote by $\mathfrak{P}(A)$ the subsystem of all intervals $X \in \mathfrak{P}$ such that $X \supset A$ and $X \neq A$. According to the conditions 1^0 and 2^0 the subsystem $\mathfrak{P}(A)$ is non-void and monotone. Hence, according to lemma 1, the system $\mathfrak{P}(A)$ is a well-ordered system with respect to the order-relation $\prec$. The system $\mathfrak{P}(A)$ will be called the ***chain*** of A. We define the ***order*** of A as an ordinal which equals the order-type of the subsystem $\mathfrak{P}(A)$. For $A = C$ we define the order $\alpha = 0$.

If A is a d-point of an interval $X \in P$ then $A \in X_1 \cap X_2$, where $X_1 \in \mathfrak{P}$, $X_2 \in \mathfrak{P}$ and $X_1 \cup X_2 = X$. Then there are two different subsystems $\mathfrak{P}_l(A) \subset \mathfrak{P}$ and $\mathfrak{P}_r(A) \subset \mathfrak{P}$ the first of which consists of all intervals of $\mathfrak{P}$ containing A inside or as a right end-point and the second consists of all intervals of $\mathfrak{P}$ containing the point A inside or as a left end-point. Both subsystems $\mathfrak{P}_l(A)$ and $\mathfrak{P}_r(A)$ are non-void and well-ordered with respect to the relation $\prec$. The system

[2]) We assume that every ordered continuum contains two different end-points. The degenerate interval which contains only one point or which is an empty set will not be counted as an interval.

[3]) More generally we could define for any natural number $p > 1$ the p-adic partition by postulating 1^0, 2^0, 4^0 and by replacing 3^0 by the following condition:

$3^{0\prime}$ For any interval $X \in \mathfrak{P}$ there are p intervals $X_i \in \mathfrak{P}$ such that $X = \bigcup X$ and that the product $X_i \cap X_{i+1}$, $i = 1, 2, \ldots, p-1$, contains exactly one point.

$\mathfrak{P}_l(A)$ will be called the *left* and $\mathfrak{P}_r(A)$ the ***right chain*** of the point A so that there are two order-types: α of the system $\mathfrak{P}_l(A)$ and β of the system $\mathfrak{P}_r(A)$. The ordinal $\max(\alpha,\beta)$ will be called the ***order of the point*** A.

The order of A will be denoted by $\alpha(\mathfrak{P},A)$ or by $\alpha(A)$ or simply by α.

The least ordinal $\alpha(\mathfrak{P})$ such that there is no interval $X\in\mathfrak{P}$ of order $\alpha(X)>\alpha(\mathfrak{P})$ will be called the ***order of the partition*** $\mathfrak{P}$. From the conditions 3^0 and 4^0 it follows that the order $\alpha(x)$ of any point $x\in C$ and the order $\alpha(\mathfrak{P})$ of any partition of the ordered continuum C are limit ordinals.

Lemma 3. *Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C and let $X_1\in\mathfrak{P}$ and $X_2\in\mathfrak{P}$ be two different intervals of the same order. Then the intervals X_1 and X_2 have at most one point in common.*

Proof. If $X_1\in\mathfrak{P}$, $X_2\in\mathfrak{P}$, $X_1\neq X_2$ and if the product contains at least two different points then according to 1^0 either $X_1\subset X_2$ or $X_2\subset X_1$. Therefore either $X_2\in\mathfrak{P}(X_1)$ or $X_1\in\mathfrak{P}(X_2)$. Consequently $\alpha(X_1)\neq\alpha(X_2)$.

Lemma 4. *Let $\mathfrak{P}$ be a partition of an ordered continuum C. Then there is a one-to-one correspondence between the system $\mathfrak{P}$ and the set $D\subset C$ of all d-points.*

Proof. Let $X\in\mathfrak{P}$ be any interval of $\mathfrak{P}$. Let us attach to X its d-point whose existence is secured by the condition 3^0. According to lemma 2 there is only one d-point of X like this. Therefore we have a mapping f of $\mathfrak{P}$ into the set D. This mapping is one-to-one. Indeed, let $X\in\mathfrak{P}$, $Y\in\mathfrak{P}$, $X\neq Y$. If the product $X\cap Y$ contains at most one point the corresponding d-points of X and Y are different from each other, one of them being inside X and the other inside Y. If $X\cap Y$ is an interval then by 1^0 we have either $X\subset Y$ or $Y\subset X$. Consider the first possibility $X\subset Y$. According to 3^0 there are two intervals $Y_1\in\mathfrak{P}$, $Y_2\in\mathfrak{P}$ such that $Y=Y_1\cup Y_2$ and the product $Y_1\cap Y_2$ contains only one point viz. the d-point of Y. With respect to the property 1^0 we have either $X\subset Y_1$ or $X\subset Y_2$ so that the d-point of Y is not inside X and therefore it is different from the d-point of X which lies inside X. The same holds true in the case for which $Y\subset X$. To complete the proof it is sufficient to remark that for every d-point $x\in C$ there is an interval $X\in\mathfrak{P}$ whose d-point is the point x. Therefore f maps $\mathfrak{P}$ onto D.

