pyramid 6(c) has been drilled out. Adjoining 6(c) therefore completes our construction, for it is easily checked that all the identifications described by f have been carried out.

Bibliography.

[1] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig 1934, p. 215.

[2] K. Borsuk, *On the third symmetric potency of the circumference*, Fund. Math. **36** (1949), p. 236.

[3] l. c. pp. 237-243.

The Existence of Pseudoconjugates on Riemann Surfaces.

By

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§ 1. Introduction. Among the characteristics of a function U which is harmonic on a Riemann surface G^* are the topological interrelations of the level lines of U. One has merely to look at the level lines of $\mathcal{R}z$, $\mathcal{R}e^z$, $\mathcal{R}\log z$, etc. to sense both complexity and order. The dual level lines of a conjugate V of U add to this order and complexity. It seems likely that outstanding problems in Riemann surface theory, such as the type problem, the nature of essential singularities, the existence of functions on the Riemann surface with restricted properties cannot be thoroughly understood in the absence of a complete analysis of the topological characteristics of these level lines.

Such a topological study properly belongs to a somewhat larger study namely that of PH (pseudoharmonic) functions and their pseudoconjugates (defined in § 2). A first problem is that of the existence of PH functions U on G^* with prescribed level sets locally topologically like families of parallel straight lines except in the neighbourhood of points of a discrete set ω of points z_0. At a point z_0 the level curves of U may cross after the manner of the level curves of a harmonic function with a critical point at z_0.

Let E be the finite z-plane. In case $G^*=E$ and $\omega=0$, and excluding all recurrent level curves other than periodic curves, Kaplan has solved the above problem in [5]. More recently Boothby has extended Kaplan's result to the case of a general ω. As explained in [4] the *a priori* exclusion of recurrent level curves other than periodic curves does not seem justified. The writers of this paper have accordingly established the existence of PH functions without this hypothesis of non-recurrence [4].

The present paper gives a solution of a second major problem in the topological theory, namely the existence of a function V, PC (pseudoconjugate) to a given PH function U. When $\omega=0$ Kaplan has affirmed that the set of level curves of U are topologically equivalent to the trajectories of a differential system of the form

$$\frac{dx}{dt}=p(x,y), \quad \frac{dy}{dt}=q(x,y) \qquad (p^2+q^2\neq 0),$$

where p and q are of class C' over E. The method of proof of this theorem as outlined by Kaplan in a few paragraphs in [6] does not seem to form an adequate basis for proving the existence of V.

We here give an explicit topological proof of the existence of a V, PC to U. The key to this proof lies in the proper use of the theory of μ-length as developed for arbitrary curves in a general metric space [8, § 27].

The results and methods of this paper will lead in a subsequent paper to theorems on the existence of PH functions and their pseudoconjugates on an arbitrary open Riemann surface.

The reader familiar with the geometric theory of dynamical systems as initiated by Poincaré [9] with its attention to singularities, periodicity, recurrence, etc., will recognize that the underlying topological theory of the level lines of a PH function is a form of 2-dimensional topological dynamics.

§ 2. Review of earlier theorems. Let S be the complex z-sphere with Z the point $z=\infty$. Let ω be a set of isolated points in $G^*=S-Z$ with no point other than Z as limit point on S.

The family F, the sets G^* and G. An arc, open arc, or topological circle in G^* is for us the $1-1$ continuous image in G^* of a closed interval, open interval, or circle respectively. These elements are to be distinguished from parameterized curves (p-curves) which are mappings, and not sets. Let F be a family of elements α, β, γ etc., which are open arcs or topological circles in $G=G^*-\omega$ and which include one and only one α meeting each point $p\in G$.

The family F shall have the following local property. Let D be the open disc ($|w|<1$) in the complex w-plane. With each point $p\in S-Z$ there shall be associated an "F-neighbourhood" X_p of p with $\overline{X}_p\subset G\cup p$, and a homeomorphic mapping T_p of $\overline{X}_p$ onto $\overline{D}$ under which p goes into $w=0$ and the maximal open arcs of $F|X_p$ go into the maximal open level arcs of $\mathcal{R}w^n$ in D when $n=1$, and in D with the origin deleted when $n>1$. We term n the *exponent* of p. The exponent of p shall be 1 for $p\in G$, and exceed 1 for $p\in\omega$. Points $p\in\omega$ are termed *F-singular points*. A value $w\in D$ is termed the *canonical parameter* of its antecedent in X_p.

The open arcs of $F|X_p$ which have limiting end points at $p\in\omega$ are termed *F-rays* of X_p incident with p. They are $2n$ in number, thereby showing that n is independent of the choice of X_p. These F-rays of X_p divide X_p-p into $2n$ open regions termed *F-sectors* incident with p.

Right N. With each $p\in G$ one can also associate a neighbourhood N_p of p with $\overline{N}_p\subset G$, and a *homeomorphic sense-preserving mapping* of $\overline{N}_p$ onto a square K: $(-1\leqslant u\leqslant 1)$ $(-1\leqslant v\leqslant 1)$ such that p goes into the origin in K and the maximal subarcs of $F|\overline{N}_p$ go into arcs $u=c$; $-1\leqslant v\leqslant 1$, where the constant c ranges over the interval $[-1,1]$. We refer to N_p as a right N of p and term u and v *canonical coordinates* of the antecedent in $\overline{N}_p$ of (u,v) in K.

Transversals. By a *transversal* λ is meant an open arc in G whose intersection with any right N with canonical coordinates (u,v) has locally the form $v=\varphi(u)$ where φ is single-valued and continuous. By the *principal transversal* of a right N is meant the open arc in N on which $v=0$, $-1<u<1$.

F-vectors. Any sensed subarc A of an $\alpha\in F$ will be called an *F-vector*. By definition a F-vector is simple, closed, and never a topological circle. The following lemma is established in [4, § 3].

Lemma 2.1. Each F-vector is in some right N.

Coherent sensing. Let each $\alpha\in F$ be given a sense. The resulting family F^s of sensed α will be called a *sensed image* of F. We shall refer to a continuous deformation Δ of an F-vector A in the space of F-vectors metricized by means of the Fréchet distance between any two sensed arcs. We shall understand that each image of A under Δ is sensed by Δ, that is that the sense of the image arc shall be determined by the images under Δ of the initial and final points of A. We say that F^s is *coherently sensed* if any continuous deformation Δ of an F-vector A, initially sensed by F^s, through F-vectors sensed by Δ, is necessarily a deformation through F-vectors sensed by F^s. In the case at hand, where $G^*=S-Z$ is simply connected, there are two distinct coherently sensed images F^s of F [4, § 5].

