pyramid 6(c) has been drilled out. Adjoining 6(c) therefore completes our construction, for it is easily checked that all the identifications described by \( f \) have been carried out.

Bibliography.


The Existence of Pseudoconjugates on Riemann Surfaces.

By

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§ 1. Introduction. Among the characteristics of a function \( U \) which is harmonic on a Riemann surface \( G \) are the topological interrelations of the level lines of \( U \). One has merely to look at the level lines of \( \mathbb{R}s, \mathbb{R}t, \mathbb{R}\log z \), etc. to sense both complexity and order. The dual level lines of a conjugate \( V \) of \( U \) add to this order and complexity. It seems likely that outstanding problems in Riemann surface theory, such as the type problem, the nature of essential singularities, the existence of functions on the Riemann surface with restricted properties cannot be thoroughly understood in the absence of a complete analysis of the topological characteristics of these level lines.

Such a topological study properly belongs to a somewhat larger study namely that of \( \text{PH} \) (pseudoharmonic) functions and their pseudoconjugates (defined in § 2). A first problem is that of the existence of \( \text{PH} \) functions \( U \) on \( G \) with prescribed level sets locally topologically like families of parallel straight lines except in the neighbourhood of points of a discrete set \( \omega \) of points \( \varepsilon_{n} \). At a point \( \varepsilon_{n} \) the level curves of \( U \) may cross after the manner of the level curves of a harmonic function with a critical point at \( \varepsilon_{n} \).

Let \( E \) be the finite \( \varepsilon \)-plane. In case \( \varepsilon^{*} = E \) and \( \omega = 0 \), and excluding all recurrent level curves other than periodic curves, Kaplan has solved the above problem in [5]. More recently Boothby has extended Kaplan’s result to the case of a general \( \omega \). As explained in [4] the \( a \text{ priori} \) exclusion of recurrent level curves other than periodic curves does not seem justified. The writers of this paper have accordingly established the existence of \( \text{PH} \) functions without this hypothesis of non-recurrence [4].
The Existence of Pseudonjugates

in $D$ with the origin deleted when $n>1$. We term $\alpha$ the exponent of $p$. The exponent of $p$ shall be 1 for $p \in G$, and exceed 1 for $p \in J$. Points $p \in \omega$ are termed $F$-singular points. A value $w \in D$ is termed the canonical parameter of its antecedent in $X_p$.

The open areas of $F|X_p$ which have limiting end points at $p \in \omega$ are termed $F$-rays of $X_p$ incident with $p$. They are $2n$ in number, thereby showing that $n$ is independent of the choice of $X_p$. These $F$-rays of $X_p$ divide $X_p$ into $2n$ open regions termed $F$-sectors incident with $p$.

Right $N$. With each $p \in G$ one can also associate a neighbourhood $X_p$ of $p$ with $X_p \subseteq G$, and a homeomorphic sense-preserving mapping of $X_p$ onto a square $K: (-1 \leq u \leq 1) (-1 \leq v \leq 1)$ such that $p$ goes into the origin in $K$ and the maximal subarcs of $F|X_p$ go into arcs $u = \pm \epsilon, -1 \leq \epsilon \leq 1$, where the constant $\epsilon$ ranges over the interval $[-1, 1]$. We refer to $X_p$ as a right $N$ of $p$ and term $\alpha$ and the canonical coordinates of the antecedent in $X_p$ of $(u, v)$ in $K$.

Transversals. By a transversal $\lambda$ is meant an open arc in $G$ whose intersection with any right $N$ with canonical coordinates $(u, v)$ has locally the form $z = g(u)$ where $g$ is single-valued and continuous. By the principal transversal of a right $N$ is meant the open arc in $N$ on which $\alpha = 0, -1 < u < 1$.

$F$-vectors. Any sensed subarc $A$ of an $a \in F$ will be called an $F$-vector. By definition an $F$-vector is simple, closed, and never a topological circle. The following lemma is established in [4, § 3].

Lemma 2.1. Each $F$-vector is in some right $N$.

Coherent sensing. Let each $a \in F$ be given a sense. The resulting family $F$ of sensed $a$ will be called a sensed image of $F$. We shall refer to a continuous deformation $a$ of an $F$-vector $A$ in the space of $F$-vectors metrized by means of the Fréchet distance between any two sensed arcs. We shall understand that each image of $A$ under $a$ is sensed by $A$, that is the sense of the image arc shall be determined by the images under $a$ of the initial and final points of $A$. We say that $a$ is coherently sensed if any continuous deformation $a$ of an $F$-vector $A$, initially sensed by $F^+$, through $F$-vectors sensed by $A$, is necessarily a deformation through $F$-vectors of $F$. In the case of a right $N$, which is simply connected, there are two distinct coherently sensed images $F^+$ of $F$ [4, § 5].
The following theorem came next in our development:

**Theorem 2.1.** When \( G^* = S - Z \) each \( \alpha \in F \) is the homeomorphic image in \( G \) of an open interval with limiting end points on \( S \), distinct unless both are coincident with \( Z \). Each finite end point is a singular point of \( F \).