Let C be an ordered continuum. Then C is dense in itself. Therefore among subsets which are dense in C there is a subset with the least power. This power will be denoted by $\mathfrak{m}(C)$. In the sequel we shall use the symbol $\mathfrak{m}(C)$ only in the meaning just stated.

Theorem 1. *Let C be an ordered continuum. Then every partition $\mathfrak{P}$ of C has the power $\mathfrak{m}(C)$. The set $D(\mathfrak{P})\subset C$ of all d-points is dense in C and has the power $\mathfrak{m}(C)$ as well.*

Proof. Let $\mathfrak{P}$ be a partition of an ordered continuum C. Suppose on the contrary that the set $D(\mathfrak{P})\subset C$ of all d-points is not a dense subset in C. Then there is an interval $I\subset C$ such that $I\cap D=0$. According to 4^0 we have $I\subset\bigcap\mathfrak{P}(I)\in\mathfrak{P}$ so that according to 3^0 the product $\bigcap\mathfrak{P}(I)=Z_1\cup Z_2$ where $Z_1\in\mathfrak{P}$, $Z_2\in\mathfrak{P}$ and the product $Z_1\cap Z_2$ consists of one d-point only. Since this d-point does not belong to I we have $I\subset Z_1$ or $I\subset Z_2$. Consequently $Z_1\in\mathfrak{P}(I)$ or $Z_2\in\mathfrak{P}(I)$ and $\bigcap\mathfrak{P}(I)\subset Z_1$ or $\bigcap\mathfrak{P}(I)\subset Z_2$. This contradicts the fact that $\bigcap\mathfrak{P}(I)=$ $=Z_1\cup Z_2$. Therefore D is dense in C.

Now, we shall prove that the power of the ordinal $\alpha(\mathfrak{P})$ cannot exceed the power $\mathfrak{m}(C)$ which will be denoted (for the present) by $\aleph_\varrho$. As a matter of fact, if the power of $\alpha(\mathfrak{P})$ were $>\aleph_{\varrho+1}$ there would exist in $\mathfrak{P}$ a chain of power $\aleph_{\varrho+1}$. Consequently there would be a decreasing or an increasing sequence of points in C of power $\aleph_{\varrho+1}>\mathfrak{m}(C)$, which is impossible. If the power of $\alpha(\mathfrak{P})$ were $\aleph_{\varrho+1}$ then we could choose a subset $D'\subset D$ of power $\mathfrak{m}(C)=\aleph_\varrho$ which is dense in C. With respect to lemma 4 there would exist an ordinal α_0 of power $\aleph_\varrho$ such that the d-points of all intervals $X\in\mathfrak{P}$ with $\alpha(X)>\alpha_0$ would belong to the set $D-D'$, the power $\aleph_{\varrho+1}$ with an isolated index being a regular power. Let $Y\in\mathfrak{P}$ and $\alpha(Y)>\alpha_0$. Since the set D' is dense in C there is a point $z'\in D'$ lying inside Y. Then z' is a d-point of a certain interval $Z\in\mathfrak{P}$ and we have $z'\in Z_1\cap Z_2$ where $Z_1\in\mathfrak{P}$, $Z_2\in\mathfrak{P}$, $Z_1\cup Z_2=Z$. As z' is inside Y and inside Z, it follows from condition 1^0 that $Y\subset Z$ or $Z\subset Y$. The first possibility $Y\subset Z=Z_1\cup Z_2$ would imply (according to 1^0) that $Y\subset Z_1$ or $Y\subset Z_2$. This is not possible because z' is an interior point of Y. Therefore $Z\subset Y$. Then $Y\in\mathfrak{P}(Z)$ or $Y=Z$ so that $\alpha_0<\alpha(Y)\leqslant\alpha(Z)$. But in this case $z'\in D-D'$ and we have a contradiction. Thus the power of $\alpha(\mathfrak{P})$ is $\leqslant\mathfrak{m}(C)$.

According to lemma 3 the cardinal number of all intervals $Y\in\mathfrak{P}$ which have the same order $\alpha(Y)=\beta$ is $\leqslant\mathfrak{m}(C)$. As we have just proved the cardinal number of all orders α of all intervals

$X \in \mathfrak{P}$ is likewise $\leqslant \mathfrak{m}(C)$. Therefore the power of the system $\mathfrak{P}$ is $\leqslant \mathfrak{m}(C)$ and according to lemma 4 the power of the set D is also $\leqslant \mathfrak{m}(C)$. On the other hand the power of the set D is $\geqslant \mathfrak{m}(C)$ because D is dense in C. Thus — with respect to lemma 4 — the proof is complete.

