A Proof of the Compactness Theorem for Arithmetical Classes *).

By

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This paper contains a simple mathematical proof of the following *compactness theorem for arithmetical classes* stated by Tarski:

(*) *If* K *is a set of arithmetical classes and* $\prod_{X\in \mathbf{K}} X=0$ [1], *then there is a finite set* $\mathbf{L}\subseteq\mathbf{K}$ *such that* $\prod_{X\in\mathbf{L}} X=0$.

The mathematical proof proposed by Tarski [1] is involved. The other proof is based on the metamathematical *completeness theorem* of Gödel[2]. The method used in this paper is a modification of the algebraic method of proving the Skolem-Löwenheim theorem[3]. A similar proof can be given for the analogous *compactness theorem for arithmetical functions*[4].

(**) *If* $\mathcal{K}$ *is a set of arithmetical functions and* $\bigcap_{F\in\mathcal{K}} F=\wedge$, *then there is a finite set* $\mathcal{L}\subseteq\mathcal{K}$ *such that* $\bigcap_{F\in\mathcal{L}} F=\wedge$.

*) Presented at the Seminar on Foundations of Mathematics in the State Institute of Mathematics in June 1951.

[1]) $\prod_{X\in\mathbf{K}} X$ and 0 denote the set-theoretical product and the empty set, respectively.

[2]) After having submitted my paper for publication I was informed by Professor A. Tarski that he has found another mathematical proof of this theorem.

[3]) See [2].

[4]) This theorem is stated in [1]. The proof of (**) differs from that of (*) by the lemmas 3.1 and 3.2. The dual ideal generated by a set $\mathcal{K}$ of arbitrary arithmetical functions does not preserve all the sums (3) (see 3.1 (ii)). By a simple modification of the lemmas 3.1 and 3.2 this difficulty may be avoided.

§ 1. Arithmetical functions and arithmetical classes [5].

Let $\mathcal{A}$ denote the set of all abstract algebras $\mathfrak{A}=\langle A,\circ\rangle$ [6], i. e. the set of all systems $\mathfrak{A}=\langle A,\circ\rangle$, where A is a non-empty set and $\circ$ is a binary operation class-closing on A. The set of all non-negative integers is denoted by ω, and the set of all infinite sequences $x=\langle x_0,x_1,...\rangle$ whose terms are in A is denoted by A^ω.

1.1. By $\boldsymbol{F}$ we shall denote the set of all functions F the domain of which is $\mathcal{A}$ (in symbols $D(F)=\mathcal{A}$) and such that $F(\mathfrak{A})\subseteq A^\omega$ for every $\mathfrak{A}=\langle A,\circ\rangle\in\mathcal{A}$.

1.2. By $I_{k,l}$ and $S_{k,l,m}$ $(k,l,m=0,1,2,...)$ we shall mean the functions defined as follows:

$$D(I_{k,l})=D(S_{k,l,m})=\mathcal{A}$$

and for every $\mathfrak{A}=\langle A,\circ\rangle$

$$I_{k,l}(\mathfrak{A})=\underset{x\in A^\omega}{E}(x_k=x_l)\ ^{7)};$$

$$S_{k,l,m}(\mathfrak{A})=\underset{x\in A^\omega}{E}(x_k\circ x_l=x_m).$$

1.3. Let $F,G\in\boldsymbol{F}$ and let $k=0,1,2,...$

(i) The *union* $F\cup G$ is the function H such that $D(H)=\mathcal{A}$ and $H(\mathfrak{A})=F(\mathfrak{A})+G(\mathfrak{A})$ for every $\mathfrak{A}\in\mathcal{A}$.

(ii) The *intersection* $F\cap G$, the *complement* $\overline{F}$, the *union* $\bigcup_{i\in I} H$ and the *intersection* $\bigcap_{i\in I} H_i$ are defined analogously (in terms of operations on sets of sequences).

(iii) The *outer cylindrification* $\bigvee_k F$ is the function H defined by the conditions: $D(H)=\mathcal{A}$ and, for every $\mathfrak{A}=\langle A,\circ\rangle$, $H(\mathfrak{A})$ is the set of all sequences $y\in A^\omega$ such that $x\in F(\mathfrak{A})$ for some sequence $x\in A^\omega$, which differs from y at most in its k-th term. In a similar way we define the *inner cylindrification* $\bigwedge_k F$, by replacing "for some sequence" with "for every sequence".

[5]) We shall use the terminology of Tarski (see [1]).

