On The Third Symmetric Potency of $S_1$

By

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(From a letter to K. Borsuk.)

In your paper "On the third symmetric potency of the circumference," Fund. Math. 35 (1949), p. 236-244, you assert that the third symmetric potency of $S_1^3$ of the circle $S_1$ is homeomorphic to the Cartesian product of $S_1$ and the two sphere $S_2$. In your proof of this statement two parts should be distinguished. In the first (p. 237-243) and main part you show that $S_1^3$ can be obtained by a suitable identification of the boundaries of two anchor rings. In the second part (p. 244) you assert that the identification on the boundaries of these anchor rings is such, that the obtained manifold is homeomorphic to $S_1 \times S_2$. But in fact the identification you have made is incorrect and in consequence your final conclusion that the obtained manifold is homeomorphic to $S_1 \times S_2$ is false. A quite simple and short argument shows that $S_1^3$ has a vanishing fundamental group whence by your first result $S_1^3$ is a simply connected lensspace [1], i.e. the three sphere $S_3$. Your final theorem should therefore be corrected to read:

Theorem. The third symmetric potency $S_1^3$ of $S_1$ is homeomorphic with the three sphere $S_3$.

Proof that the fundamental group of $S_1^3$ vanishes.

If $X$ is a metric space, $X^3$ shall denote the space of sets of 3 or less points of $X$, topologized as in your paper [2].

Let $I$ be the unit interval. We obtain a model of $I^3$ by considering triplets of real numbers $(x,y,z)$ with $0 \leq x \leq y \leq z \leq 1$, where triplets of the type $(a,a,b)$ are to be identified with triplets of the type $(a,b,b)$. Hence the simplex of fig. 1 with the identifications

indicated represents $I^3$. $(0,1,3)$ is identified with $(0,2,3)$ by the simplicial map which sends

0 → 0,
1 → 2,
3 → 3.

Clearly $I^3$ is a 3 cell.

$S_1$ is obtained from $I$ by identifying the number 0 with the number 1. This induces an identification in our model of $I^3$ given by:

$(0,a,b) \sim (a,b,1)$; or equivalently the face $(0,1,2)$ is identified with the face $(1,2,3)$ by the simplicial map which sends

0 → 1,
1 → 2,
2 → 3.

In fig. 2 we have indicated the model obtained after these additional inductions.

We observe that the resulting complex consists of one 3 cell $e_3$, two 2 cells $e_2^1$ and $e_2^2$, two 1 cells $a$ and $\beta$, and a simple point $p$.

Going around what used to be the face $(0,1,2)$ we see that $a = a^2$ ($a^3 = 1$ (represents the identity of the fundamental group).

Hence $\beta$ does too, $(\beta = a^2!)$ and it follows that the fundamental group of $S_1^3$ is trivial.

You might be interested in the following proof that $S_1^3$ can be obtained by a suitable identification.
of the boundaries of two anchor rings, which is possibly a little simpler than yours [3].

Let $K$ denote the simplex of fig. 1 without any identifications and let $ABC$ (fig. 3) be the vertices of a triangle similar to the face 123 in the face (123). If $A', B', C'$ are the images of $A, B, C,$ under the simplicial map

$$
\begin{align*}
0 &\rightarrow 1, \\
1 &\rightarrow 2, \\
2 &\rightarrow 3,
\end{align*}
$$

denote by $U$ the convex hull of the points $\{A, B, C, A', B', C'\}$ in $K$. It is then obvious that the image of $U$ under $f$, the identification map which maps $K$ onto $S^3$, is an anchor ring. To see that $f(K-U)$ is again an anchor ring we proceed as follows. We first cut $W=K-U$ into the three cells shown in fig. 4. Fig. 4a shows the convex hull of the points $(1'B'C'B'E)$, fig. 4b the convex hull of the points $(O'A'B'C'O'C)$, while fig 4c describes the convex hull of $(O'A'C'B'O')$. Under $f$ the faces (013) and (OAC0) of fig. 4b are identified with the faces (023), and (OAC0) of fig. 4c respectively. If we perform these identifications in 3 space we obtain a 3 cell $W'$ which after

![Fig. 5.](image)

![Fig. 6.](image)
pyramid 6(c) has been drilled out. Adjoining 6(c) therefore completes our construction, for it is easily checked that all the identifications described by \( f \) have been carried out.

Bibliography.


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The Existence of Pseudoconjugates on Riemann Surfaces.

By

M. Morse and J. Jenkins.

§ 1. Introduction. Among the characteristics of a function \( U \) which is harmonic on a Riemann surface \( G^2 \) are the topological interrelations of the level lines of \( U \). One has merely to look at the level lines of \( \mathcal{R} \), \( \mathcal{R}^\prime \), \( \mathcal{R} \log \pi \), etc. to sense both complexity and order. The dual level lines of a conjugate \( V \) of \( U \) add to this order and complexity. It seems likely that outstanding problems in Riemann surface theory, such as the type problem, the nature of essential singularities, the existence of functions on the Riemann surface with restricted properties cannot be thoroughly understood in the absence of a complete analysis of the topological characteristics of these level lines.

Such a topological study properly belongs to a somewhat larger study namely that of \( \Phi \) (pseudoharmonic) functions and their pseudoconjugates (defined in § 2). A first problem is that of the existence of \( \Phi \) functions \( U \) on \( G^2 \) with prescribed level sets locally topologically like families of parallel straight lines except in the neighbourhood of points of a discrete set \( \omega \) of points \( z_0 \). At a point \( z_0 \) the level curves of \( U \) may cross after the manner of the level curves of a harmonic function with a critical point at \( z_0 \).

Let \( E \) be the finite \( z \)-plane. In case \( G^2 = E \) and \( \omega = 0 \), and excluding all recurrent level curves other than periodic curves, Kaplan has solved the above problem in [5]. More recently Boothby has extended Kaplan's result to the case of a general \( \omega \). As explained in [4] the a priori exclusion of recurrent level curves other than periodic curves does not seem justified. The writers of this paper have accordingly established the existence of \( \Phi \) functions without this hypothesis of non-recurrence [4].