Now, let $\mathfrak{P}$ be a partition of an ordered continuum C. The system $\mathfrak{P}$ is partially ordered with respect to the relation $\lesssim$. Another partial order can be introduced into $\mathfrak{P}$ by defining $X \cdot < Y$ for $X \in \mathfrak{P}$, $Y \in \mathfrak{P}$, whenever the product $X \cap Y$ contains at most one point and if there are two points $x \in X$ and $y \in Y$ such that $x < y$ (in C). It is easy to see that for any two intervals $X \in \mathfrak{P}$, $Y \in \mathfrak{P}$ one and only one of the four relations holds: $X \cdot < Y$, $X = Y$, $Y \cdot < X$, $X \| Y$ and that $X \cdot < Y$, $Y \cdot < Z$ implies $X \cdot < Z$.

Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C. Let $\boldsymbol{C}^*$ be the system of all chains of all points $x \in C$. Now, some points $x \in C$ have only one chain, the other points — the d-points — have two different chains. Let $\mathfrak{P}(x_1) \subset \mathfrak{P}$ and $\mathfrak{P}(x_2) \subset \mathfrak{P}$ be two different chains; in the case $x = x_1 = x_2$ that is in the case in which x is a d-point, we shall understand by $\mathfrak{P}(x_1)$ the left and by $\mathfrak{P}(x_2)$ the right chain of the point x. It is evident that neither $\mathfrak{P}(x_1) \subset \mathfrak{P}(x_2)$ nor $\mathfrak{P}(x_2) \subset \mathfrak{P}(x_1)$ holds. Thus there exist two intervals $I_1 \in \mathfrak{P}(x_1) - \mathfrak{P}(x_2)$ and $I_2 \in \mathfrak{P}(x_2) - \mathfrak{P}(x_1)$ which have — by 1^0 — at most one point in common. Therefore we can define the order in $\boldsymbol{C}^*$ in the following manner: $\mathfrak{P}(x_1) < \mathfrak{P}(x_2)$ if $I_1 \cdot < I_2$. Then $\boldsymbol{C}^*$ is an ordered system and $<$ is an order-relation in it. Indeed, we may easily see that for two different chains $\mathfrak{P}(x_1)$ and $\mathfrak{P}(x_2)$ either $\mathfrak{P}(x_1) < \mathfrak{P}(x_2)$ or $\mathfrak{P}(x_2) < \mathfrak{P}(x_1)$ and moreover the rule of transitivity holds true: If $\mathfrak{P}(x_1) < \mathfrak{P}(x_2)$ and $\mathfrak{P}(x_2) < \mathfrak{P}(x_3)$ then $\mathfrak{P}(x_1) < \mathfrak{P}(x_3)$.

Every d-point has two different chains which are the neighbour-elements in $\boldsymbol{C}^*$. If we consider only the right chains $\mathfrak{P}_r(x)$ of d-points $x \in C$ we get an ordered system $\boldsymbol{C}$ without neighbour-elements. This follows from:

Theorem 2. *Let $\mathfrak{P}$ be a dyadic partition of an ordered continuum C. Let $\boldsymbol{C}$ be the system of all chains in $\mathfrak{P}$ of points $x \in C$ from which all left chains have been omitted. Then C is similar to $\boldsymbol{C}$.*

Proof. After omitting all left chains from the system $\boldsymbol{C}^*$ we get a system $\boldsymbol{C}$ such that every point $x \in C$ has only one chain in $\boldsymbol{C}$. There is a one-valued mapping of C onto $\boldsymbol{C}$. If $x_1 < x_2$, $x_1 \in C$, $x_2 \in C$ then there are two different chains $\mathfrak{P}(x_1) \in \boldsymbol{C}$ and $\mathfrak{P}(x_2) \in \boldsymbol{C}$. Therefore there are two intervals $I_1 \in \mathfrak{P}(x_1) - \mathfrak{P}(x_2)$ and $I_2 \in \mathfrak{P}(x_2) - \mathfrak{P}(x_1)$. With respect to 1^0 we have $I_1 \cdot < I_2$ so that $\mathfrak{P}(x_1) < \mathfrak{P}(x_2)$. Thus the mapping is a similarity function.