[6]) To avoid any appearence of antinomial construction we can consider only algebras $\mathfrak{A}=\langle A,\circ\rangle$ in which A is a subset of a certain infinite set $\bigvee^*$ fixed in advance. See [1].

[7]) The symbol $\underset{x\in A^\omega}{E}(x_k=x_l)$ denotes the set of all $x\in A^\omega$ such that $x_k=x_l$. The meaning of $\underset{x\in A^\omega}{E}(x_k\circ x_l=x_m)$ is analogous.

1.4. $\bigwedge$ and $\bigvee$ are functions such that $D(\bigwedge)=D(\bigvee)=\mathcal{A}$, and $\bigwedge(\mathfrak{A})=0$, $\bigvee(\mathfrak{A})=A^{\omega}$ for every $\mathfrak{A}=\langle A,\circ\rangle$.

1.5. We shall write $F\subseteq G$ if $F(\mathfrak{A})\subseteq G(\mathfrak{A})$ for every $\mathfrak{A}\in\mathcal{A}$.

1.6. The set $\boldsymbol{AF}$ of the *arithmetical functions* is the least set including $I_{k,l}$, $S_{k,l,m}$ for $k,l,m=0,1,2,\ldots$, and closed under the operations $\cup, -$ and $\bigvee_k$ for $k=0,1,2,\ldots$ It is easy to see that $\boldsymbol{AF}$ is likewise closed under the operations $\cap$ and $\bigwedge_k$ for $k=0,1,2,\ldots$

1.7. Given $F\in\boldsymbol{AF}$, let $\mathrm{Cl}(F)$ denote the set of all algebras $\mathfrak{A}=\langle A,\circ\rangle$ such that $F(\mathfrak{A})=A^{\omega}$.

1.8. By an *arithmetical class* we shall mean a set $S\subseteq\mathcal{A}$ such that $S=\mathrm{Cl}(F)$ for some $F\in\boldsymbol{AF}$. The set of all arithmetical classes will be denoted by $\boldsymbol{AC}$.

1.9. Given an arithmetical function F let $(F)^l_k$ $(k,l=0,1,2,\ldots)$ denote a function defined as follows: $D((F)^l_k)=\mathcal{A}$ and for every $\mathfrak{A}=\langle A,\circ\rangle$, $(F)^l_k(\mathfrak{A})$ is the set of all sequences $y\in A^{\omega}$ such that $x\in F(\mathfrak{A})$ for the sequences $x\in A^{\omega}$ defined by conditions $x_k=y_l$ and $x_i=y_i$ for $i\neq k$.

1.10. By *index* of $F\in\boldsymbol{AF}$ we shall understand the set $\mathrm{Ind}\,F$ of positive integers defined as follows:

$$\mathrm{Ind}\,F=\mathop{E}_{k\in\omega}(\bigvee_k F\neq F).$$

It is easy to see that $\mathrm{Ind}\,F=\mathop{E}_{k\in\omega}(\bigwedge_k F\neq F)$.

The following lemmas either are cited in [1] or are very simply derivable.

1.11. *The system* $\mathfrak{B}_0=\langle\boldsymbol{AF},\cup,\cap,-\rangle$ *is a denumerable Boolean algebra* [8]) (i. e. the power of $\mathfrak{B}_0$ is $\aleph_0$). $\bigwedge$ and $\bigvee$ are the *zero element* and the *unit element* of $\mathfrak{B}_0$, respectively; $\subseteq$ is the *inclusion relation* in $\mathfrak{B}_0$.

1.12. *Given* $F\in\boldsymbol{AF}$, *there is a* $G\in\boldsymbol{AF}$ *such that* $\mathrm{Cl}(F)=\mathrm{Cl}(G)$, *and* $\bigwedge_k G=G$ *for every* $k=0,1,2,\ldots$ This function will be called a *simple function*.

1.13. $I_{k,k}=\bigvee$, $I_{k,l}=I_{l,k}$, $I_{k,l}\cap I_{l,m}\subseteq I_{k,m}$.

[8]) The operations $\cup$, $\cap$, $-$ correspond to the Boolean operations of *join*, *meet*, and *complement*, respectively.

For every $F,G\in\boldsymbol{AF}$ *and* $k,l,m,n,p\in\omega$ *we have*

1.14. $\bigwedge_k F\cap\bigwedge_k G=\bigwedge_k(F\cap G)$.

1.15. $F\subseteq G$ *implies* $\bigwedge_k F\subseteq\bigwedge_k G$.

1.16. $\bigwedge_k(\overline{F}\cup G)\subseteq\overline{\bigvee_k F}\cup G$ *if* $k\,\overline{\in}\,\mathrm{Ind}\,G$.