The following theorem came next in our development:

Theorem 2.1. *When $G^*=S-\omega$ each $a \in F$ is the homeomorphic image in G of an open interval with limiting end points on S, distinct unless both are coincident with Z. Each finite end point is a singular point of F.*

The family F^*. It is shown in [4, § 7] that when $G^*=S-Z$ there are no topological circles in G^* formed from the union of the closures of a finite set of $a \in F$. In this case we define F^* as the set of all open arcs h, k, m, etc. in G^* with the following properties. If p is an F-non-singular point of h, h shall contain the $a_p \in F$ meeting p, and any finite limiting end point or end points of a_p; if an F-singular point q is in h, h shall contain just two of the $a \in F$ with q as a limiting end point. An $h \in F^*$ may be identical with an $a \in F$, or it may be formed from a sequence of $a \in F$ of one of the forms

$$\ldots a_{-2}\, a_{-1}\, a_0\, a_1\, a_2 \ldots$$
$$\ldots a_{-2}\, a_{-1}\, a_0;\;\; a_0\, a_1\, a_2 \ldots$$
$$a_0\, a_1 \ldots a_n.$$

The following theorem bears on the behaviour of subarcs of $h \in F^*$ [4, Cor. 7.5].

Theorem 2.2. *When $G^*=S-Z$ an arc g in $S-Z$ which is the closure of a finite sequence of $a \in F$ intersects the closure of an F-neighbourhood X_p or of a right N, if at all, in a single arc.*

Corollary 2.2. *When $G^*=S-Z$ each $h \in F^*$ has Z as a limiting end point in both senses.*

Bands $R(N)$. Given a right N the union of the sets $a \in F$ which intersect N will be called the *band* $R(N)$. In accordance with Lemma 8.1 [4] each band $R(N)$ is simply connected when $G^*=S-Z$, and in accordance with Theorem 8.1 [4] the boundary $\beta R(N)$ of $R(N)$ in S is the union of Z and at most a countable set of non-intersecting open arcs $h \in F^*$ of which at most a finite number have diameters on S which exceed a finite constant d. An $h \in F^*$ in $\beta R(N)$ is either *concave towards* $R(N)$ in that $R(N)$ contains just one F-sector incident with each singular point of F in h, or *semi-concave towards* $R(N)$ in that $R(N)$ includes just one F-sector incident with each singular point of F in h except for one singular point q of F in h; corresponding to q, $R(N)$ contains just two F-sectors incident with q and the F-ray incident with q between the two F-sectors incident with q. An $h \in F^* | \beta R(N)$ which meets βN is always concave towards $R(N)$ (Cf. Theorem 8.1 [4]).

A singular point p of F in $\beta R(N)$ is termed of *type* 1 or *type* 2 relative to $R(N)$ if $R(N)$ contains one or two F-sectors respectively incident with p.

An entrance to $R(N)$. An open arc η in $\beta R(N)$ will be called an *entrance to* $R(N)$ if η is the union of a finite or countably infinite set of $a \in F$ each joined to its successor or predecessor at a singular point in F, and if η intersects βN in an open arc. An entrance η to $R(N)$ is an open subarc of an $h \in F^*$ in $\beta R(N)$ such that h intersects βN and is accordingly concave towards $R(N)$ [4, Th. 8.1].

We shall make use of the following decomposition of G^* [4, Th. 9.1].

Theorem 2.3. *When $G^*=S-Z$ there exists a sequence of non-intersecting bands $R(N_r)$, $r=1,2,\ldots$, and for each $R(N_r)$ with $r>1$ an open entrance η_r to $R(N_r)$, such that the η_r do not intersect each other or any of the bands, and such that $G^*=\text{Union } \Sigma_n$, $n=1,2,\ldots$, where $\Sigma_1=R(N_1)$ and for $n=2,3,\ldots$*

$$\Sigma_n=\Sigma_{n-1}\cup R(N_n)\cup \eta_n; \quad \bar{\Sigma}_{n-1}\cap \bar{R}(N_n)=\bar{\eta}_n\cup Z \tag{2.1}$$

and where the N_r are so chosen that any compact subset of G^* is included in Σ_n for n sufficiently large.

We note that each Σ_n is open and simply connected. If Σ_n contains a point $p \in G$, it contains the $a \in F$ which meets p.

Let U_0 be harmonic in a neighbourhood H of $q \in G^*$ and not identically constant. Let Ψ be a topological mapping of a neighbourhood H_0 of q onto H. Then $U_0\Psi$ is termed *PH (pseudoharmonic)* at q. A function U is termed *PH over a region* $R\subset G^*$ if U is PH at each point of R.

Let φ be analytic in a neighbourhood H of $q \in G^*$, and not identically constant, and let Ψ be a sense-preserving topological mapping of a neighbourhood H_0 of q onto H. Then $\varphi\Psi$ is termed *interior at* q. A function f is termed *interior over* a region $R\subset G^*$ if f is interior at each point of R.

If U is PH over a region R and V real, and if $U+iV$ is interior over R then V is said to be *PC (pseudoconjugate) to* U *over* R.

§ 3. Bands $R(N)$ conformally represented. Let U be given as PH over $R(N)$ with the $a \in F$ as level lines. It is relatively easy to use signed μ-length along the $a \in F | R(N)$ measured algebraically from the principal transversal λ of N to obtain a function V PC to U. To extend the definition of V to $\beta R(N)$ is more difficult because $\beta R(N)$ is not readily given as a p-curve. The arcs of $\beta R(N)$ meeting Z require a definite parameterization which is related to the parameterization of the $a \in F | R(N)$. To this end we find it useful to map a unit disc $D | (|z|<1)$ directly conformally onto $R(N)$, and then by continuously extending the mapping of D over $\bar{D}$ to obtain the desired parameterization of arcs in $\beta R(N)$.