Theorem 3. *Let C be an ordered continuum. Then there exists at least one partition $\mathfrak{P}$ of C.*

Proof. Let $B \subset C$ be a subset which is dense in C. Let us choose a point $b \in B$ inside C. Then b is an end-point of two intervals $I_0 \cdot < I_1$ such that $I_0 \cup I_1 = C$ and $b \in I_0 \cap I_1$. Having already constructed intervals $I_{i_0 i_1 \dots i_\xi \dots (\xi < \lambda)} \subset C$ for all ordinals $\lambda < \alpha$ whereby $i_\xi = 0$ or $= 1$ we shall define the intervals $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$ in the following way: If α is an isolated ordinal we choose inside every interval $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha - 1)}$ a point of the set B which divides the interval into two closed subintervals $I_{i_0 i_1 \dots i_\xi \dots 0 (\xi < \alpha)} \cdot < I_{i_0 i_1 \dots i_\xi \dots 1 (\xi < \alpha)}$ the first of which is to the left of the other. If α is a limit ordinal we form all products $\bigcap_{\lambda < \alpha} I_{i_0 i_1 \dots i_\xi \dots (\xi < \lambda)}$ where $i_\xi = 0$ or $= 1$ for $\xi < \alpha$. Each such product is either a closed interval or a point; in the former case we shall denote it by $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$. We continue this construction as long as there is the least ordinal ϑ for which there does not exist any interval $I_{i_0 i_1 \dots i_\xi \dots (\xi < \vartheta)}$. Such an ordinal does exist. As a matter of fact the number of all possible intervals of the continuum C equals the number of different couples of points of C whose cardinal number equals the cardinal number of C.

Denote the system of all intervals $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$ where $\alpha < \vartheta$, including[4]) the interval C, by the symbol $\mathfrak{S}$. Our task will be to prove that $\mathfrak{S}$ is a partition of the continuum C.

From the construction it follows at once that the conditions 2^0, 3^0 and 4^0 are satisfied. It remains to prove that the system $\mathfrak{S}$ also satisfies the condition 1^0. Let $X = I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$ and $Y = I_{j_0 j_1 \dots j_\xi \dots (\xi < \beta)}$ be two different intervals of $\mathfrak{S}$. Three cases are possible:

1) $\alpha \leqslant \beta$ and $i_\xi = j_\xi$ for $\xi < \alpha$. Using the method of transfinite induction it is easy to prove that $I_{j_0 j_1 \dots j_\xi \dots (\xi < \beta)} \subset I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$.

2) $\beta \leqslant \alpha$ and $i_\xi = j_\xi$ for $\xi < \beta$. Then according to 1) we get the inclusion $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)} \subset I_{j_0 j_1 \dots j_\xi \dots (\xi < \beta)}$.

3) there exists the least ordinal $\delta < \min(\alpha, \beta)$ such that $i_\xi = j_\xi$ for $\xi < \delta$ whereas $i_\delta \neq j_\delta$. Since $\delta < \alpha$ and $\delta < \beta$ we have $X \subset I_{i_0 i_1 \dots i_\xi \dots (\xi < \delta + 1)} \in \mathfrak{S}$ and $Y \subset I_{j_0 j_1 \dots j_\xi \dots (\xi < \delta + 1)} \in \mathfrak{S}$ and the product $I_{i_0 i_1 \dots i_\xi \dots (\xi < \delta + 1)} \cap I_{j_0 j_1 \dots j_\xi \dots (\xi < \delta + 1)}$ contains only one point so that the

---

[4]) For $\alpha = 0$ we understand by $I_{i_0 i_1 \dots i_\xi \dots (\xi < \alpha)}$ the whole continuum C.

product $X \cap Y$ contains at most one point. Thus the proof is complete.

Theorem 4. *Let C be an ordered continuum. Then there exists* [5] *an ordinal ϑ of power $\leqslant \mathfrak{m}(C)$ such that C is similar to a lexicographically ordered set whose elements are transfinite sequences of zeros and ones of order-types $\leqslant \vartheta$.*

Proof. According to theorem 3 there is a partition $\mathfrak{P}$ of the given ordered continuum C whose elements can be denoted by symbols $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)}$ where $i_\xi = 0$ or $=1$. Let $\boldsymbol{C}^*$ be the system of all chains $\mathfrak{P}(x)$ of all points $x \in C$ and let $\mathfrak{P}(x) \in \boldsymbol{C}^*$ be any chain. Then, using the method of transfinite induction we can easily prove that $(x) = \cap \mathfrak{P}(x) = \bigcap_{\lambda<\alpha} I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\delta)}$ where $\alpha \leqslant \alpha(x)$. Therefore we can attach to every chain $\mathfrak{P}(x) \in \boldsymbol{C}^*$ a transfinite sequence $i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)$ of zeros and ones such that $x \in I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\lambda)} \in \mathfrak{P}(x)$ for every $\lambda < \alpha$. Let us denote the set of all such sequences by $\mathfrak{C}^*$. If $\mathfrak{P}(x) < \mathfrak{P}(y)$ are two different chains and $(i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha))$, $(j_0 j_1 \ldots j_\xi \ldots (\xi < \beta))$ two corresponding transfinite sequences then there exists a least ordinal δ such that $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\delta+1)} \neq I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\delta+1)}$, the first interval belonging to $\mathfrak{P}(x)$ and the second to $\mathfrak{P}(y)$ so that $i_\xi = j_\xi$ for $\xi < \delta$, $i_\delta \neq j_\delta$ and [4] $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\delta)} = I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\delta+1)} \cup I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\delta+1)}$. As $\mathfrak{P}(x) < \mathfrak{P}(y)$ we have $I_{i_0 i_1 \ldots i_\xi \ldots (\xi+\delta<1)} \cdot < I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\delta+1)}$. Therefore $i_\delta = 0$ and $j_\delta = 1$ so that $(i_0 i_1 \ldots i_\xi \ldots (\xi < \delta)) < (j_0 j_1 \ldots j_\xi \ldots (\xi < \beta))$, $<$ denoting here the lexicographical order-relation in the set $\mathfrak{C}^*$. From this it follows that $\boldsymbol{C}^*$ is similar to $\mathfrak{C}^*$. Evidently $(i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)) \in \mathfrak{C}^*$ implies that $\alpha \leqslant \vartheta$, where $\vartheta = \alpha(\mathfrak{P})$. According to Theorem 1 the power of the ordinal ϑ is $\leqslant \mathfrak{m}(C)$.