1.17. $\bigwedge_k F=\overline{\bigvee_k\overline{F}}$.

1.18. $\bigvee_p((F)^p_k)=\bigvee_k F$ *if* $p\,\overline{\in}\,\mathrm{Ind}\,F$.

1.19. $\bigvee_m((F)^p_k)=(\bigvee_m F)^p_k$ *if* $p\neq m$ *and* $k\neq m$.

1.20. *Given* $F\in\boldsymbol{AF}$ *and* $k_0,l_0,\ldots,k_n,l_n\in\omega$, $(((F)^{l_0}_{k_0})\ldots)^{l_n}_{k_n}$ *is an arithmetical function*. (The proof by induction on the length of F is based on 1.18 and 1.19).

1.21. $\bigvee_m S_{k,l,m}=\bigvee$.

1.22. $S_{k,l,m}\cap S_{k,l,n}\subseteq I_{m,n}$.

1.23. $(F)^p_k\subseteq\bigvee_k F$ *for every* $p\in\omega$.

1.24. *If* $(F)^p_k\subseteq G$ *for every* $p\in\omega$, *then* $\bigvee_k F\subseteq G$.

1.25. $\bigvee_k F=\sum_{p\in\omega}(F)^p_k$ [9]) [from 1.23 and 1.24].

§ 2. Lemma on the existence of prime ideals in Boolean algebras [10]).

Let $\mathfrak{i}$ be a dual ideal [11]) of a Boolean algebra $\mathfrak{B}=\langle B,\cup,\cap,-\rangle$, let $a,a_\tau\in B$ for $\tau\in T$, and

$$a=\sum_{\tau\in T}a_\tau\quad\text{in}\quad\mathfrak{B}. \tag{1}$$

We shall say that the ideal $\mathfrak{i}$ *preserves* the sum (1) if $[a]=\sum_{\tau\in T}[a_\tau]$ in $\mathfrak{B}/\mathfrak{i}$, where, for every $b\in B$, $[b]$ is the element (of the quotient algebra $\mathfrak{B}/\mathfrak{i}$) determined by b.

[9]) $\sum_{p\in\omega}(F)^p_k$ denotes the Boolean sum in Boolean algebra $\mathfrak{B}_0$.

[10]) This lemma is due to R. Sikorski.

[11]) A *dual ideal* of a Bolean algebra $\mathfrak{B}=\langle B,\cup,\cap,-\rangle$ is a subset $\mathfrak{i}\subseteq B$ such that 1° if $a,b\in\mathfrak{i}$ then $a\cap b\in\mathfrak{i}$, 2° if $a\subseteq b$ and $a\in\mathfrak{i}$ then $b\in\mathfrak{i}$. The ideal $\mathfrak{i}$ is *proper* if $\mathfrak{i}\neq B$.

2.1. *Let* a_n *and* $a_{n,\tau}$ ($\tau \in T_n$, where T_n is an arbitrary set, $n = 0,1,2,\ldots$) *be elements of a Boolean algebra* $\mathfrak{B} = \langle B, \cup, \cap, - \rangle$ *such that*

(2) $$a_n = \sum_{\tau \in T_n} a_{n,\tau} \quad \text{in} \quad \mathfrak{B}$$

Then every proper dual ideal $\mathfrak{i}$ *of* $\mathfrak{B}$ *preserving all the sums* (2) *is contained in a prime dual ideal* $\mathfrak{p}$ *preserving all the sums* (2).

In fact, by hypothesis we have

$$[a_n] = \sum_{\tau \in T_n} [a_{n,\tau}] \quad \text{in} \quad \mathfrak{B}/\mathfrak{i}.$$

On account of lemma (iv) in [3] there is a prime dual ideal $\mathfrak{p}_0$ of $\mathfrak{B}/\mathfrak{i}$ which preserves all the sums (2). The prime dual ideal $\mathfrak{p}$ formed of all $a \in B$ such that $[a] \in \mathfrak{p}_0$, is the required one.

§ 3. Fundamental lemmas.

3.1. *Let* $\mathcal{K}$ *be a set of simple arithmetical functions. Let* $\mathfrak{i}$ *be the dual ideal of the Boolean algebra* $\mathfrak{B}_0 = \langle \boldsymbol{AF}, \cup, \cap, - \rangle$ *generated by* $\mathcal{K}$. *Then*

(i) *for every arithmetical function* G *the condition* $G \in \mathfrak{i}$ *implies* $\bigwedge_k G \in \mathfrak{i}$ ($k = 0,1,2,\ldots$),

(ii) $\mathfrak{i}$ *preserves all the sums*

(3) $$\sum_{p \in \omega} (F)_k^p = \bigvee_k F \qquad (F \in \boldsymbol{AF},\ k = 0,1,2,\ldots).$$

Proof. The remark (i) follows from the definition of the dual ideal and from 1.12, 1.14, 1.15.