Canonical coordinates (u,v) in a right N. Let (u,v) be canonical coordinates in a right N. Set $u+iv=w$. To each point $z \in \bar{N}$ corresponds a complex canonical parameter $w=\Psi(z)$ given with N. The mapping Ψ is given as 1—1 continuous and sense-preserving in N. The value of U at the point $w \in \bar{N}$ reduces to a strictly monotone function of u.

Convention. *In this paper we shall choose the canonical coordinate u in each right N so that $U(u)$ is strictly increasing. Let F^s be a coherently sensed image of F so chosen that in each right N the canonical coordinate v increases in the positive sense of $F^s | \bar{N}$ along each level arc of U.*

This relation of F^s and U to the canonical coordinates (u,v) in one right N implies a similar relation to the canonical coordinates in an arbitrary right N. This follows from the orientability of S and the fact that F^s is a coherently sensed image of F. In this connection recall that canonical coordinates (u,v) in a right N are to be chosen so that the mapping $u+iv=\varphi(z)$ which carries $z \in N$ into the canonical point (u,v) is an interior transformation.

The open arc β_u. Let λ be the principal transversal of N. Let the point on $\bar{\lambda}$ with canonical coordinates $(u,0)$ in $\bar{N}$ be denoted by $\lambda(u)$. The coordinate u ranges on an interval $I=[-1,1]$. Let $\beta_u \in F^s | (u \in I)$ meet $\bar{\lambda}$ in the point $\lambda(u)$. In particular β_1 and β_{-1} are in F^s and on $\beta R(N)$.

The conformal mapping f. Let f be a direct conformal mapping of the open unit disc D onto $R(N)$. An arc $h \in F^*$ in $\beta R(N)$ is a *free boundary arc* of $R(N)$ in the sense that it is an open arc which does not meet the closure of $\beta R(N)-h$. The mapping f can be extended to a homeomorphism which maps an open arc h^D on βD onto h [2, p. 86]. If f is further extended so as to map each point in the set

$$Q=\beta D - \text{Union } h^D \qquad (h \in F^* | \beta R(N)) \tag{3.0}$$

into Z, then the extended f maps $\bar{D}$ continuously into S.

Indeed if $\{z_n\}$ is a sequence of points in $\bar{D}$ tending to a point in Q, the image sequence $\{f(z_n)\}$ can have no accumulation point in $R(N)$ or on an arc $h \in F^*$ in $\beta R(N)$. Thus $f(z_n) \to Z$ as $n \uparrow \infty$ and f maps $\bar{D}$ continuously into S.

Without loss of generality we may assume that f carries $z=+1$ and $z=-1$ in βD into $\lambda(1)$ and $\lambda(-1)$ in $\beta R(N)$ respectively.

If Y is any set in $\bar{R}(N)$ we will denote $f^{-1}(Y)$ by Y^D. The sensed antecedents β_1^D and β_{-1}^D in βD of β_1 and β_{-1} respectively in $\beta R(N)$ are open arcs in βD which appear in βD in counter-clockwise and clockwise sense respectively, with $z=1$ an inner point of β_1^D and $z=-1$ an inner point of β_{-1}^D.

Let (a) be the set of $a \in F | \beta R(N)$, and (a^D) the set of the respective antecedents of $a \in (a)$.

Let $d(a)$ and $d(a^D)$ be respectively the diameter in S of $a \in (a)$ and of a^D in (a^D). As stated in § 2 there are at most a finite number of the $a \in (a)$ with $d(a)>1/n$. The same is true of the $a^D \in (a^D)$ since $\Sigma d(a^D) \leqslant 2\pi$, the length of βD. It follows that $d(a^D) \to 0$ if $d(a) \to 0$, and conversely that $d(a) \to 0$ if $d(a^D) \to 0$.

The antecedent Z^D. This antecedent of $Z \in S$ is closed. It is nowhere dense in βD since it can contain no subarc of βD.

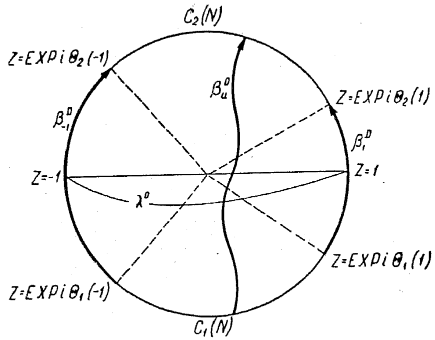
The antecedent β_u^D. Since β_{-1} and β_1 are in $\beta R(N)$, β_{-1}^D and β_1^D are in βD. For $-1<u<1$, β_u^D is in D since β_u is in $R(N)$. The point $(r,\Theta)=(1,0)$ in βD is in β_1^D. We shall restrict $\Theta = \text{arc}\, z | (z \in \bar{D})$ to values on the interval $0 \leqslant \Theta < 2\pi$.

The end points $\text{Exp}\,(i\Theta_1(u))$ and $\text{Exp}\,(i\Theta_2(u))$ of β_u^D. For $z \in \beta_u^D | (u \in I)$, $\Theta = \text{arc}\, z$ has limiting initial and final values which we denote by $\Theta_1(u)$ and $\Theta_2(u)$ respectively. That $\Theta_1(u)$ and $\Theta_2(u)$ exist is immediate when $u=\pm 1$. If β_u^D $(-1<u<1)$ did not have unique limiting end points in βD, $\bar{\beta}_u^D - \beta_u^D$ would include some subarc of an $a^D \in (a^D)$ since Z^D is nowhere dense in βD. Since f is continuous over $\bar{D}$, $\bar{\beta}_u - \beta_u$ would then include some subarc of an $a \in (a)$ contrary to the fact that $\bar{\beta}_u - \beta_u$ contains at most two points of S. Hence the limits $\Theta_1(u)$ and $\Theta_2(u)$ exist and are unique. The limiting end points of β_u^D are then $\text{Exp}\,(i\Theta_1(u))$ and $\text{Exp}\,(i\Theta_2(u))$.