Now, if we consider the set $\boldsymbol{C} \subset \boldsymbol{C}^*$ of all chains $\mathfrak{P}(x) \subset \mathfrak{P}$ containing no left chain we see that the corresponding set $\mathfrak{C} \subset \mathfrak{C}^*$ of transfinite sequences $(i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)) \in \mathfrak{C}$ is a lexicographically ordered set which is — according to Theorem 2 — similar to C.

The example of the interval $C = \langle 0, 1 \rangle$ of real numbers shows that the supremum $\mathfrak{m}(C)$ of the ordinal ϑ cannot be lessened. On the other hand, in some special cases, the power of ϑ can be $< \mathfrak{m}(C)$. For instance, let us consider the lexicographically ordered continuum $C = \langle 0,1 \rangle \times \langle 0,1 \rangle$. Then there exists a partition $\mathfrak{P}$ whose elements are intervals $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)}$, where $\alpha < \omega 2$ so that $\vartheta = \omega 2$.

[5]) I do not know whether there always exists a partition $\mathfrak{P}$ of order ϑ where ϑ is the first ordinal of power $\mathfrak{m}(C)$.

The power of the system $\mathfrak{P}$ is $2^{\aleph_0}$. From Theorem 1 it follows that $\mathfrak{m}(C) = 2^{\aleph_0}$. Nevertheless the power of ϑ is $\aleph_0$ which is $< \mathfrak{m}(C)$.

Theorem 5. *Every ordered continuum of power $\aleph_\sigma$ contains at least one point with character c_{00} if and only if $\aleph_\sigma < 2^{\aleph_1}$.*

Proof. The condition is necessary. In fact, let $\aleph_\sigma \geqslant 2^{\aleph_1}$ and let $C = P \times Q$ be a lexicographically ordered continuum, where P is the set of all ordinals $\leqslant \omega_\sigma$ and Q is a lexicographically ordered set whose elements are transfinite sequences $i_0 i_1 \ldots i_\xi \ldots (\xi < \omega_1)$ where $i_\xi = 0$ or $=1$ for $\xi < \omega_1$, in which, i. e. in C, the identifications of every two neighbour elements have been made. Since [6]) in Q there is no point with character c_{00} and because there are no countable decreasing or increasing sequences of points in C converging to the end-point $a \in Q$ or $b \in Q$, it is easy to see that in C there is no point with character c_{00}. As the power of Q is $2^{\aleph_1}$ we conclude that the power of C is $\aleph_\sigma$.

The condition is sufficient. As a matter of fact, if we let $\aleph_\sigma < 2^{\aleph_1}$ and suppose that, on the contrary, there exists an ordered continuum C of power $\aleph_\sigma$ containing no point of character c_{00} then according to Theorem 3 there exists a partition $\mathfrak{P}$ of the continuum C whose elements can be denoted by symbols $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)}$ where $i_\xi = 0$ or $=1$.