The family \( F^* \). It is shown in [4, § 7] that when \( G^* = S - Z \) there are no topological circles in \( G^* \) formed from the union of the closures of a finite set of \( \alpha \in F \). In this case we define \( F^* \) as the set of all open arcs \( h, b, m, \) etc. in \( G^* \) with the following properties. If \( p \) is a \( F \)-non-singular point of \( h \), \( h \) shall contain the \( a \in F \) meeting \( p \), and any finite limiting end point or end points of \( a \); if \( F \)-singular point \( q \) is in \( h \), \( h \) shall contain just two of the \( \alpha \in F \) with \( q \) as a limiting end point. An \( h \in F^* \) may be identical with an \( \alpha \in F \), or it may be formed from a sequence of \( \alpha \in F \) of one of the forms

\[
\ldots h = a \circ a \circ a \circ a \ldots \\
\alpha \circ a \circ a \circ a \circ a \ldots \\
a \circ a \circ a \circ a \circ a \ldots
\]

The following theorem bears on the behaviour of subarcs of \( h \in F^* \) [4, Cor. 7.3].

**Theorem 2.2.** When \( G^* = S - Z \) an arc \( g \) in \( S - Z \) which is the closure of a finite sequence of \( \alpha \in F \) intersects the closure of an \( F \)-neighbourhood \( X_g \) of or a right \( N \), if at all, in a single arc.

**Corollary 2.3.** When \( G^* = S - Z \) each \( h \in F^* \) has \( Z \) as a limiting end point in both senses.

 Bands \( R(N) \). Given a right \( N \) the union of the sets \( \alpha \in F \) which intersect \( N \) will be called the band \( R(N) \). In accordance with Lemma 8.1 [4] each band \( R(N) \) is simply connected when \( G^* = S - Z \), and in accordance with Theorem 8.1 [4] the boundary \( \delta R(N) \) of \( R(N) \) in \( S \) is the union of \( Z \) and at most a countable set of non-intersecting open arcs \( \lambda \in F^* \) of which at most a finite number have diameters on \( S \) which exceed a finite constant \( d \). An \( h \in F^* \) in \( \delta R(N) \) is either concave towards \( R(N) \) in that \( R(N) \) contains just one \( F \)-sector incident with each singular point of \( F \) in \( h \), or semi-concave towards \( R(N) \) in that \( R(N) \) includes just one \( F \)-sector incident with each singular point of \( F \) in \( h \) except for one singular point \( q \) of \( F \) in \( h \); corresponding to \( q \), \( R(N) \) contains just two \( F \)-sectors incident with \( q \) and the \( F \)-ray incident with \( q \) between the two \( F \)-sectors incident with \( q \). An \( h \in F^* \) in \( \delta R(N) \) which meets \( \delta N \) is always concave towards \( R(N) \) (cf. Theorem 8.1 [4]).

A singular point \( p \) of \( F \) in \( \delta R(N) \) is termed of type 1 or type 2 relative to \( R(N) \) if \( R(N) \) contains one or two \( F \)-sectors respectively incident with \( p \).

An entrance to \( R(N) \). An open arc \( g \) in \( \delta R(N) \) will be called an entrance to \( \delta R(N) \) if \( \eta \) is the union of a finite or countably infinite set of \( \alpha \in F \) each joined to its successor or predecessor at a singular point in \( F \), and if \( \eta \) intersects \( \delta N \) in an open arc. An entrance \( \eta \) to \( R(N) \) is an open subarc of an \( h \in F^* \) in \( \delta R(N) \) such that \( h \) intersects \( \delta N \) and is accordingly concave towards \( R(N) \) [4, Th. 8.1].

We shall make use of the following decomposition of \( G^* \) [4, Th. 9.1].

**Theorem 2.3.** When \( G^* = S - Z \) there exists a sequence of non-intersecting bands \( B(N_i), i = 1, 2, \ldots \), and for each \( B(N_i) \) with \( i > 1 \) an open entrance \( \eta_i \) to \( B(N_i) \), such that the \( \eta_i \) do not intersect each other or any of the bands, and such that \( G^* = \bigcup_i B(N_i) \) and for \( n = 2, 3, \ldots \)

\[
\Sigma_n = \Sigma_{n-1} \cup B(N_n) \cup \eta_n; \quad \Sigma_{n+1} \cap B(N_n) = \eta_n \cup B(N_n) = \Sigma_n \cup Z
\]

and where the \( \eta_n \) are so chosen that any compact subset of \( G^* \) is included in \( \Sigma_n \) for \( n \) sufficiently large.

We note that each \( \Sigma_n \) is open and simply connected. If \( \Sigma_n \) contains a point \( p \in F \), it contains the \( \alpha \in F \) which meets \( p \).

Let \( U \) be harmonic in a neighbourhood \( H \) of \( g \in G^* \) and not identically constant. Let \( \Phi \) be a topological mapping of a neighbourhood \( H \) of \( g \) onto \( H' \). Then \( U \circ \Phi \) is termed \( \Phi H \) \( (\text{pseudo-harmonic}) \) at \( g \). A function \( U \) is termed \( \Phi H \) \( \text{over a region } BC \subset G^* \) if \( U \) is \( \Phi H \) at each point of \( E \).