Lemma 3.1. *The functions Θ_1 and Θ_2 are monotone on the interval I: $-1\leqslant u\leqslant 1$, with Θ_1 increasing, Θ_2 decreasing, and*

$$0<\Theta_2(u)<\pi, \quad \pi<\Theta_1(u)<2\pi. \tag{3.1}$$

The principal transversal λ of N is in $R(N)$ and has an antecedent λ^D in D which is a cross cut of D with initial and final end points $z=-1$ and $z=1$ in βD. Each $\beta_u^D|(u\in I)$ is divided by λ^D



into a half open arc with initial point $\lambda^D(u)$ in λ^D and final point $\operatorname{Exp}(i\Theta_2(u))$ in βD, and a half open arc with final point $\lambda^D(u)$ in λ^D and initial point $\operatorname{Exp}(i\Theta_1(u))$ in βD. The relations (3.1) clearly hold for $u=\pm 1$ and hence hold for each $u\in I$.

For $-1<u<1$, β_u is in $R(N)$ and has limiting end points on $\beta R(N)$. Since $R(N)$ is simply connected β_u separates $R(N)$. In particular if $u'<u''<u'''$ in I, $\beta_{u''}$ separates $\beta_{u'}$ and $\beta_{u'''}$ in $R(N)$, for otherwise $\lambda(u')$ and $\lambda(u''')$ would be connected on $R(N)-\beta_{u''}$ and $\beta_{u''}$ would not separate $R(N)$. The monotonicity of Θ_1 and Θ_2 follows. It is clear moreover that

$$0<\Theta_2(1)\leqslant\Theta_2(-1)<\pi<\Theta_1(-1)\leqslant\Theta(1)<2\pi$$

from which we infer that Θ_1 is increasing and Θ_2 decreasing.

Let $C_1(N)$ and $C_2(N)$ be subarcs of βD conditioned as follows:

$$C_2(N):\quad \Theta_2(-1)\geqslant\Theta\geqslant\Theta_2(1), \tag{3.2$'$}$$

$$C_1(N):\quad \Theta_1(1)\geqslant\Theta\geqslant\Theta_1(-1). \tag{3.2$''$}$$

The arc $C_2(N)$ bears all the final end points of the $\beta_u^D|(u\in I)$, and $C_1(N)$ all the initial end points. Then $C_2(N)$ and $C_1(N)$ clearly contain each $a^D\in(a^D)$ except β_1^D and β_{-1}^D. We shall be more explicit in the case of $C_2(N)$.

Lemma 3.2. *Corresponding to each $a\in F$ with $a^D\in C_2(N)$ there is a unique value $u\in I$ such that a^D lies in an arc on βD of one of the forms*

$$\Theta_2(u-)>\Theta>\Theta_2(u), \tag{3.3}$$

$$\Theta_2(u)>\Theta>\Theta_2(u+). \tag{3.4}$$

Let J be the interval of values of Θ such that $\operatorname{Exp}(i\Theta)$ is in a^D. Then J is open and contains no value $\Theta_2(s)|(s\in I)$; for the final end point $\operatorname{Exp}(i\Theta_2(s))$ of β_s^D has an f-image which is either an F-singular point or Z, and accordingly not in a. Hence J must be in an interval of the type (3.3) or (3.4). Since no two intervals of type (3.3) or (3.4) intersect, the interval (3.3) or (3.4) containing J must be unique.

The p-curve B_u in S. Let the half open arc of β_u with initial point $\lambda(u)$ be closed by its final end point in S, and be parameterized by μ-length in S to define a p-curve B_u.

If g is a p-curve it is convenient to introduce the carrier $|g|$ of g defined as the union of the points in the range of g.

The extensions B_{u+} and B_{u-} of B_u in S. Let B_u^D be extended as a p-curve by the arc of points $\operatorname{Exp}(i\Theta)$ of βD on which

$$\Theta_2(u)\geqslant\Theta\geqslant\Theta_2(u+) \qquad (-1\leqslant u<1) \tag{3.5}$$

taken in the sense of decreasing Θ and admitting the possibility that (3.5) may reduce to a point and B_u^D coincide with its extension. Let B_{u+}^D be the resultant extension of B_u^D as a p-curve, and $B_{u+}=f(B_{u+}^D)$ its p-curve image in $\bar{R}(N)$. Recall that a p-curve is not a set, and that $|B_{u+}^D|$ is in general not the full antecedent of the set $|B_{u+}|$, since Z is in $|B_{u+}|$ (see Lemma 3.3) and $|B_{u+}^D|$ will not in general contain the whole set Z^D. We shall suppose that B_{u+} is parameterized by μ-length measured along B_{u+} from $\lambda(u)$, and note that B_{u+} is a p-curve in $\bar{R}(N)$ extending B_u.

Let B_u^D be similarly extended in $\bar{D}$ as a p-curve by adding the arc of βD (if any exists) on which

$$\Theta_2(u-)\geqslant\Theta\geqslant\Theta_2(u) \qquad (-1<u\leqslant 1) \tag{3.6}$$

taken in the sense of increasing Θ. Let B^D_{u-} be this extension of B^D_u. Set $B_{u-}=f(B^D_{u-})$, parameterizing B_{u-} by means of μ-length in S to form a p-curve B_{u-} extending B_u.

We understand that $\Theta_2(u+)$ and B_{u+} are defined for $-1\leqslant u<1$, and that $\Theta_2(u-)$ and B_{u-} are defined for $-1<u\leqslant 1$.

We shall need the following lemma.

Lemma 3.3. *When $G^*=S-Z$ the terminal point of B_{u+} or B_{u-} is Z.*

The f-image of the point $\operatorname{Exp}[i\Theta_2(u)]$ in βD is either Z or a singular point of F. Since the singular points of F have Z as their sole limit point in S the f-image of $\operatorname{Exp}(i\Theta_2(u+))$ or $\operatorname{Exp}(i\Theta_2(u-))$ must be Z. But these points are respectively the terminal points of B_{u+} and B_{u-} so that the lemma follows.

Note that the relation $\Theta_2(u)=\Theta_2(u+)$ implies $B_u=B_{u+}$ and the relation $\Theta_2(u)=\Theta_2(u-)$ implies that $B_u=B_{u-}$. In either case the terminal point of B_u is Z by virtue of Lemma 3.3.

§ 4. B_{u+} and B_{u-} as Fréchet limits. We treat the case of B_{u+}. The case of B_{u-} is similar.