Let $m_0 m_1 \ldots m_\xi \ldots (\xi < \alpha)$, where $\alpha \leqslant \omega_1$ is a limit ordinal, be a transfinite sequence of natural numbers $m_\xi \geqslant 2$. Let us attach to this sequence a transfinite sequence $i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)$ of zeros and ones in the following manner: For $\xi = \omega\mu + p$, where p is a non-negative integer, we put $i_\xi = 1$ if and only if $\xi = \omega\mu + m_{\omega\mu} + m_{\omega\mu+1} + \ldots + m_{\omega\mu+p'}$ where p' is a suitable non-negative integer. For example if we have the sequence 3222... the corresponding sequence will be 0001010101... For a sequence $m_0 m_1 \ldots m_\xi \ldots (\xi < \alpha)$ there is only one corresponding sequence $i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)$. Otherwise, if there were two different corresponding sequences $i_0 i_1 \ldots i_\xi \ldots (\xi < \alpha)$ and $j_0 j_1 \ldots j_\xi \ldots (\xi < \alpha)$, where $\alpha \leqslant \omega_1$, then there would be the least index $\delta = \omega\nu + q$, $q \geqslant 0$, such that $i_\delta = 0$, $j_\delta = 1$, say. As $j_\delta = 1$ we have $\delta = \omega\nu + m_{\omega\nu} + m_{\omega\nu+1} + \ldots + m_{\omega\nu+q'}$ for a suitable $q' \geqslant 0$ so that $i_\delta = 1$,

[6]) See J. Novák, *On some ordered continua of power $2^{\aleph_0}$ containing a dense subset of power $\aleph_1$*. Časopis pro pěstování matematiky (Чехословацкий математический журнал, т. 1 (76) 1951, стр. 82 (Лемма 2). Czechoslovak Mathematical Journal, Vol. 1 (76) 1951, p. 67 (Lemma 2)).

which would be a contradiction. Therefore we have a single-valued correspondence $f(m_0 m_1 \ldots m_\xi \ldots (\xi<\alpha)) = i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)$.

Now, let $m_0 m_1 \ldots m_\xi \ldots (\xi<\alpha)$ and $n_0 n_1 \ldots n_\xi \ldots (\xi<\beta)$, where $\alpha \leqslant \omega_1$ and $\beta \leqslant \omega_1$ are two limit ordinals, be two different transfinite sequences of natural numbers $m_\xi \geqslant 2$ and $n_\xi \geqslant 2$. Then $f(m_0 m_1 \ldots m_\xi \ldots (\xi<\alpha)) = i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)$ and $f(n_0 n_1 \ldots n_\xi \ldots (\xi<\beta)) = j_0 j_1 \ldots j_\xi \ldots (\xi<\beta)$ are two different corresponding sequences. Indeed, let us consider two cases. In the first case we have $\alpha=\beta$ and there exists the least ordinal $\varrho=\omega\mu+r<\alpha$, $r\geqslant 0$, such that $m_\varrho \neq n_\varrho$ that is $m_\varrho<n_\varrho$ or $m_\varrho>n_\varrho$ whereas $m_\xi=n_\xi$ for $\xi<\varrho$. Consider the first possibility $m_\varrho<n_\varrho$. Denote by $\sigma=\omega\mu+m_{\omega\mu}+m_{\omega\mu+1}+\ldots+m_{\omega\mu+r}$ so that $i_\sigma=1$. As $m_{\omega\mu+i}=n_{\omega\mu+i}$ for $0\leqslant i<r$ whereas $m_{\omega\mu+r}<n_{\omega\mu+r}$ the index σ cannot occur among ordinals of the form: $\omega\nu+n_{\omega\nu}+n_{\omega\nu+1}+\ldots+n_{\omega\nu+p}$ where $p\geqslant 0$. Therefore $j_\sigma=0$ and the sequences $i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)$ and $j_0 j_1 \ldots j_\xi \ldots (\xi<\alpha)$ are different from one another. If we consider the second possibility $m_\varrho>n_\varrho$ we get the same result. In the second case we have $\alpha\neq\beta$ that is either $\alpha<\beta$ or $\alpha>\beta$. Suppose $\alpha<\beta$. Since α and β are limit ordinals, then $j_\delta=1$ for $\delta=\alpha+n_\alpha$ whereas i_δ does not exists at all. We can get the same conclusion for $\alpha>\beta$. Therefore, also in the second case, the sequences are different from each other. Thus we have proved that the correspondence f is one-to-one.

Now, we shall prove the following statement: Let $m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)$ be any transfinite uncountable sequence of natural numbers $m_\xi\geqslant 2$. Let $f(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)) = i_0 i_1 \ldots i_\xi \ldots (\xi<\omega_1)$. Then $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)} \in \mathfrak{P}$ for every ordinal $\alpha<\omega_1$.