By 1.23 we have

(4) $$[(F)_k^p] \subseteq [\bigvee_k F].$$

Suppose $[(F)_k^p] \subseteq [G]$ for every $p \in \omega$. Hence $\overline{(F)_k^p} \cup G \in \mathfrak{i}$ and by (i), $\bigwedge_p (\overline{(F)_k^p} \cup G) \in \mathfrak{i}$. In particular, this holds for such integers p, that p belong neither to Ind F nor to Ind G. We then have by 1.16, $\bigwedge_p (\overline{(F)_k^p} \cup G) \subseteq \overline{\bigvee_p ((F)_k^p)} \cup G$. Therefore $\overline{\bigvee_p ((F)_k^p)} \cup G \in \mathfrak{i}$. By 1.18, $\overline{\bigvee_k F} \cup G \in \mathfrak{i}$. Consequently,

(5) $$[\bigvee_k F] \subseteq [G].$$

(4) and (5) imply $[\bigvee_k F] = \sum_{p \in \omega} [(F)_k^p]$, which proves (ii).

3.2. *Let* $\mathcal{K}$ *be a set of arithmetical functions. If* $\mathcal{K}$ *is contained in a prime dual ideal* $\mathfrak{p}$ *of* $\mathfrak{B}_0 = \langle \boldsymbol{AF}, \cup, \cap, - \rangle$ *preserving all the sums* (3), *then* $\bigcap_{F \in \mathcal{K}} F \neq \Lambda$.

Proof. Suppose the assumptions of 3.2 are satisfied. Given an arithmetical function F, let $C(F, \mathfrak{A}; x)$ denote the characteristic function of the set $F(\mathfrak{A})$ (where $\mathfrak{A} = \langle A, \circ \rangle$), i. e.

$$C(F, \mathfrak{A}; x) = \begin{cases} 1 & \text{if} \quad x \in F(\mathfrak{A}) \\ 0 & \text{if} \quad x \,\bar{\in}\, F(\mathfrak{A}) \end{cases} \quad \text{for every } x \in A^{\omega}.$$

Clearly C may be interpreted as a function whose values belong to the two-element Boolean algebra $\mathfrak{B}_0/\mathfrak{p}$. Obviously

(6) $$C(F \cup G, \mathfrak{A}; x) = C(F, \mathfrak{A}; x) \cup C(G, \mathfrak{A}; x),$$

(7) $$C(\overline{F}, \mathfrak{A}; x) = -C(F, \mathfrak{A}; x),$$

(8) $$C(\bigvee_k F, \mathfrak{A}; x) = C(\sum_{p \in \omega} (F)_k^p, \mathfrak{A}; x) = \sum_{p \in \omega} C((F)_k^p, \mathfrak{A}; x).$$

Let k, l be arbitrary non-negative integers. By saying that $k \simeq l$, we shall mean that $I_{k,l} \in \mathfrak{p}$. By 1.13 the relation $\simeq$ is a congruence relation. Let $|k|$ denote the class of all $n \in \omega$ such that $n \simeq k$. Let A^* denote the class of all $|k|$, where $k \in \omega$. For every $|k|, |l|, |m| \in A^*$, let

$$|k| \circ |l| = |m| \quad \text{if and only if} \quad [S_{k,l,m}] = 1\ ^{12)} \quad \text{in} \quad \mathfrak{B}_0/\mathfrak{p}.$$

Obviously we have $|k| = |l|$ if and only if $[I_{k,l}] = 1$ in $\mathfrak{B}_0/\mathfrak{p}$. Making use of 1.21, 1.22 and of the fact that the ideal $\mathfrak{p}$ preserves all the sums (3), it is easy to show that the system $\mathfrak{A}^* = \langle A^*, \circ \rangle$ is an algebra. Consequently $\mathfrak{A}^* \in \mathcal{A}$.

Let $x^* \in A^{*\omega}$ denote the sequence $x_n = |n|$. Clearly

(9) $$C(S_{k,l,m}, \mathfrak{A}^*; x^*) = [S_{k,l,m}],$$

(10) $$C(I_{k,l}, \mathfrak{A}^*; x^*) = [I_{k,l}].$$

Since the ideal $\mathfrak{p}$ preserves all the sums (3), it is easy to show by a simple induction argument (making use of (6)-(10)) that $C(F, \mathfrak{A}^*; x^*) = [F]$ for every $F \in \boldsymbol{AF}$. In particular, if $F \in \mathcal{K}$ then $C(F, \mathfrak{A}^*; x^*) = 1$. Hence $x^* \in F(\mathfrak{A}^*)$, for every $F \in \mathcal{K}$. Consequently, $x^* \in \bigcap_{F \in \mathcal{K}} F(\mathfrak{A}^*)$. Therefore $\bigcap_{F \in \mathcal{K}} F \neq \Lambda$.

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12) 1 denotes the unit element of the quotient algebra $\mathfrak{B}_0/\mathfrak{p}$. For every $F \in \boldsymbol{AF}$, $[F]$ denotes the element of $\mathfrak{B}_0/\mathfrak{p}$ determined by F.

§ 4. Proof of the compactness theorem.

Theorem (*) is the immediate consequence of the following theorem:

(***) *If $\mathcal{K}$ is a set of simple arithmetical functions and* $\bigcap_{F\in\mathcal{K}} F=\wedge$, *then there is a finite set* $\mathcal{L}\subseteq\mathcal{K}$ *such that* $\bigcap_{F\in\mathcal{L}} F=\wedge$.

Proof. Suppose $\bigcap_{F\in\mathcal{K}} F=\wedge$ and for every finite $\mathcal{L}\subseteq\mathcal{K}$, $\bigcap_{F\in\mathcal{L}} F\neq\wedge$. Hence, for every finite $\mathcal{L}\subseteq\mathcal{K}$ the dual ideal of the algebra $\mathfrak{B}_0=\langle AF, \cup,\cap,-\rangle$ generated by $\mathcal{L}$ is proper. Consequently, the ideal generated by $\mathcal{K}$ is proper. By 3.1 this ideal preserves all the sums (3). Hence, by 2.1, it is contained in a prime dual ideal $\mathfrak{p}$ of $\mathfrak{B}_0$ preserving all the sums (3). In consequence, by 3.2, $\bigcap_{F\in\mathcal{K}} F\neq\wedge$, contrary to supposition.

References.

[1] A. Tarski[13]), *Some Notions and Methods on the Borderlinie of Algebra and Metamathematics*, Proceedings of the International Congress of Mathematicians **1** (1950), pp. 705-720.

[2] H. Rasiowa and R. Sikorski, *A Proof of the Skolem-Löwenheim Theorem*, Fundamenta Mathematicae **38** (1951), pp. 230-232.

[3] — *A Proof of the Completeness Theorem of Gödel*, Fundamenta Mathematicae **37** (1950), pp. 193-200.

[13]) I wish to thank Professor A. Tarski for the opportunity he gave me to see the manuscript of his paper.

Sur un problème concernant les coupures des régions par des continus.

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I. Préliminaires. Nous nous occupons dans cette note du problème suivant.

Imaginons sur la surface sphérique $\mathcal{S}_2$ (= plan euclidien augmenté du point à l'infini) k continus

(1) $$K_1, K_2, \ldots, K_k$$

et n régions (= ensembles ouverts connexes)

(2) $$R_1, R_2, \ldots, R_n.$$

Admettons que:

(i) les continus K_i sont disjoints deux à deux,

(ii) les régions R_j sont disjointes deux à deux,

(iii) pour tout couple i, j on a $K_i \cdot R_j \neq 0$,

(iv) aucune région R_j n'est une coupure de $\mathcal{S}_2$ (c'est-à-dire que l'ensemble $\mathcal{S}_2 - R_j$ est connexe).

Envisageons tout couple i, j tel que l'ensemble $R_j - K_i$ n'est pas connexe (c'est-à-dire que le continu K_i coupe la région R_j) et désignons par $s_{k,n}$ le nombre minimum de ces couples (pour K_i et R_j variables).

Il s'agit de calculer le nombre $s_{k,n}$.

L'hypothèse de M. Zarankiewicz est que

(3) $$s_{k,n}=(k-2)(n-2) \quad \text{pour} \quad k\geqslant 2 \text{ et } n\geqslant 2$$ [1]).

Nous nous proposons de démontrer la formule (3) pour le cas particulier où, *soit* $k\leqslant 4$, *soit* $n\leqslant 4$. Dans le cas général, le problème reste ouvert.

[1]) Le problème a été posé par M. Zarankiewicz pour le cas où $n=k$. La forme actuelle du problème est due à M. A. Rényi.