Lemma 4.1. *Given $e>0$ and u in the interval $-1\leqslant u<1$, an e-neighbourhood of $|B^D_{u+}|$ contains $|B^D_t|$ provided $t>u$, and $t-u$ is sufficiently small.*

(a) If the lemma were false there would exist a sequence of values t_n tending to u from above as $n\uparrow\infty$, and points $z_n\in|B^D_{t_n}|$ such that $z_n\to z_0$ not in $|B^D_{u+}|$ as $n\uparrow\infty$. Two cases are distinguished.

Case (i). $z_0\in D$. Set $p_n=f(z_n)$, $n=0,1,\ldots$, where f is the conformal mapping defined in § 3. For a suitable $s\in(-1,1)$, $\beta_s\in F|R(N)$ meets p_0. The point $p_0\in\beta_s$ is a limit point of points p_n only if $t_n\to s$. Hence $s=u$. But p_0 is not in $|B_u|\subset|B_{u+}|$ by hypothesis (a). Hence p_0 is in $\beta_u-|B_u|$. Hence p_0 is not a limit point of points in $|B_{t_n}|$ as $t_n\to u$, so that z_0 is not a limit point of the sequence $z_1,z_2,\ldots$ We infer that case (i) is impossible.

Case (ii). $z_0\in\beta D$. We shall show here that z_0 must lie in $|B^D_{u+}|$ contrary to hypothesis (a).

For $u<t<1$ let λ_t be the subarc of λ on which $z=\lambda(s)$ $(u\leqslant s\leqslant t)$. Let η_t be the arc or point of βD on which $\Theta_2(u+)\geqslant\Theta\geqslant\Theta_2(t)$. Let C_t be the topological circle on $\bar D$ formed by the circular sequence of the four arcs (with η_t possibly a point)

$$|B^D_{u+}|,\quad \eta_t,\quad |B^D_t|,\quad \lambda_t.$$

Let X_t be the closure of the Jordan region bounded by C_t. For $u<s<t$, B^D_s is in X_t. As $t\downarrow u$, X_t contracts on itself as a point set, and $C_t\cap\beta D$ is an arc of $X_t\cap\beta D$ which contracts on itself to $|B^D_{u+}|\cap\beta D$ as a limiting arc or point. The point z_0 is in X_t for each $t>u$. In case (ii) z_0 must then be in $C_t\cap\beta D$ for each $t>u$ and hence in $|B^D_{u+}|\cap\beta D$. From this contradiction to (a) we infer the truth of the lemma.

A right neighbourhood relative to $\bar R(N)$. Let g be an arc in $\beta R(N)-Z$ whose end points are F-non-singular and whose F-singular points are finite in number and of type 1 relative to $R(N)$.

Lemma 4.2. *There exists a neighbourhood M of g relative to $\bar R(N)$ such that $\bar M$ is the homeomorph of a square* $(0\leqslant u\leqslant 1)$ $(0\leqslant v\leqslant 1)$ *in which the open edge $u=0$, $0<v<1$, contains the image of g and each arc $u=\text{const}\neq 0$, $0\leqslant v\leqslant 1$, corresponds to an F-vector in $\bar R(N)$.*

The lemma is true if there are no singular points of F on g, by virtue of Lemma 2.1. It is also clearly true if g is the closure of two F-rays on the boundary of an F-sector. The lemma is readily established in the general case following the pattern of the proof of Lemma 3.1 of [4].

The neighbourhood of g in Lemma 4.2 is termed a *right neighbourhood of g relative to $\bar R(N)$*.

In the proof of Theorem 4.1 use will be made of various distance functions. If A and B are non-empty subsets of S let $d(A,B)$ denote the inferior limit of the distances $d(p,q)$ between $p\in A$ and $q\in B$. Let

$$\delta(A,B)=\sup_{p\in A} d(p,B).$$

If A, B, C are subsets of S,

$$\delta(A,C)\leqslant\delta(A,B)+\delta(B,C). \tag{4.1}$$

If $B=B_1\cup B_2$,

$$\delta(A,B)\geqslant\min[\delta(A,B_1),\ d(A,B_2)]. \tag{4.2}$$

If g and h are p-curves in S, $\varrho(g,h)$ shall denote the *Fréchet distance* between g and h.

Theorem 4.1 is basic in the proof of the existence of a V pseudoconjugate to U. We prepare for its proof.

An e-decomposition of B_{u+}. We suppose $u\in[-1,1)$. Given $e>0$ the number of open arcs $\gamma\in F^*$ in B_{u+} such that $\delta(\gamma,Z)\geqslant e/4$ is finite (see § 3). Let $\gamma_1,\ldots,\gamma_m$ be these open arcs ordered as they appear in the p-curve B_{u+}. No one of the γ_i, $i=1,\dots,m$ intersects B_u. Let γ_0 be the maximal initial half open arc of B_{u+} which does not meet Z. The initial point of γ_0 is $\lambda(u)$ and $\bar{\gamma}_0 \supset |B_u|$. Let

(4.3) $$g_0, g_1, \dots, g_m$$

be subarcs of $\gamma_0, \gamma_1, \dots, \gamma_m$ respectively, whose end points are F-nonsingular and are such that the initial point of g_0 is $\lambda(u)$ and

$$\delta(\gamma_r - g_r, Z) < e/4 \qquad (r=0,\dots,m).$$

Parameterizing g_r as in B_{u+} it appears that B_{u+} is a sequence of sub-p-curves

(4.4) $$g_0, k_0, g_1, k_1, \dots, g_m, k_m.$$

No p-curve g_r intersects any other p-curve g_i or any p-curve k_i.

A right neighbourhood M_r of g_r. Recall that $-1 \leqslant u < 1$. The set $N - \beta_u$ is the union of two open sets M^u and N^u of which $\overline{N^u}$ contains $\lambda(1)$. Then $R(N^u) \subset R(N)$. Moreover $\gamma_1, \dots, \gamma_m$ are in $\beta R(N) \cap \beta R(N^u)$ while γ_0 is in $\beta R(N^u)$. Each F-singular point of γ_i, $i=1,\dots,m$, is of type 1 relative to $R(N)$ and $R(N^u)$, by virtue of the definition of the extension B_{u+} of B_u and nature of Θ_2. Each F-singular point of γ_0 is of type 1 relative to $R(N^u)$ by virtue of Theorem 8.1 (b) [4]. By Lemma 4.2 then, the respective g_r in (4.4) have right neighbourhoods

(4.5) $$M_0, M_1, \dots, M_m,$$

relative to $\bar{R}(N^u)$. Each M_r in (4.5) will be so restricted that each maximal sub-p-curve of B_t in $\bar{M}_r$ (when $B_t \cap M_r \neq 0$) is at a Fréchet distance less than e from g_r (sensed as in F^s), and that $\bar{M}_r \cap \bar{M}_i = 0$ for $r \neq i$.