Suppose that, on the contrary, there exists the least ordinal $\alpha_0<\omega_1$ such that $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha_0)}$ is no interval belonging to $\mathfrak{P}$ whereas $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha')} \in \mathfrak{P}$ for every $\alpha'<\alpha_0$. With respect to the conditions 2⁰ and 4⁰ the product $\bigcap_{\alpha'<\alpha_0} I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha')}$ contains only one point $x_0 \in C$. From the condition 1⁰ it follows that α_0 is a limit ordinal >0. Thus there exists an ordinal μ such that $\alpha_0=\omega\mu+\omega$ or a limit ordinal ν such that $\alpha_0=\omega\nu$. In the first case when $\alpha_0=\omega\mu+\omega$ we put $\xi_n=\omega\mu+m_{\omega\mu}+m_{\omega\mu+1}+\ldots+m_{\omega\mu+n}$ for $n>0$ so that $i_{\xi_n}=1$. Because, by our hypothesis, $m_\xi\geqslant 2$ for all $\xi<\omega_1$, we have $i_{\xi_n-1}=0$. In the second case when $\alpha_0=\omega\nu$ there is an ordinary increasing sequence $\{\nu_n\}_1^\infty$ of ordinals ν_n converging to ν and we also have $i_{\xi_n}=1$ for $\xi_n=\omega\nu_n+m_{\omega\nu_n}$ and $i_{\xi_n-1}=0$. In both cases we obtain an ordinary increasing sequence $\{\xi_n\}$ of isolated ordinals ξ_n converging to α_0 and such that $i_{\xi_n}=1$ whereas $i_{\xi_n-1}=0$. Then $I_{i_0 i_1 \ldots i_\xi \ldots 0(\xi<\xi_{n+1})} \subset$

$\subset I_{i_0 i_1 \ldots i_\xi \ldots 0(\xi<\xi_n)}$ and $I_{i_0 i_1 \ldots i_\xi \ldots 1(\xi<\xi_{n+1}+1)} \subset I_{i_0 i_1 \ldots i_\xi \ldots 1(\xi<\xi_n+1)}$ Since $\xi_n<\xi_{n+1}$, the four intervals are different from one another. Thus we can conclude that the right end-point of the first interval precedes the right end-point of the second interval and that the left end-point of the fourth interval precedes the left end-point of the third interval. From this it follows the existence of an ordinary decreasing sequence of right end-points of intervals $I_{i_0 i_1 \ldots i_\xi \ldots 0(\xi<\xi_n)}$ right converging to the point x_0 and the existence of an ordinary infinite increasing sequence of left end-points of intervals $I_{i_0 i_1 \ldots i_\xi \ldots 1(\xi<\xi_n+1)}$ left converging to the point x_0. Therefore the character of the point x_0 is c_{00}. This is contrary to our supposition. Consequently $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha_0)} \in \mathfrak{P}$ and our statement is proved.

Let $f(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)) = i_0 i_1 \ldots i_\xi \ldots (\xi<\omega_1)$. From our statement and from the condition 4⁰ it follows that the product $\bigcap_{\alpha<\omega_1} I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)}$ exists and that it is either a single point or an interval. In this product we can choose a point $x=g(i_0 i_1 \ldots i_\xi \ldots (\xi<\omega_1)) = g(f(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)))$. Let $i_0 i_1 \ldots i_\xi \ldots (\xi<\omega_1)$ and $j_0 j_1 \ldots j_\xi \ldots (\xi<\omega_1)$ be two different sequences of zeros and ones corresponding to two sequences $m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)$ and $n_0 n_1 \ldots n_\xi \ldots (\xi<\omega_1)$ and let $\delta=\omega\mu+p$, where p is a non-negative integer, be the least ordinal such that $i_\delta\neq j_\delta$. Consider the case $i_\delta<j_\delta$, that is $i_\delta=0$, $j_\delta=1$. According to Lemma 3 the product $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\delta+1)} \cap I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\delta+1)}$ contains at most one point in common. Now $j_\eta=1$ for $\eta=\omega\mu+n_{\omega\mu}+n_{\omega\mu+1}+\ldots+n_{\omega\mu+p}$ and $\eta>\delta$ so that the left end-point of the interval $I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\delta+1)}$ precedes the left end-point of the interval $I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\eta+1)}$. Therefore $I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\eta+1)} \cap I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\eta+1)} = 0$. Since $\bigcap_{\alpha<\omega_1} I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)} \subset I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\eta+1)}$ and $\bigcap_{\alpha<\omega_1} I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\alpha)} \subset I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\eta+1)}$ we have $\bigcap_{\alpha<\omega_1} I_{i_0 i_1 \ldots i_\xi \ldots (\xi<\alpha)} \cap \bigcap_{\alpha<\omega_1} I_{j_0 j_1 \ldots j_\xi \ldots (\xi<\alpha)} = 0$ so that the points $g(i_0 i_1 \ldots i_\xi \ldots (\xi<\omega_1))$ and $g(j_0 j_1 \ldots j_\xi \ldots (\xi<\omega_1))$ differ from one another. Thus the correspondence g is one-to-one. Since the correspondence f is one-to-one, as well, the composed correspondence $g(f)$ between the transfinite sequences $(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1))$ where m_ξ are $\geqslant 2$, and the chosen points $g(f(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)))$ is one-to-one. From this it follows that the cardinal number of all points $g(f(m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1))) \in C$ is the same as the cardinal number of all sequences $m_0 m_1 \ldots m_\xi \ldots (\xi<\omega_1)$ viz. $\aleph_0^{\aleph_1} = 2^{\aleph_1}$. But then $\aleph_\sigma \geqslant 2^{\aleph_1}$. This is a contradiction.