Let \( \varphi \) be analytic in a neighbourhood \( H \) of \( g \in G^* \), and not identically constant, and let \( \Phi \) be a sense-preserving topological mapping of a neighbourhood \( H \) of \( g \) onto \( H' \). Then \( \varphi \circ \Phi \) is termed \( \Phi H \) \( \text{over a region } BC \subset G^* \) if \( \Phi \) is interior at each point of \( E \).

If \( U \) is \( \Phi H \) over a region \( R \) and \( V \) real, and if \( U + iV \) is interior over \( R \) then \( V \) is said to be \( \Phi U \) \( \text{(pseudo-harmonic)} \) to \( V \) over \( R \).
§ 3. Hands $\beta(R(N))$ conformally represented. Let $U$ be given as the $\alpha$ of $\beta(R(N))$ with $\alpha \in F$ as level lines. It is relatively easy to use signed $\mu$-length along the $\alpha \in F(R(N))$ measured algebraically from the principal transversal $\mu$ of $\alpha$ to obtain a function $\beta(R(N))$. To extend the definition of $\beta(R(N))$ is more difficult because $\beta(R(N))$ is not readily given as a $\mu$-curve. The arcs of $\beta(R(N))$ meeting $Z$ require a definite parameterization which is related to the parameterization of the $\alpha \epsilon F(R(N))$. To this end we find it useful to map a unit disc $D(\{x|<1\})$ directly conformally onto $R(N)$, and then by continuously extending the mapping of $D$ over $\beta(R)$ to obtain the desired parameterization of $\alpha \in F(R(N))$.

Canonical coordinates $(u,v)$ in a right $\alpha$. Let $(u,v)$ be canonical coordinates in a right $\alpha$. Set $u + iv = w$. To each point $z \in N$ corresponds a complex canonical parameter $w = \mathcal{F}(z)$ given with $N$. The mapping $\mathcal{F}$ is given as $1 - 1$ continuous and sense-preserving in $N$. The value of $U$ at the point $w \in N$ reduces to a strictly monotone function of $u$.

Convention. In this paper we shall choose the canonical coordinate $u$ in each right $\alpha$ so that $U(u)$ is strictly increasing.

Let $\alpha \in F$ be a coherently sensed image of $F$ so chosen that in each right $\alpha$ the canonical coordinate $u$ increases in the positive sense of $F^*|N$ along each level arc of $U$.

This relation of $F^*$ and $U$ to the canonical coordinates $(u,v)$ in one right $\alpha$ implies a similar relation to the canonical coordinates in an arbitrary right $\alpha$. This follows from the orientability of $S$ and the fact that $F^*$ is a coherently sensed image of $F$. In this connection recall that canonical coordinates $(u,v)$ in a right $\alpha$ are to be chosen so that the mapping $u + iv = \phi(z)$ which carries $z \in N$ into the canonical point $(u,v)$ is an interior transformation.

The open arc $\beta_a$. Let $\lambda$ be the principal transversal of $N$. Let the point $\lambda$ with canonical coordinates $(u,0)$ in $N$ be denoted by $\lambda(u)$. The coordinate $u$ ranges on an interval $I = [-1,1]$. Let $\beta_a F^*|(u)\lambda(u)$ meet $\lambda$ at the point $\lambda(u)$. In particular $\beta_a$ and $\beta_a$ are in $F^*$ and on $\beta(R(N))$.

The conformal mapping $f$. Let $f$ be a direct conformal mapping of the open unit disc $D$ onto $R(N)$. An arc $h \epsilon F$ in $\beta(R(N))$ is a free boundary arc of $R(N)$ in the sense that it is an open arc which does not meet the closure of $\beta(R(N))$. The mapping $f$ can be extended to a homeomorphism which maps an open arc $h^0$ on $\beta(R(N))$ onto $h$ [2, p. 86]. If $f$ is further extended so as to map each point in the set

$$Q = \beta D \cup \beta h^0 \quad (h \epsilon F^*|\beta(R(N)))$$

into $Z$, then the extended $f$ maps $D$ continuously into $S$.

Indeed if $x_n$ is a sequence of points in $D$ tending to a point in $Q$, the image sequence $f(x_n)$ can have no accumulation point in $R(N)$ or on an arc $h \epsilon F^*$ in $\beta(R(N))$. Thus $f(x_n) \rightarrow Z$ as $n \rightarrow \infty$ and $f$ maps $D$ continuously into $S$.

Without loss of generality we may assume that $f$ carries $z = +1$ and $z = -1$ in $\beta(D)$ into $\lambda(1)$ and $\lambda(-1)$ in $\beta(R(N))$ respectively.

If $F$ is any set in $R(N)$ we will denote $F^*|(X)$ by $Y^*$. The sensed antecedents $\beta_a$ and $\beta_a$ in $\beta(D)$ of $\beta_a$ and $\beta_a$ respectively in $\beta(R(N))$ are open arcs in $\beta(D)$ which appear in $\beta(D)$ in counter-clockwise and clockwise sense respectively, with $z = 1$ an inner point of $\beta_a$ and $z = -1$ an inner point of $\beta_a$.

Let $(a)$ be the set of all $\beta_a \epsilon F(R(N))$, and $(a)$ the set of the respective antecedents of $\alpha \epsilon (a)$.