Lemma 4.3. *Corresponding to the given $u \in [-1,1)$ and $e>0$ a constant t_1 $(1>t_1>u)$ for which t_1-u is sufficiently small has the following property. Each simple p-curve B_t for which $u<t<t_1$ admits a decomposition*

(4.6) $$g_0^t,\ k_0^t,\ g_1^t,\ k_1^t, \dots, g_m^t,\ k_m^t$$

into successive non-intersecting p-curves such that $\varrho(g_r^t, g_r) < e$, and, setting

(4.7) $$K^t = \text{Union}\ |k_r^t|, \quad K = \text{Union}\ |k_r| \qquad (r=0,\dots,m);$$
$$\delta(K^t, K) < e/4.$$

We shall refer to $\bar{D}$ and make use of the fact that B_{u+}^D is simple. Set

$$\operatorname{Exp}[i\Theta_2(u+)] = q.$$

The point q is the final end point of B_{u+}^D. As $t \downarrow u$ the final end point $\operatorname{Exp}[i\Theta_2(t)]$ of B_t^D tends to q while the initial end point $\lambda^D(t)$ of B_t^D tends to the initial end point $\lambda^D(u)$ of B_{u+}^D; at the same time

$$\delta(|B_t^D|, |B_{u+}^D|) \to 0$$

in accordance with Lemma 4.2. The simple arc B_{u+}^D meets the disjoint sets

(4.8) $$M_0^D, M_1^D, \dots, M_m^D, q$$

in the order written. It follows from Lemma 4.2 that if t_1-u is sufficiently small and if $u<t<t_1<1$, B_t^D will meet each of the sets $\bar{M}_r^D$ in (4.8) in at least one maximal p-curve $(g_r^t)^D$ and by virtue of Theorem 2.2 in at most one p-curve, there will be a terminal p-curve $(k_m^t)^D$ of B_t^D which intersects no $\bar{M}_r^D$, and the initial point $\lambda^D(t)$ of B_t^D will be in M_0^D. We suppose t_1 and t so conditioned.

It follows from this choice of g_r^t and M_r that $\varrho(g_r^t, g_r) < e$. The residual p-curves k_r^t in (4.6) are determined. Set $\underset{r}{\text{Union}}\ |g_r| = \Gamma$, $r=0,1,\dots,m$. In the relation

(4.9) $$\delta(K^t, K \cup \Gamma) \geqslant \min[\delta(K^t, K), d(K^t, \Gamma)] \qquad \text{[Cf. (4.2)]}$$

the left member tends to 0 as $t \downarrow u$ by virtue of Lemma 4.2, while $d(K^t, \Gamma)$ remains bounded from 0. Hence for t_1-u sufficiently small and $u<t<t_1$

(4.10) $$\delta(K^t, K) \leqslant \delta(K^t, K \cup \Gamma).$$

Relation (4.7) then follows for suitable choice of t_1.

Theorem 4.1. *If $t \downarrow u \in [-1,1)$ then B_t tends to B_{u+} in the sense of Fréchet.*

Corresponding to $e>0$ let B_{u+} be given the decomposition (4.4), where $\delta(K, Z) < e/4$. If t_1 is chosen as in Lemma 4.3 and if $u<t<t_1$, then $\delta(K^t, K) < e/4$. Moreover by (4.1)

$$\delta(k_r^t, Z) \leqslant \delta(K^t, Z) \leqslant \delta(K^t, K) + \delta(K, Z) < e/2 \quad (r=0,\dots,m).$$

Hence both k_r^t and k_r are within a distance $e/2$ of Z so that $\varrho(k_r^t, k_r) < e$. By virtue of Lemma 4.3 $\varrho(g_r^t, g_r) < e$. We infer that

$$\varrho(B_t, B_{u+}) < e \qquad (u<t<t_1)$$

and the theorem follows.

§ 5. Definition of $V|(\bar{R}(N)-Z)$**.** We are supposing that U is PH over $S-Z$, each $\alpha \in F$ a level set of U and that each F-singular point in ω is a topological critical point of U.

Lemma 5.1. *Let N_0 be a right neighbourhood with canonical coordinates (u,v). If V is continuous over N_0 and strictly increasing with v for u constant, then V is* PC *to U over N_0.*

Since U has each $\alpha \in F$ as level set the value of U at the point (u,v) in N_0 depends only on u. Let $U_1(u)$ be this value and let $V_1(u,v)$ be the value of V at (u,v). By hypotheses $V_1(u,v)$ increases in a strict sense with v for u constant and $u \in (-1,1)$, so that $U_1(u)+iV_1(u,v)$ defines an interior transformation from the $(u+iv)$-plane to the (U_1+iV_1)-plane. The mapping from the $(u+iv)$-plane to the z-plane in which $(u+iv)$ corresponds to the point $z \in N$ represented by (u,v) is interior by convention as to admissible coordinates (u,v). The mapping from N_0 in the z-plane into the $(U+iV)$-plane is accordingly interior, and hence V is PC to U.

Lemma 5.2. *Given a band $R(N)$ there exists a function V,* PH *and bounded over $R(N)$,* PC *to U with boundary values on $\beta R(N)-Z$ which are strictly monotone on each $\alpha \in F$ in $\beta R(N)$, increasing in the positive sense of α as derived from F^s, and such that V is continuous over $\bar{R}(N)-Z$.*

Definition of $V|R(N)$**.** We refer to the principal transversal λ of N with its 1—1 continuous parameterization $\lambda(u)$, $-1<u<1$, and to the open arc β_u of F^s meeting λ in the point $\lambda(u)$. For $z \in \beta_u$ let $V(z)$ denote the signed μ-distance measured from $\lambda(u)$ along β_u to z; taking $V(z)>0$ when z follows $\lambda(u)$ on β_u, and negative when z precedes $\lambda(u)$ on β_u. The parameter μ is bounded.