Now, we can answer the[7] question whether there exists an ordered continuum C of power $2^{\aleph_0}$, where $\mathfrak{m}(C)\leqslant\aleph_1$, without containing points with character c_{00}. The answer to this question is negative. As a matter of fact, suppose on the contrary that C is such an ordered continuum. Let $\mathfrak{P}$ be a partition of C. Since C does not contain any point with character c_{00}, every symbol $I_{i_0 i_1 \dots i_n \dots (n<\omega)}$, where $i_n=0$ and $i_{n'}=1$ for infinitely many naturals n and n', denotes an interval of C belonging to $\mathfrak{P}$. The cardinal number of all intervals like these is $2^{\aleph_0}$. According to lemma 3 every two intervals of this sort have at most one point in common. As $\mathfrak{m}(C)\leqslant\aleph_1$ we have $2^{\aleph_0}=\aleph_1$ and consequently $2^{\aleph_0}<2^{\aleph_1}$. Thus the above supposition contradicts Theorem 5.

Corollary. *Every ordered continuum C with power $2^{\aleph_0}$ and with $\mathfrak{m}(C)=\aleph_1$ contains a subset of power $2^{\aleph_0}$ which is dense in C and whose points have character c_{00}.*

Proof. Every[2]) interval $J\subset C$ has the power $2^{\aleph_0}$. Since $\mathfrak{m}(J)\leqslant\aleph_1$ there is a point in J — as we have just shown — with character c_{00}. Consequently, the subset $A_{00}\subset C$ of all points with character c_{00} is dense in C and the power of A_{00} is $\geqslant\aleph_1$ and $\leqslant 2^{\aleph_0}$ at the same time. Therefore if the power of A_{00} were $<2^{\aleph_0}$ then we should have $\aleph_1<2^{\aleph_0}$. Now, let $\mathfrak{P}$ be a partition of C. The cardinal number of the system of all intervals of $\mathfrak{P}$ of order ω and of all points of C of the same order ω is $2^{\aleph_0}$. According to Lemma 3 and because $\mathfrak{m}(C)=\aleph_1$ the system of all intervals of $\mathfrak{P}$ of order ω has the power $\leqslant\aleph_1$. Therefore the cardinal number of all points in C of order ω is $2^{\aleph_0}$. The power of A_{00} is $\geqslant 2^{\aleph_0}$, every point of order ω belonging to the set A_{00}. Thus we should get a contradiction.

Remarks. I do not know whether or not there exists an ordered continuum of power $2^{\aleph_0}$ with $\mathfrak{m}(C)=\aleph_1$ such that the power of $C-A_{00}$ is $2^{\aleph_0}$.

From Theorem 5 it follows that the more general question whether there exists an ordered continuum of power $2^{\aleph_0}$ without points of character c_{00} is equivalent to the question whether $2^{\aleph_0}=2^{\aleph_1}$.

---

[7]) See J. Novák, l. c. ad [6]), p. 79.

Characterization of Types of Order Satisfying $\alpha_0+\alpha_1=\alpha_1+\alpha_0$.

By

N. Aronszajn (Lawrence, Kansas, U.S.A.).

Introduction. The question of what types of order α_0 and α_1 are commutative with respect to their addition is one of several seemingly simple questions in the theory of ordered sets. A few cases of commutative types of orders have been known for a long time:

(a) $\alpha_0=\delta m$, $\alpha_1=\delta n$, with some type of order δ, m and n being natural numbers,

(b) $\alpha_0=\alpha_1\omega+\delta+\alpha_1\omega^*$ or
$\alpha_1=\alpha_0\omega+\delta+\alpha_0\omega^*$ with some type of order δ.

A. Tarski has communicated to the author that in the middle thirties he proved that (a) and (b) represent all the commutative types if one of them is assumed to be either enumerable or dispersed. He also made a conjecture that these two cases exhaust all possible commutative types, but a counter-example was constructed by A. Lindenbaum. (None of these results have been published).

In the present paper we shall give a complete characterization of all commutative pairs α_0, α_1 of types of order. The characterization is obtained by using the theory of partitions of ordered sets, and also by introducing the new notion of semi-similarity between ordered sets. We are thus able to attack the problem in a much more general form.

We consider a class of types of order $\{\alpha_i\}$, $i\in I$, where I is an arbitrary set of indices. In I we introduce two order relations which make of I two different ordered sets I' and I''. The problem is then one of characterizing all the classes $\{\alpha_i\}$ satisfying the "generalized commutativity" equation

$$\sum_{i\in I'}\alpha_i=\sum_{i\in I''}\alpha_i.$$