Let $(a)$ and $(a)$ be respectively the diameter in $S$ of $\alpha \epsilon (a)$ and of $a \epsilon (a)$. As stated in § 2 there are at most a finite number of the $\alpha \epsilon (a)$ with $d(a) > 1/n$. The same is true of the $a \epsilon (a)$ since $2\pi(n) \leq 2\pi$, the length of $\beta(D)$. It follows that $d(a) \rightarrow 0$, and conversely that $d(a) \rightarrow 0$ if $d(a) \rightarrow 0$.

The antecedent $Z^*$. This antecedent of $Z$ is closed. It is nowhere dense in $\beta(D)$ since it can contain no subarc of $\beta(D).

The antecedent $F^*$. Since $\beta_a$ and $\beta_a$ are in $\beta(R(N))$, $\beta_a$ and $\beta_a$ are in $\beta(D)$. For $-1 < u < 1$, $\beta_a$ is in $D$ since $\beta_a$ is in $D$. The point $(r, \theta) = (1,0)$ in $\beta(D)$ is in $\beta_a$. We shall restrict $\theta = x_1(x_1 D)$ to values on the interval $0 \leq \theta < 2\pi$.

The end points $\exp((\Theta_1) + u)$ and $\exp((\Theta_2) + u)$ of $\beta_a$. For $z \epsilon F^*|(u \epsilon I)$, $\Theta = x_1$ has limiting initial and final values which we denote by $\Theta_1(u)$ and $\Theta_2(u)$ respectively. That $\Theta_1(u)$ and $\Theta_2(u)$ exist is immediate when $a = -1$. If $\beta_a^{(u)} (-1 < u < 1)$ did not have unique limiting end points in $\beta(D)$, $\beta_a^{(u)}$ would include some subarc of an $\alpha \epsilon (a)$ since $Z^*$ is nowhere dense in $\beta(D)$. Since $f$ is continuous over $\beta(D)$, $\beta_a^{(u)}$ contains at most two points of $S$. Hence the limits $\Theta_1(u)$ and $\Theta_2(u)$ exist and are unique. The limiting end points of $\beta_a^{(u)}$ are then $\exp((\Theta_1) + u)$ and $\exp((\Theta_2) + u))$. 18
The Existence of Pseudoconjuncts

The arc $C_d(N)$ bears all the final end points of the $\beta^u_2 \{ u = 1 \}$, and $G_d(N)$ all the initial end points. Then $C_d(N)$ and $G_d(N)$ clearly contain each $a^u \in (D^u)$ except $\beta_d^u$ and $\beta_d^u$. We shall be more explicit in the case of $G_d(N)$.

**Lemma 3.2.** Corresponding to each $a \in F$ with $a^u \in C_d(N)$ there is a unique value $u = I$ such that $a^u$ lies in an arc on $\beta D$ of one of the forms

(3.3) \[ \Theta \theta(u) > \theta > \Theta \theta(u) \]

(3.4) \[ \Theta \theta(u) > \theta > \Theta \theta(u) \]

Let $J$ be the interval of values of $\theta$ such that $\exp (i \theta)$ is in $a^u$. Then $J$ is open and contains no value $\Theta \theta(u) \{ u \in I \}$ for the final end point $\exp (i \Theta \theta(u))$ of $\beta^u_d$ has an $f$-image which is either an $F$-singular point or $Z$, and accordingly not in $a$. Hence $J$ must be in the interval of the type (3.3) or (3.4). Since no two intervals of type (3.3) or (3.4) intersect, the interval (3.3) or (3.4) containing $J$ must be unique.

The $p$-curve $B_a$ in $S$. Let the half open arc of $\beta_d$ with initial point $\lambda(u)$ be closed by its final end point in $S$, and be parameterized by $\mu$-length in $S$ to define a $p$-curve $B_a$.

If $g$ is a $p$-curve it is convenient to introduce the carrier $|g|$ of $g$ defined as the union of the points in the range of $g$.

The extensions $B_a$ and $B_{a^u}$ of $B_a$ in $S$. Let $B_d^u$ be extended as a $p$-curve by the arc of points $\exp (i \theta)$ of $\beta D$ on which

(3.5) \[ \Theta \theta(u) > \theta > \Theta \theta(u) \quad (-1 < u < 1) \]

taken in the sense of decreasing $\theta$ and admitting the possibility that (3.3) may reduce to a point and $B_d^u$ coincide with its extension. Let $B_{d^u}$ be the resultant extension of $B_d^u$ as a $p$-curve, and $B_{d^u} = (B_d^u)$ its $p$-curve image in $R(N)$. Recall that a $p$-curve is not a set, and that $\exp (i \theta)$ is in general not the full antecedent of the set $[B_{d^u}]$, since $S$ is in $[B_{d^u}]$ (see Lemma 3.3) and $\exp (i \theta)$ will not in general contain the whole set $\exp (i \theta_d)$. We shall suppose that $B_{d^u}$ is parameterized by $\mu$-length measured along $B_{d^u}$ from $\lambda(u)$, and note that $B_{d^u}$ is a $p$-curve in $R(N)$ extending $B_d$.