The mapping

$$U(z)+iV(z)=\varphi(z)=w' \qquad (z \in R(N)) \tag{5.1}$$

of $R(N)$ into the $w'=U+iV$ plane is continuous over $R(N)$, since U and V are continuous over $R(N)$. It is 1—1 since the value of $U(z)=u$ uniquely determines the open arc β_u on which z lies, and the signed μ-length $V(z)$ then uniquely determines the point $z \in \beta_u$.

The mapping φ from the z-plane to the w'-plane is also sense preserving, as we shall now prove. It is sufficient to prove that $\varphi|N$ is sense preserving. Let (u,v) be canonical coordinates in N as previously. By convention as to canonical coordinates v increases in the positive sense of F^s along each arc $u=$ constant in N. But V as

defined above similarly increases in the positive sense of F^s along each arc $u=$ constant in N. It follows from Lemma 5.1 that $U+iV$ is interior over N. Hence $U+iV$ is interior over $R(N)$ and V is PC to U.

Definition of $V|(\beta R(N)-Z)$**.** For $z \in \beta_1$ or β_{-1} $V(z)$ is defined as the algebraic μ-value at z on β_1 or β_{-1} as measured from $\lambda(1)$ or $\lambda(-1)$ respectively. The continuity of $V|\bar{R}(N)$ at such a point z is clear. There remain points z on $f[C_2(N)]-Z$ or $f[C_1(N)]-Z$. Suppose $z \in f[C_2(N)]-Z$. Then z is either in an $\alpha \in F^s|\beta R(N)$ or is an F-singular point $z \in \omega \cap \beta R(N)$.

I. Consider $\alpha \in F^s|\beta R(N)$. Such an α is in a unique p-curve B_{u+} or B_{u-}, say B_{u+}. Parameterize B_{u+} by μ-length measured from $\lambda(u)$. At each point $z \in \alpha$ whose parameter in B_{u+} is μ set $V(z)=\mu$. Since $B_t|(t>u)$ converges in the sense of Fréchet to B_{u+} as $t \downarrow u$, it follows from the theory of μ-length that V, as just defined over $R(N)$, takes on continuous boundary values on α. The boundary values of V on α as given by μ-length on B_{u+} are strictly increasing in the positive sense of α as derived from F^s. The case of an $\alpha \in F^s|B_{u-}$ is similar.

II. Consider $z \in \omega \cap \beta R(N)$ with z of type 1 relative to $R(N)$. In this case z is the common end point of two $\alpha \in F^s|\beta R(N)$, say α' and α''. Then any sufficiently restricted open subarc of $\alpha' \cup z \cup \alpha''$ containing z is in a unique p-curve B_{u+} or B_{u-}. One defines $V(z)$ as in *I* and proves that $V|\bar{R}(N)$, as defined on a neighbourhood of z relative to $\bar{R}(N)$ is continuous at z.

III. Consider $z \in \omega \cap \beta R(N)$ with z of type 2 relative to $R(N)$. In this case there exists a unique $u \in (-1,1)$ such that z is the terminal point of β_u. With N^u defined as in § 4 let M^u be $N-(N^u \cup \beta_u)$. Then z is in

$$\beta R(N^u) \cap \beta R(M^u) = \bar{R}(N^u) \cap \bar{R}(M^u) = \bar{\beta}_u \cup Z.$$

But $V|\beta_u$ is already defined, and extends both $V|R(N^u)$ and $V|R(M^u)$ continuously. As in I and II unique extensions of $V|R(N^u)$ and $V|R(M^u)$ over neighbourhoods of z relative to $\bar{R}(N^u)$ and $\bar{R}(M^u)$ respectively exist and make V continuous over such neighbourhoods. It follows that these two definitions of $V|\beta_u$ and hence of $V(z)$ agree, and make $V|\bar{R}(N)$ continuous over a neighbourhood of z relative to $\bar{R}(N)$.

The case of a $z \in f[C_1(N)]-Z$ is similar and this completes the proof of Lemma 5.2.

§ 6. The definition of V over $S-Z$. An $h \in F^*$ will be said to have a *sense derived* from F^s if each $\alpha \in F^s$ in h has the sense in h which is proper to it in F^s. Clearly an $h \in F^*$ can derive a sense from F^s only if at each F-singular point p in h the two F-rays in h incident with p are separated among F-rays incident with p by an even number of F-rays incident with p, including a null set of such F-rays. In this connection we shall use the following lemma.

Lemma 6.1. *Each $h \in F^*$ in $\beta R(N)$ which is concave towards $R(N)$ admits a sense derived from F^s. On h so sensed, the function V, defined in § 5 over $\bar{R}(N)-Z$, is strictly increasing.*

The function V, as defined in $R(N)$ in § 5, increases in the positive sense of each $\alpha \in F^s|R(N)$ and so by virtue of Lemma 5.2 increases in the positive sense of each $\alpha \in F^s|\beta R(N)$. Since the h in Lemma 6.1 clearly admits a sense derived from F^s, Lemma 6.1 follows.

Once V has been formally defined over $S-Z$ it is necessary to show that V is PH, and PC to U. For this purpose Lemma 6.2 is needed.

The following lemma has been established in [3 p. 26].

Lemma 6.2. *If a mapping φ from S to a w-plane is continuous in a neighbourhood H of a point $p \in S$ and interior at each point of $H-p$, then φ is interior at p.*

Definition of $V|\Sigma_n$. Use will be made of the notation of Theorem 2.3 beginning with N_1. From § 5 a definition of $V|(\bar{R}(N_1)-Z)$ is obtained on setting $N=N_1$. Proceeding inductively we shall assume that this definition of V has been extended over $\bar{\Sigma}_{n-1}-Z$ for $n>1$, so that V is PH over Σ_{n-1} and PC to U over Σ_{n-1}, continuous and bounded over $\bar{\Sigma}_{n-1}-Z$ with values on each $\alpha \in F$ on which V has been defined which are strictly increasing in the positive sense of F^s. This induction will be completed at the end of § 6.

Since $\bar{\Sigma}_{n-1} \supset \bar{\eta}_n$ by virtue of Theorem 2.3, $V|(\bar{\eta}_n-Z)$ is defined. Let V^* be defined over $\bar{R}(N_n)-Z$ as in § 5, so as to be PH and bounded over $R(N_n)$ and PC to U. The open arc η_n serving as an "entrance" to $R(N_n)$ is a subarc of an open arc $h \in F^*$ in $\beta R(N_n)$ and by definition meets βN_n. According to Theorem 8.1 (b) of [4], h is then concave toward $R(N_n)$ and hence admits a sense derived from F^s. In accordance with Lemma 6.1 the entrance η_n may be parameterized by the values $t(z)=V(z)|(z \in \eta_n)$. These are values already assigned to η_n by our inductive hypothesis.

The open arc η_n, sensed by F^s, can be equally well parameterized by the values

$$\mu(z)=V^*(z)|(z \in \eta_n),$$

where $\mu(z)$ is the signed μ-length, measured along η_n from the intersection of η_n with the closure of the principal transversal of N_n. Values $\mu(z)$ and $t(z)$ which parameterize the same point $z \in \eta_n$ stand in a relation

$$t=T(\mu) \qquad (\mu_1<\mu<\mu_2),$$

where (μ_1,μ_2) is the finite range of $V^*(z)$ over η_n, and T is a continuous strictly increasing function mapping (μ_1,μ_2) onto the finite range (t_1,t_2) of $t(z)$ over η_n. Let T as defined over (μ_1,μ_2) be extended over the μ-axis so as to map the μ-axis homeomorphically onto the t-axis. Set

$$V(z)=TV^*(z) \qquad (z \in \bar{R}(N_n)-Z).$$

In particular this definition of V over $\bar{\eta}_n-Z$ agrees with the definition of V over $\bar{\eta}_n-Z$ as derived from the earlier definition of $V|(\bar{\Sigma}_{n-1}-Z)$. This completes the definition of $V|(\bar{\Sigma}_n-Z)$. It is clear that V is bounded and continuous over $\bar{\Sigma}_n-Z$ with values on each $\alpha \in F$ on which V has been defined which are strictly increasing in the sense of F^s. A definition of $V|(S-Z)$ is implied.

Theorem 6.1. *Corresponding to any function U given as* PH *over $S-Z$ there exists a function V,* PH *over $S-Z$ and* PC *to U.*

The function V as just defined over $S-Z$ is clearly PH over each $R(N_n)$ and there PC to U. It remains to consider V in a neighbourhood of a point $p \in \eta_n$. If p is F-non-singular we introduce a right N_0 of p with canonical coordinates (u,v). We note that $\eta_n \cap N_0$ satisfies the condition $u=0$. As defined V is increasing over each $\alpha \in F^s|(\bar{\Sigma}_{n-1}-Z)$ in the positive sense of α, by our inductive hypothesis. By construction V is increasing over each $\alpha \in F^s|(\bar{R}(N_n)-Z)$ in the positive sense of α. In N_0 this is the sense of increasing v by virtue of our choice of F^s. It follows from Lemma 5.1 that V is PC to U on N_0.

If $p \in \eta_n$ is F-singular the preceding analysis shows that $U+iV$ is continuous in a neighbourhood X_p of p and interior at each point of X_p-p. It follows from Lemma 6.2 that $U+iV$ is interior at p. This completes the induction and the proof of the theorem.

§ 7. Uniformization of PH functions. A PH function is obtained locally from a harmonic function by composition with a topological transformation. The question naturally arises whether a similar result holds in the large. This is the case for a PH function when defined in a simply connected domain as stated in Theorem 7.1.

Theorem 7.1. *Let U be PH in a simply connected domain J_z in the z-plane. There then exists a homeomorphism φ from J_z onto a domain J_w in the w-plane and a function U^* harmonic on J_w such that $U=U^*\varphi$.*

Indeed J_z is homemorphic to $S-Z$ under a transformation ψ from J_z to $S-Z$. Hence $U'=U\psi^{-1}$ is PH over $S-Z$. By Theorem 6.1 there exists V', PH over $S-Z$ and PC to U'. Then $V=V'\psi$ is PH over J_z and PC to $U=U'\psi$ [3, p. 39].

The function $U+iV=g$ is interior on J_z and thus $\zeta=g(z)$ maps J_z on a simply connected Riemann surface R_ζ spread over the $\zeta=(U+iV)$-plane. By the general uniformization theorem there exists a domain J_w in the w-plane and a function f analytic on J_w such that $\zeta=f(w)$ maps J_w onto R_ζ. The mapping $f^{-1}g$ is a homeomorphism φ of J_z^1 onto J_w and if one sets $f=U^*+iV^*$ with U^* and V^* real, we have $g=f\varphi$ and $U=U^*\varphi$. This proves Theorem 7.1.

It is clear that the theorem holds for a U which is PH in any simply connected Riemann domain.

Recall that V' and thus V may be constructed so as to be bounded. If this is done the Riemann surface R_ζ is of hyperbolic type. From this we deduce the following corollary.

Corollary 7.1. *If J_z is of hyperbolic type the mapping φ of Theorem 7.1 may be chosen so as to map J_z onto itself.*

Bibliography.

[1] W. M. Boothby, *The topology of the level curves of harmonic functions with critical points*, American Journal of Mathematics **73** (1951), pp. 512-538.

[2] C. Carathéodory, *Conformal representation*, Cambridge 1932.

[3] J. A. Jenkins and M. Morse, *Contour equivalent pseudoharmonic functions and pseudoconjugates*, American Journal of Mathematics **74** (1952), pp. 23—51.

[4] — *Topological methods on Riemann surfaces. Pseudoharmonic functions*, Annals of Mathematics Studies, Princeton 1952.

[5] W. Kaplan, *Regular curve families filling the plane*, Duke Mathematical Journal I **7** (1940), pp. 154-185, II **8** (1941), pp. 11-45.

[6] — *Differentiability of regular curve families on the sphere. Lectures in Topology*, Ann Arbor 1941, pp. 299-301.

[7] — *Topology of level curves of harmonic functions*, Transactions of the American Mathematical Society **63** (1948), pp. 514-522.

[8] M. Morse, *Topological methods in the theory of functions of a complex variable*, Annals of Mathematics Studies, Princeton 1947.

[9] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, Journal de Mathématique **7**, 3-rd series, (1881), pp. 375-422, and **8** (1882), pp. 251-296.

[10] S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris 1938.

[11] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Lectures, New York 1942.

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