Let $B_d^u$ be similarly extended in $D$ as a $p$-curve by adding the arc of $\beta D$ (if any exists) on which

(3.6) \[ \Theta \theta(u) > \theta > \Theta \theta(u) \quad (-1 < u < 1) \]
The Existence of Pseudoconjugates

Let $X$ be the closure of the Jordan region bounded by $G$. For $0 < s < 1$, $B_s$ is in $X$. As $t \downarrow 0$, $X$ contracts itself as a point set, and $G \cap \partial D$ is an arc of $X \cap \partial D$ which contracts on itself to $|B_s| \cap \partial D$ as a limiting arc or point. The point $x_0$ is in $X$ for each $t > 0$. In case (ii), $x_0$ must then be in $G \cap \partial D$ for each $t > 0$ and hence in $|B_s| \cap \partial D$. From this contradiction to (a) we infer the truth of the lemma.

A right neighbourhood relative to $R(X)$. Let $g$ be an arc in $B(\mathbb{R}) - Z$ whose end points are $F$-non-singular and whose $F$-singular points are finite in number and of type 1 relative to $R(X)$.

**Lemma 4.2.** There exists a neighbourhood $M$ of $g$ relative to $R(X)$ such that $M$ is the homeomorphic of a square $(0 \leq u \leq 1)$ $(0 \leq v \leq 1)$ in which the open edge $u = 0$, $0 < v < 1$, contains the image of $g$ and each arc $u = \text{const} \neq 0$, $0 < v < 1$, corresponds to an $F$-vector in $R(X)$.

The lemma is true if there are no singular points of $F$ on $g$; by virtue of Lemma 2.1. It is also clearly true if $g$ is the closure of two $F$-rays on the boundary of an $F$-sector. The lemma is readily established in the general case following the pattern of the proof of Lemma 3.1 of [4].

The neighbourhood of $g$ in Lemma 4.2 is termed a right neigh-
bourhood of $g$ relative to $R(X)$.

In the proof of Theorem 4.1 use will be made of various distance functions. If $A$ and $B$ are non-empty subsets of $S$ let $d(A, B)$ denote the inferior limit of the distances $d(p, q)$ between $p \in A$ and $q \in B$. Let

$$d(A, B) = \sup_{p \in A} \inf_{q \in B} d(p, q).$$

If $A, B, C$ are subsets of $S$,

(4.1) $d(A, C) \leq d(A, B) + d(B, C)$.

(4.2) $d(A, B) \geq \min \{d(A, B_1), d(A, B_2)\}$.

If $g$ and $h$ are $p$-curves in $S$, $d(g, h)$ shall denote the Fréchet distance between $g$ and $h$.

Theorem 4.1 is basic in the proof of the existence of a $F$ pseudo-
conjugate to $U$. We prepare for its proof.

An $e$-decomposition of $B_1$. We suppose $u \in [-1, 1]$. Given $e > 0$ the number of open arcs $\gamma \in \mathbb{R}$ in $B_1$ such that $\gamma(Z) \supseteq e/4$ is finite (see §3). Let $\gamma_1, ..., \gamma_m$ be these open arcs ordered as they
appear in the p-curve $B_{u+}$. No one of the $\gamma_i$, $i=1, \ldots, m$ intersects $B_u$. Let $\gamma_u$ be the maximal initial half open arc of $B_{u+}$ which does not meet $Z$. The initial point of $\gamma_u$ is $\lambda(u)$ and $\gamma_u \subset B_u$. Let $g_u, g_1, \ldots, g_m$ be subarcs of $\gamma_u, \gamma_1, \ldots, \gamma_m$ respectively, whose end points are $F$-non-singular and are such that the initial point of $g_k$ is $\lambda(u)$ and
\[
\delta(\gamma_k, Z) < \epsilon / k \quad (r = 0, \ldots, m).
\]
Parameterizing $g_r$ as in $B_{u+}$ it appears that $B_{u+}$ is a sequence of sub-p-curves
\[
g_u, g_1, g_2, \ldots, g_m, g_m.
\]
No p-curve $g_r$ intersects any other p-curve $g_k$ or any p-curve $k_r$.
A right neighbourhood $M_r$ of $g_r$. Recall that $-1 < \tau < 1$. The set $N \cap g_r$ is the union of two open sets $M^o$ and $N^o$ of which $N^o$ contains $\lambda(1)$. Then $F(N^o) \subset F(N)$. Moreover $g_1, \ldots, g_m$ are in $\beta F(N) \cap \beta F(N)$ while $g_2$ is in $\beta F(N)$. Each $F$-singular point of $\gamma_i$, $i=1, \ldots, m$, is of type 1 relative to $F(N)$ and $F(N)$, by virtue of the definition of the extension $B_{u+}$ of $B_u$ and nature of $\beta F$. Each $F$-singular point of $\gamma_k$ is of type 1 relative to $F(N)$, by virtue of Theorem 8.1 (b) [1]. By Lemma 4.2 then, the respective $g_r$ in (4.4) have right neighbourhoods
\[
M_0, M_1, \ldots, M_m,
\]
relative to $F(N)$. Each $M_r$ in (4.5) will be so restricted that each maximal sub-p-curve of $B_r$ in $M_r$ (when $B_r \cap M_r = 0$) is at a Fréchet distance less than $\epsilon$ from $g_r$ (sensed as in $F^o$), and that $M_r \cap M_{r'} = 0$ for $r \neq r'$.

Lemma 4.3. Corresponding to the given $u \in [-1, 1]$ and $\epsilon > 0$ a constant $t_u (1 > t_u > u)$ such that $u - t_u < u$ is sufficiently small has the following property. Each simple p-curve $B_r$ for which $u - t_u < t_u$ admits a decomposition
\[
g_r, g_r, g_r, g_r, \ldots, g_r, g_r
\]
into successive non-intersecting p-curves such that $g(g_r, g_r) < \epsilon$, and, setting
\[
K' = \text{Union } [k_r], \quad K = \text{Union } [k_r] \quad (r = 0, \ldots, m);
\]
\[
\delta(K', K) < \epsilon / 4.
\]
We shall refer to $K$ and make use of the fact that $B_{u+}$ is simple. Set
\[
\exp(i \theta_k(u-\epsilon)) = q.
\]
The point $q$ is the final end point of $B_{u+}$. As $t \downarrow u$ the final end point $\exp(i \theta_k(t))$ of $B_{u+}$ tends to $q$ while the initial point $\lambda^p(t)$ of $B_{u+}$ tends to the initial end point $\lambda^p(u)$ of $B_{u+}$; at the same time
\[
\delta([B_{u+}]^p|B_{u+}^p) \to 0
\]
in accordance with Lemma 4.2. The simple arc $B_{u+}$ meets the disjoint sets
\[
M_0, M_1, \ldots, M_m, q
\]
in the ordered written. It follows from Lemma 4.2 that if $t_u - u$ is sufficiently small and if $u < t < t_u < 1$, $B_{u+}$ will meet each of the sets $M^p_r$ in (4.8) in at least one maximal p-curve $(g^p_r)$ and by virtue of Theorem 2.2 in at most one p-curve, there will be a terminal p-curve $(g^p_r)$ of $B_{u+}$ which intersects no $M^p_r$, and the initial point $\lambda^p(t_u)$ of $B_{u+}$ will be in $M^p_r$. We suppose $t_u$ and $u$ so conditioned.
It follows from this choice of $g_r$ and $M_r$ that $g(g_r, g_r) < \epsilon$. The residual p-curves $g_r$ in (4.6) are determined. Set $\bigcup M_r = \Gamma$, $r = 0, \ldots, m$. In the relation
\[
\delta(K', K') \geq \min \{\delta(K', K), \delta(K', K')\} \quad (4.9)
\]
the left member tends to 0 as $t \downarrow u$ by virtue of Lemma 4.2, while $\delta(K', K')$ remains bounded from 0. Hence for $t_u - u$ sufficiently small and $u < t < t_u$
\[
(4.10)
\]
Relation (4.7) then follows for suitable choice of $t_u$.

Theorem 4.1. If $t \downarrow u \in [-1, 1]$ then $B_r$ tends to $B_{u+}$ in the sense of Fréchet.

Corresponding to $\epsilon > 0$ let $B_{u+}$ be the decomposition (4.4), where $\delta(K, Z) < \epsilon / 4$. If $t_u$ is chosen as in Lemma 4.3 and if $u < t < t_u$, then $\delta(K', K) < \epsilon / 4$. Moreover by (4.1)
\[
\delta(g_r, Z) < \delta(K', Z) + \delta(K', Z) < \epsilon / 2 \quad (r = 0, \ldots, m).
\]
Hence both $g_r$ and $k_r$ are within a distance $\epsilon / 2$ of $Z$ so that $g(g_r, k_r) < \epsilon$. By virtue of Lemma 4.3 $g(g_r, g_r) < \epsilon$. We infer that
\[
\delta(B_u, B_{u+}) < \epsilon \quad (u < t < t_u)
\]
and the theorem follows.
§ 5. Definition of $\varphi(N) - Z$. We are supposing that $U$ is $\text{PB}$ over $S - Z$, each $x \in F$ a level set of $U$ and that each $F$-singular point in $\mathcal{N}$ is a topological critical point of $U$.

**Lemma 5.1.** Let $N_x$ be a right neighbourhood with canonical coordinates $(u, v)$. If $V$ is continuous over $N_x$ and strictly increasing with $v$ for $u$ constant, then $V$ is $\text{PC}$ over $N_x$.

Since $U$ has each $x \in F$ as level set the value of $U$ at the point $(u, v)$ in $N_x$ only depends on $u$. Let $U_1(u)$ be this value and let $V_s(u, v)$ be the value of $V$ at $(u, v)$. By hypothesis $V_s(u, v)$ increases in a strict sense with $v$ for $u$ constant and $u \neq -1, 1$, so that $U_1(u) = V_s(u, v)$ defines an interior transformation from the $(u + iv)$-plane to the $(U_1 + iv)$-plane. The mapping from the $(u + iv)$-plane to the $z$-plane in which $(u + iv)$ corresponds to the point $z \in N$ represented by $(u, v)$ is interior by convention as to adhesive coordinates $(u, v)$. The mapping from $N_x$ in the $z$-plane into the $(U + iv)$-plane is accordingly interior, and hence $V$ is $\text{PC}$ to $U$.

**Lemma 5.2.** Given a band $R(N)$ there exists a function $V$, $\text{PH}$ and bounded over $R(N)$, $\text{PC}$ to $U$ with boundary values on $\partial R(N) - \mathcal{Z}$ which are strictly monotone on each $x \in F$ in $R(N)$, increasing in the positive sense of $u$ as derived from $F^*$, and such that $V$ is continuous over $(R(N) - \mathcal{Z})$.

**Definition of $\varphi(N)$**. We refer to the principal transversal $\lambda$ of $N$ with its $-1 - 1$ continuous parameterization $\lambda(u), -1 < u < 1$, and to the open arc $\beta_u$ of $F^*$ meeting $\lambda$ in the point $\lambda(u)$. For $x \in \beta_u$ let $V(z)$ denote the signed $\mu$-distance measured from $\lambda(u)$ along $\beta_u$ to $z$; taking $V(z) > 0$ when $z$ follows $\lambda(u)$ on $\beta_u$, and negative when $z$ precedes $\lambda(u)$ on $\beta_u$. The parameter $\mu$ is bounded.

The mapping $U(z) + iV(z) = \varphi(z) = w$ of $R(N)$ into the $w = U + iv$ plane is continuous over $R(N)$, since $U$ and $V$ are continuous over $R(N)$. It is $1 - 1$ since the value of $U(z)$ uniquely determines the open $\beta_u$ on which $z$ lies, and the signed $\mu$-length $V(z)$ then uniquely determines the point $z \in \beta_u$.

The mapping $\varphi$ from the $z$-plane to the $w$-plane is also sense preserving, as we shall now prove. It is sufficient to prove that $\varphi(N)$ is sense preserving. Let $(u, v)$ be canonical coordinates in $N$ as previously. By convention as to canonical coordinates $v$ increases in the positive sense of $F^*$ along each arc $u = \text{constant in } N$. But $V$ as defined above similarly increases in the positive sense of $F^*$ along each arc $u = \text{constant in } N$. It follows from Lemma 5.1 that $U + iv$ is interior over $N$. Hence $U + iv$ is interior over $R(N)$ and $\varphi$ is $\text{PC}$ to $U$.

**Definition of $\varphi(R(N) - \mathcal{Z})$**. For $x \in \beta_u$ or $\beta_{-1}$, $V(z)$ is defined as the algebraic $\mu$-value at $z$ on $\beta_u$ or $\beta_{-1}$ as measured from $\lambda(1)$ or $\lambda(-1)$ respectively. The continuity of $V(R(N))$ at such a point $z$ is clear. There remain points on $f(N) - Z$ or $f(N) - Z$. Suppose $x \in f(N) - Z$. Then $z$ is either in an $x \in F^*$ or is on a $F$-singular point $x \in \partial R(N)$.

I. Consider $x \in F^* \beta(N)$, such an $x$ is in a unique $\mu$-curve $B_{\beta}$, say $B_{\beta_+}$. Parameterize $B_{\beta}$ by $\mu$-length measured from $\lambda(u)$ at each point $z \in \beta$ whose parameter in $B_{\beta_+}$ is $\mu$ set $V(z) = \mu$. Since $B_{\beta_+}(t > u)$ converges in the sense of Fréchet to $B_{\beta_+}$ as $t \downarrow u$, it follows from the theory of $\mu$-length that $V(z)$ as just defined over $R(N)$ takes on continuous boundary values on $\partial$. The boundary values of $V$ on $\partial$ as given by $\mu$-length on $B_{\beta_+}$ are strictly increasing in the positive sense of $u$ as derived from $F^*$. The case of an $x \in F^* B_{\beta_-}$ is similar.

II. Consider $x \in \partial \beta(N)$ with $z$ of type 1 relative to $R(N)$. In this case $z$ is the common end point of two $x \in F^* \beta(N)$, say $x'$ and $x''$. Then any sufficiently restricted open subarc of $x' \cup x''$ containing $z$ is in a unique $\mu$-curve $B_{\beta_+}$ or $B_{\beta_-}$ One defines $\varphi$ as in I and proves that $V(R(N))$, as defined on a neighbourhood of $z$ relative to $R(N)$ is continuous at $z$.

III. Consider $x \in \partial \beta(N)$ with $z$ of type 2 relative to $R(N)$. In this case there exists a unique $x \in (-1, 1)$ such that $z$ is the terminal point of $\beta_u$. With $N^*$ defined as in § 4 let $M^*$ be $N^* - (\mathcal{Z} \cup \beta_u)$. Then $z$ is in $\beta(N)^* \cap \beta(M^*) = \beta(N)^* \cup \beta(M^*) = \beta(N^*)$. But $V(\beta_u)$ is already defined, and extends both $\partial V(N^*)$ and $\partial V(M^*)$ continuously. As in I and II unique extensions of $\partial V(R(N))$ and $\partial V(M^*)$ over neighbourhoods of $z$ relative to $R(N^*)$ and $R(M^*)$ respectively exist and make $V$ continuous over such neighbourhoods. It follows that these two definitions of $V(\beta_u)$ and hence of $\varphi(z)$ agree, and make $\varphi(R(N))$ continuous over a neighbourhood of $z$ relative to $R(N)$.

The case of a $x \in f(N) - Z$ is similar and this completes the proof of Lemma 5.2.
The Existence of Pseudoconjugates

The open arc \( \eta_a \), sensed by \( F^a \), can be equally well parameterized by the values

\[ \mu(z) = \Gamma^a(z) |[z \in \eta_a], \]

where \( \mu(z) \) is the signed \( \mu \)-length, measured along \( \eta_a \) from the intersection of \( \eta_a \) with the closure of the principal transversal of \( \Sigma_a \). Values \( \mu(z) \) and \( t(z) \) which parameterize the same point \( z \in \eta_a \) stand in a relation

\[ t = T(\mu) \quad (\mu_1 < \mu < \mu_2), \]

where \( (\mu_1, \mu_2) \) is the finite range of \( V^a(z) \) over \( \eta_a \), and \( T \) is a continuous strictly increasing function mapping \( (\mu_1, \mu_2) \) onto the finite range \( (t_1, t_2) \) of \( t(z) \) over \( \eta_a \). Let \( T \) as defined over \( (\mu_1, \mu_2) \) be extended over the \( \mu \)-axis so as to map the \( \mu \)-axis homeomorphically onto the \( t \)-axis. Set

\[ V(z) = T(\mu) \quad (z \in \Sigma_a). \]

In particular this definition of \( V \) over \( \eta_a \) agrees with the definition of \( V \) over \( \eta_a \) as derived from the earlier definition of \( V(\Sigma_a) \). This completes the definition of \( V(\Sigma_a) \). It is clear that \( V \) is bounded and continuous over \( \Sigma_a \) with values on each \( \alpha \in F \) on which \( V \) has been defined which are strictly increasing in the sense of \( F^a \). A definition of \( V(\Sigma) \) is implied.

Theorem 6.1. Corresponding to any function \( U \) given as \( \Phi \) over \( \Sigma_a \), there exists a function \( V \), \( \Phi \) over \( \Sigma_a \) and \( PC \) to \( U \).

The function \( V \) as just defined over \( \Sigma_a \) is clearly \( \Phi \) over each \( \Sigma_a \) and there to \( U \). It remains to consider \( V \) in a neighbourhood of a point \( p \in \Sigma_a \). If \( p \) is \( F \)-non-singular we introduce a right \( \Sigma_a \) of \( p \) with canonical coordinates \( (u, v) \). We note that \( \eta_a \in N_u \) satisfies the condition \( u = 0 \). As defined \( V \) is increasing over each \( \alpha \in F^a(\Sigma_a) \) in the positive sense of \( \alpha \), by our inductive hypothesis. By construction \( V \) is increasing over each \( \alpha \in F^a(\Sigma_a) \) in the positive sense of \( \alpha \). In \( \Sigma_a \) this is the sense of increasing \( t \) by virtue of our choice of \( F^a \). It follows from Lemma 6.1 that \( V \) is \( PC \) to \( U \) on \( N_u \).

If \( p \in \eta_a \) is \( F \)-singular the preceding analysis shows that \( U + iW \) is continuous in a neighbourhood \( X_a \) of \( p \) and interior at each point of \( X_p \). It follows from Lemma 6.2 that \( U + iW \) is interior at \( p \). This completes the induction and the proof of the theorem.
§ 7. Uniformization of PH functions. A PH function is obtained locally from a harmonic function by composition with a topological transformation. The question naturally arises whether a similar result holds in the large. This is the case for a PH function when defined in a simply connected domain as stated in Theorem 7.1.

Theorem 7.1. Let $U$ be PH in a simply connected domain $J_u$ in the $z$-plane. There exists a homeomorphism $\psi$ from $J_u$ onto a domain $J_{u'}$ in the $w$-plane and a function $U^*$ harmonic on $J_{u'}$ such that $U = U^* \psi$.

Indeed $J_u$ is homeomorphic to $S - Z$ under a transformation $\psi$ from $J_u$ to $S - Z$. Hence $U = U^* \psi$ is PH over $S - Z$. By Theorem 6.1 there exists $V'$, PH over $S - Z$ and PC to $U'$. Then $V = V' \psi$ is PH over $J_u$ and PC to $U = U^* \psi$ [3, p. 39].

The function $U + iV = g$ is interior on $J_u$ and thus $z = g(z)$ maps $J_u$ on a simply connected Riemann surface $R_2$ spread over the $\zeta = (U + iV)$-plane. By the general uniformization theorem there exists a domain $J_{u'}$ in the $w$-plane and a function $f$ analytic on $J_{u'}$ such that $z = f(w)$ maps $J_{u'}$ onto $R_2$. The mapping $f^{-1}g$ is a homeomorphism $\phi$ of $J_{u'}$ onto $J_{u''}$ and if one sets $f = U^* + iV^*$ with $U^*$ and $V^*$ real, we have $g = f\phi$ and $U = U^* \phi$. This proves Theorem 7.1. It is clear that the theorem holds for a $U$ which is PH in any simply connected Riemann domain.

Recall that $V'$ and thus $V$ may be constructed so as to be bounded. If this is done the Riemann surface $R_2$ is of hyperbolic type. From this we deduce the following corollary.

Corollary 7.1. If $J_u$ is of hyperbolic type the mapping $\psi$ of Theorem 7.1 may be chosen so as to map $J_u$ onto itself.

Bibliography: