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Products of Abstract Algebras.

By

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There are four known operations on abstract algebras of a fixed type, each of which permits the construction of new algebras from given algebras. These operations are:

1^0 the taking of a subalgebra of an algebra,

2^0 the forming of a homomorphic image of an algebra by means of a congruence relation,

3^0 the forming of the direct union of algebras,

4^0 the forming of the limit algebra of an inverse or direct system of algebras.

In case of certain special algebras we also make other operations different from those in 1^0, 2^0, 3^0, 4^0. For instance, fields and σ-fields of sets may be considered as abstract algebras with respect to the set-theoretical operations. In Measure Theory we form some products and σ-products of fields or of σ-fields of sets respectively [1]); the forming of these products is different from the operations 1^0, 2^0, 3^0, 4^0. These products have been generalized by me for the case of Boolean algebras [2]). Topological spaces may also be considered as abstract algebras, called *closure algebras* [3]). The operation of forming of the Cartesian product of topological spaces is different from the general operations mentioned in 1^0, 2^0, 3^0, 4^0.

In this paper I shall define a new general operation on abstract algebras: the forming of the *product* of a family of abstract algebras of a fixed type (§ 3). In the case of fields of sets or σ-fields of sets this definition yields the usual products from Measure Theory. However, it may also be applied to groups, rings, lattices, Boolean algebras, etc. The notion of the product is related to the notion

[1]) See *e.g.* Halmos [1], Chapter VII.

[2]) Sikorski [5].

[3]) See *e.g.* Sikorski [6].

of a free algebra. The definition of the product can be so generalized that in the case of topological spaces it coincides with the usual definition of the Cartesian product of these spaces (§§ 5, 14, 15).

The first part of this paper contains general remarks on the product of abstract algebras. In the second are examined special kinds of algebras: groups, rings, lattices, fields of sets, Boolean algebras, closure algebras, etc.

I. General remarks.

1. *Abstract algebras.* We recall the fundamental notions[4]).

Let A be a non-empty set and let α be an ordinal. A transformation o which associates with every sequence $\boldsymbol{a}=\{a_\xi\}_{\xi<\alpha}$ $(a_\xi \epsilon A)$ an element $o(\boldsymbol{a}) \epsilon A$ is called an *operation of the type* α.

An *abstract algebra* is, by definition, an ordered system $\langle A; o_1, o_2, \dots, o_n\rangle$ where A is a non-empty set and $o_1, o_2, \dots, o_n$ are operations in A of the types $\alpha_1, \alpha_2, \dots, \alpha_n$ respectively[5]). Instead of "an abstract algebra $\langle A; o_1, \dots, o_n\rangle$" we shall often write, for brevity, "an algebra A", *i. e.* we shall denote an abstract algebra by the same letter as the set of its elements.

Two abstract algebras $\langle A; o_1, \dots, o_n\rangle$ and $\langle B; \bar{o}_1, \dots, \bar{o}_m\rangle$ are said to be *similar* if $n=m$ and if the corresponding operations o_i and $\bar{o}_i$ have the same type $(i=1,\dots,n)$. If A and B are similar, we shall denote the corresponding operations by the same letter o_i.

Let $\langle A; o_1, \dots, o_n\rangle$ and $\langle B; o_1, \dots, o_n\rangle$ be two similar algebras. A mapping h of A into B is said to be a *homomorphism* if $h(o_i(\boldsymbol{a}))=o_i(h(\boldsymbol{a}))$[6]) for $i=1,\dots,n$ and for all sequences $\boldsymbol{a}$ of the type α_i (= the type of o_i). A one-one homomorphism of A onto B is called an *isomorphism*. If it exists, A and B are said to be *isomorphic*.

If a set $S \subset A$ is closed under the operations $o_1, \dots, o_n$ (*i. e.* $o_i(\boldsymbol{a}) \epsilon S$ whenever all elements of $\boldsymbol{a}$ are in S), then $\langle S; o_1, \dots, o_n\rangle$ forms an abstract algebra called a *subalgebra* of $\langle A; o_1, \dots, o_n\rangle$.

A set $K \subset A$ is said to *generate a subalgebra* S of A if S is the least subalgebra of A such that $K \subset S$.

[4]) See Birkhoff [1], pp. vii-viii.

[5]) The hypothesis that the number of algebraical operations o_i is finite is not essential.

[6]) If $\boldsymbol{a}=\{a_\xi\}$, then $h(\boldsymbol{a})$ denotes the sequence $\{h(a_\xi)\}$.

(i) *If a set K generates a subalgebra A_0 of an algebra A, and if f is a mapping of K into an algebra B similar to A, then there is at most one homomorphism h of A_0 into B which is an extension of f, i. e. $h(a)=f(a)$ for $a \epsilon K$.*

(ii)[7]) *If sets K and L generate two similar algebras A and B respectively, and if f is a one-one mapping of K onto L such that*

(a) f may be extended to a homomorphism h of A into B,

(b) f^{-1} may be extended to a homomorphism g of B into A,

then h is an isomorphism of A onto B and $g=h^{-1}$.

2. *Factor algebras*[4]). Let $\langle B; o_1, \dots, o_n\rangle$ be an abstract algebra. A *congruence relation* in B is an equivalence relation $a\equiv b$ $(a, b \epsilon B)$ preserving all the operations $o_1, \dots, o_n$, *i. e.* if $\boldsymbol{a}=\{a_\xi\}$, $\boldsymbol{b}=\{b_\xi\}$ and $a_\xi \equiv b_\xi$ for all ξ, then $o_i(\boldsymbol{a})\equiv o_i(\boldsymbol{b})$ $(i=1,\dots,n)$.

Let $a\equiv b$ be a congruence relation in B. Denote by $\tilde{a}$ the set of all b such that $a\equiv b$. If $\boldsymbol{a}=\{a_\xi\}$ is a sequence of elements in A, then $\tilde{\boldsymbol{a}}$ denotes the sequence $\{\tilde{a}_\xi\}$. Let $\tilde{B}$ be the class of all $\tilde{a}$ $(a \epsilon B)$. Clearly $\langle \tilde{B}; o_1, \dots, o_n\rangle$ is an algebra with the following definition of operations

$$o_i(\tilde{\boldsymbol{a}}) = \widetilde{o_i(\boldsymbol{a})} \qquad (i=1,\dots,n).$$

$\tilde{B}$ is similar to B and the mapping $a\to\tilde{a}$ is a homomorphism of B onto $\tilde{B}$. $\tilde{B}$ is called the *factor algebra* determined by B and the congruence relation $\equiv$.

Now let $\langle B; o_1, \dots, o_n\rangle$ be an abstract algebra and let E be a subset of the Cartesian product[8]) $B\times B$. Let $\boldsymbol{R}$ be the class of all sets $R \subset B\times B$ such that

(a) $E \subset R$,

(b) the relation $(a,b) \epsilon R$ is a congruence relation in B.

The class $\boldsymbol{R}$ is non-empty since $B\times B \epsilon \boldsymbol{R}$. The intersection R_0 of all sets $R \epsilon \boldsymbol{R}$ also satisfies conditions (a) and (b), hence $R_0 \epsilon \boldsymbol{R}$. The congruence relation defined by the condition

$$a\equiv b \quad \text{if and only if} \quad (a,b) \epsilon R_0$$

is called the *congruence relation generated by the set* $E \subset B\times B$.

[7]) The very simple proof of (ii) is the same as that of Lemma 1.4 in Sikorski [5].

[8]) $B\times B$ is the set (no algebra!) of all pairs (a,b) where $a,b \epsilon B$.

3. *Products of abstract algebras.* Let $\mathfrak{A}$ be a class of similar abstract algebras $\langle A; o_1, \ldots, o_n \rangle$.

Let $\{A_\tau\}_{\tau \in T}$ be a family [9]) of abstract algebras in $\mathfrak{A}$. We shall say that an *algebra B is the $\mathfrak{A}$-product of all the algebras A_τ* $(\tau \in T)$ if $B \in \mathfrak{A}$ and if there is a family $\{B_\tau\}_{\tau \in T}$ of subalgebras of B such that

(A) the union of the sets B_τ $(\tau \in T)$ generates B;

(B) for every $\tau \in T$, B_τ is isomorphic to A_τ;

(C) if, for every $\tau \in T$, h_τ is a homomorphism of B_τ into any algebra $C \in \mathfrak{A}$, then there is a homomorphism h of B into C which is a common extension of all the homomorphisms h_τ (*i.e.*, $h_\tau(b) = h(b)$ for $b \in B_\tau$).

The $\mathfrak{A}$-product of all A_τ is not uniquely defined. If B is the $\mathfrak{A}$-product of all A_τ, and if B' is isomorphic to B, then B' is also the $\mathfrak{A}$-product of all A_τ. The converse statement is also true and follows easily from (ii) and (A-C):

(iii) *If B and B' are $\mathfrak{A}$-products of all A_τ, then B and B' are isomorphic.*

Thus the $\mathfrak{A}$-product B is defined up to the isomorphism type. Clearly the isomorphism type of the $\mathfrak{A}$-product B depends only on the isomorphism type of algebras A_τ. It may happen that the $\mathfrak{A}$-product of a given family $\{A_\tau\}_{\tau \in T}$ does not exist. A criterion for the existence of the $\mathfrak{A}$-product will be given in § 4, (viii).

The $\mathfrak{A}$-product is completely commutative and associative:

(iv) *If $t = t(\tau)$ is a one-one transformation of T onto itself, and if B and B' are $\mathfrak{A}$-products of families $\{A_\tau\}_{\tau \in T}$ and $\{A_{t(\tau)}\}_{\tau \in T}$ respectively, then B is isomorphic to B'.*

(v) [10]) *If $T = \sum_{u \in U} T_u$ $(T_u \neq 0)$, and if B_u is the $\mathfrak{A}$-product of $\{A_\tau\}_{\tau \in T_u}$, then the $\mathfrak{A}$-product of $\{B_u\}_{u \in U}$ is isomorphic to the $\mathfrak{A}$-product of $\{A_\tau\}_{\tau \in T}$* (whenever these products exist).

The exact proof is based on (ii).

It follows immediately from the definition that

(vi) *If $\mathfrak{A}_0 \subset \mathfrak{A}$, $A_\tau \in \mathfrak{A}_0$ $(\tau \in T)$, $A \in \mathfrak{A}_0$, and if A is the $\mathfrak{A}$-product of all A_τ, then A is also the $\mathfrak{A}_0$-product of all A_τ.*

[9]) It is not supposed that $A_\tau \neq A_{\tau'}$, for $\tau \neq \tau'$. The letter T always denotes a fixed non-empty set.

[10]) The simple proof of (v) is similar to that of Theorem 7.2 in Sikorski [5].

4. *Free algebras.* Let $\mathfrak{A}$ be a class of similar algebras.

An algebra $B \in \mathfrak{A}$ is said to be *$\mathfrak{A}$-free* [11]) if there is a set $K \subset B$ such that

(A) K generates B;

(B) every mapping of K into any algebra $C \in \mathfrak{A}$ can be extended to a homomorphism of B into C.

Elements $a \in K$ are called *free generators* of the $\mathfrak{A}$-free algebra B.

Clearly, if $K_0 \subset K$ and $B_0 \subset B$ is the subalgebra generated by K_0, then K_0 and B_0 satisfy also the condition (B).

The following two theorems show the connection between the notion of the $\mathfrak{A}$-free algebra and of the $\mathfrak{A}$-product of algebras.

(vii) *Suppose there exists an $\mathfrak{A}$-free algebra A_0 with one free generator. An algebra $B \in \mathfrak{A}$ is $\mathfrak{A}$-free with $\mathfrak{m}$ free generators if and only if it is the $\mathfrak{A}$-product of $\mathfrak{m}$ replicas of the algebra A_0.*

The easy proof is left to the reader.

(viii) *Let $\{A_\tau\}_{\tau \in T}$ be a family of algebras in $\mathfrak{A}$, each algebra A_τ being generated by a set $A_\tau^0 \subset A_\tau$, $\overline{\overline{A_\tau^0}} = \mathfrak{m}_\tau$. If*

(a) [12]) *there is an algebra $D \in \mathfrak{A}$ which, for every $\tau \in T$, contains a subalgebra D_τ isomorphic to A_τ;*

(b) *there is an $\mathfrak{A}$-free algebra B with $\mathfrak{m}$ free generators, $\mathfrak{m} \geqslant \sum_{\tau \in T} \mathfrak{m}_\tau$;*

(c) *every factor algebra $\widetilde{B}$ determined by B and a congruence relation is in $\mathfrak{A}$;*

then the $\mathfrak{A}$-product of all A_τ $(\tau \in T)$ exists.

Let B^0 be the set of free generators of B, $B^0 = \sum_{\tau \in T} B_\tau^0$, $\overline{\overline{B_\tau^0}} \geqslant \mathfrak{m}_\tau$, $B_\tau^0 \cdot B_{\tau'}^0 = 0$ if $\tau \neq \tau'$, and let B_τ be a subalgebra of B generated by B_τ^0. Let f_τ^0 be a mapping of B_τ^0 onto A_τ^0. The mapping f_τ^0 may be extended to a homomorphism f_τ of B_τ onto A_τ. Let E_τ be the set of all pairs $(a, b) \in B_\tau \times B_\tau \subset B \times B$ such that $f_\tau(a) = f_\tau(b)$. Let $E = \sum_{\tau \in T} E_\tau$, and let $\equiv$ be the congruence relation generated in B by the set E (see § 2).

[11]) See Birkhoff [1], pp. viii-ix.

[12]) This condition is essential. *E. g.* if $\mathfrak{B}$ is the class of all Boolean algebras, $A_1, A_2 \in \mathfrak{B}$, A_1 is degenerate and A_2 is not degenerate, than the $\mathfrak{B}$-product of A_1 and A_2 does not exist (see § 11 and footnote [13])). However the conditions (b) and (c) are fulfilled.

We shall prove that the factor algebra $\widetilde{B}$ determined by B and the congruence relation $\equiv$ is the $\mathfrak{A}$-product of all A_τ. Let $\widetilde{B}_\tau$ be the class of all $\widetilde{b}\in\widetilde{B}$ such that $b\in B_\tau$. It is sufficient to prove that

1^0 arbitrary homomorphisms h_τ of $\widetilde{B}_\tau$ into an algebra $C\in\mathfrak{A}$ can be extended to a homomorphism h of $\widetilde{B}$ into C;

2^0 $\widetilde{B}_\tau$ is isomorphic to A_τ.

Ad 1^0. Let $g_\tau^0(b)=h_\tau(\widetilde{b})$ for $b\in B_\tau^0$. There is a homomorphism g of B into C such that

$$g(b)=g_\tau^0(b)\quad \text{for}\quad b\in B_\tau^0.$$

Clearly, by (i),

$$g(b)=h_\tau(\widetilde{b})\quad \text{for}\quad b\in B_\tau.$$

The set R of all $(a,b)\in B\times B$ such that $g(a)=g(b)$ satisfies the condition (b) from § 2. If $(a,b)\in E_\tau$, then $\widetilde{a}=\widetilde{b}$ and

$$g(a)=h_\tau(\widetilde{a})=h_\tau(\widetilde{b})=g(b).$$

Hence $E_\tau\subset R$ for every $\tau\in T$, and consequently $E\subset R$, *i. e.* R also satisfies condition (a) of § 2. Therefore

$$\text{if}\quad a\equiv b,\quad \text{then}\quad g(a)=g(b).$$

Putting $h(\widetilde{b})=g(b)$ we obtain a homomorphism h of $\widetilde{B}$ into C which is a common extension of all h_τ.

Ad 2^0. Let h_τ be an isomorphism of A_τ onto D_τ and let

$$g_\tau^0(b)=h_\tau(f_\tau^0(b))\quad \text{for}\quad b\in B_\tau^0.$$

There is a homomorphism g of B into D such that $g(b)=g_\tau^0(b)$ for $b\in B_\tau^0$. Consequently, by (i),

$$g(b)=h_\tau^{-}(f_\tau(b))\quad \text{for every}\quad b\in B_\tau.$$

Let R be the set of all $(a,b)\in B\times B$ such that $g(a)=g(b)$. If $(a,b)\in E_\tau$, *i. e.* if $f_\tau(a)=f_\tau(b)$, then $g(a)=g(b)$. Hence $E_\tau\subset R$ for every $\tau\in T$, and consequently $E\subset R$, *i. e.* R satisfies conditions (a) and (b) of § 2. Therefore if $a\equiv b$, then $g(a)=g(b)$. Hence the formula

$$\widetilde{g}(\widetilde{b})=g(b)\quad \text{for}\quad b\in B$$

defines a homomorphism $\widetilde{g}$ of $\widetilde{B}$ into D. Clearly $\widetilde{g}_\tau=\widetilde{g}|\widetilde{B}_\tau$ maps $\widetilde{B}_\tau$ onto D_τ since the set $h_\tau(A_\tau^0)=g(B_\tau^0)$ generates D_τ.

Consequently $k_\tau=h_\tau^{-1}\widetilde{g}_\tau$ is a homomorphism of $\widetilde{B}_\tau$ onto A_τ such that $k_\tau(\widetilde{b})=f_\tau(b)$ for $b\in B_\tau$. If $f_\tau(a)=f_\tau(b)$ $(a,b\in B_\tau)$, then $a\equiv b$. Consequently, if $\widetilde{a}\neq\widetilde{b}$, *i. e.* if $a\not\equiv b$, then $f_\tau(a)\neq f_\tau(b)$, and finally $k_\tau(\widetilde{a})\neq k_\tau(\widetilde{b})$. We infer that k_τ is an isomorphism of $\widetilde{B}_\tau$ onto A_τ, Q. E. D.

A class $\mathfrak{A}$ of similar algebras $\langle A;o_1,\dots,o_n\rangle$ is said to be *closed* if it is the class of all abstract algebras satisfying a set of axioms, each of which is an equation between two polynomials formed by means of the operations $o_1,\dots,o_n$.

If $\mathfrak{A}$ is closed, then (see Birkhoff [2]):

(1) for any cardinal $\mathfrak{m}$, there is an $\mathfrak{A}$-free algebra with $\mathfrak{m}$ free generators [13];

(2) every factor algebra $\widetilde{B}$ determined by an algebra $B\in\mathfrak{A}$ and by any congruence relation also belongs to $\mathfrak{A}$;

(3) every subalgebra of an algebra $A\in\mathfrak{A}$ also belongs to $\mathfrak{A}$;

(4) the full direct union D of a family of algebras $A_\tau\in\mathfrak{A}$ $(\tau\in T)$ also belongs to $\mathfrak{A}$ (D is the class of all systems $\{a^\tau\}_{\tau\in T}$ (where $a^\tau\in A_\tau$) with the operations:

$$o_i(\{\boldsymbol{a}^\tau\})=\{o_i(\boldsymbol{a}^\tau)\}$$

where $\boldsymbol{a}^\tau=\{a_\xi^\tau\}_{\xi<\alpha_i}$).

It follows immediately from (viii) that

(ix) *If $\mathfrak{A}$ is closed, and if every algebra $A_\tau\in\mathfrak{A}$ $(\tau\in T)$ contains a one-element subalgebra (e_τ), then the $\mathfrak{A}$-product of all A_τ $(\tau\in T)$ exists.*

It is sufficient to remark that the full direct union D of all A_τ has subalgebras D_τ isomorphic to A_τ. In fact, the subalgebra D_{τ_0} formed of all $\{a^\tau\}$ where $a^\tau=e_\tau$ for $\tau\neq\tau_0$ is isomorphic to A_{τ_0}.

5. *The* $(\mathfrak{A},\Phi)$*-product.* Let $\mathfrak{A}$ be a class of similar algebras, and let Φ be a class of transformations such that

(a) each $f\in\Phi$ maps a subalgebra of an algebra $A\in\mathfrak{A}$ into an algebra $B\in\mathfrak{A}$;

(b) if f maps A onto B $(A,B\in\mathfrak{A})$, and if $f\in\Phi$ and $f^{-1}\in\Phi$, then f is an isomorphism of A onto B;

(c) if $f,g\in\Phi$, then the superposition $fg\in\Phi$ also (whenever this superposition is feasible);

(d) if h is an isomorphism of an algebra $A\in\mathfrak{A}$ onto a subalgebra $B_0\subset B\in\mathfrak{A}$, then $h\in\Phi$ and $h^{-1}\in\Phi$;

[13]) For the case of finite operations, see Birkhoff [1], p. viii. The construction described there holds also for infinite operations.

(e) for every mapping f of a subset $K \subset A \in \mathfrak{A}$ into an algebra $B \in \mathfrak{A}$ there is at most one transformation $g \in \Phi$ which is an extension of f over the least subalgebra generated by K.

Let $\{A_\tau\}_{\tau \in T}$ be a family of abstract algebras in $\mathfrak{A}$. We shall say that an *algebra B is the $(\mathfrak{A},\Phi)$-product of all A_τ* $(\tau \in T)$ if $B \in \mathfrak{A}$ and if there is a family $\{B_\tau\}_{\tau \in T}$ of subalgebras of B such that

(A) the union of all sets B_τ generates B;

(B) for every $\tau \in T$, B_τ is isomorphic to A_τ;

(C) if, for every $\tau \in T$, $f_\tau \in \Phi$ is a transformation of B_τ into any algebra $C \in \mathfrak{A}$, then there is a transformation $f \in \Phi$ which is the common extension of all f_τ.

It is easy to see that if Φ is the class of all homomorphisms between subalgebras of algebras in $\mathfrak{A}$, then the $(\mathfrak{A},\Phi)$-product coincides with the $\mathfrak{A}$-product.

The $(\mathfrak{A},\Phi)$-product is determined up to the isomorphism type only.

It follows from (a-e) that Theorem (ii) remains true after replacing the word "a homomorphism" by "a mapping in Φ". Consequently, Theorems (iii), (iv), (v) and (vi) are also true for $(\mathfrak{A},\Phi)$-products.

II. Examples.

6. ***Groups.*** Let $\mathfrak{G}$ and $\mathfrak{G}_a$ be respectively the class of all groups and the class of all abelian groups. The $\mathfrak{G}$-product and the $\mathfrak{G}_a$-product always exist, by (ix).

It is easy to see that the $\mathfrak{G}_a$-product of a family of abelian groups $\{G_\tau\}_{\tau \in T}$ (written additively) is the weak direct union of the groups G_τ, *i. e.* the class of all finite sums[14])

$$a_1 + \ldots + a_n \qquad (a_i \in G_{\tau_i},\ \tau_i \neq \tau_j \text{ for } i \neq j),$$

where the order of the summands makes no difference. The group operations are defined in the obvious way.

The $\mathfrak{G}$-product of arbitrary groups $\{G_\tau\}_{\tau \in T}$ (written multiplicatively) is the free product of these groups, *i. e.* the class of all finite words[14])

$$a_1 \cdot \ldots \cdot a_n \qquad (a_i \in G_{\tau_i},\ \tau_i \neq \tau_{i+1}),$$

where $ab \neq ba$ if $e \neq a \in G_\tau$, $e \neq b \in G_{\tau'}$, $\tau \neq \tau'$. The group operations are defined in the obvious way.

[14]) We assume here, for simplicity, that, for $\tau \neq \tau'$, $G_\tau \cdot G_{\tau'} = (e)$ where e is the zero (unit) element common to all groups.

7. ***Algebraic rings.*** Let $\mathfrak{R}$ and $\mathfrak{R}_a$ be respectively the class of all algebraic rings and the class of all commutative algebraic rings. The $\mathfrak{R}$-product and the $\mathfrak{R}_a$-product always exist on account of (ix).

The $\mathfrak{R}_a$-product of a family $\{K_\tau\}_{\tau \in T}$ of commutative rings is the set of all finite sums of products[15])

$$a_1 \cdot \ldots \cdot a_n \qquad (a_i \in K_{\tau_i}),$$

where the order of factors and summands is of no consequence.

The $\mathfrak{R}$-product of arbitrary rings can be defined in a similar way.

8. ***Lattices.*** Let $\mathfrak{L}$ be the class of all lattices. The $\mathfrak{L}$-product of any family $\{L_\tau\}_{\tau \in T}$ of lattices exists by (ix).

9. ***Distributive lattices.*** Let $\mathfrak{L}_d$ be the class of all distributive lattices. The $\mathfrak{L}_d$-product of any family $\{L_\tau\}_{\tau \in T}$ of distributive lattices exists on account of (ix). It can be described immediately as follows.

We shall assume the following notations. Let $\{\mathcal{Y}_\tau\}_{\tau \in T}$ be a family of non-empty sets. The letter $\mathcal{Y}$ will always denote the Cartesian product of all sets $\mathcal{Y}_\tau$. If $Y \subset \mathcal{Y}_\tau$, then Y^* will denote the set of all points in $\mathcal{Y}$ whose τ-th coordinate is in Y. If $\boldsymbol{Y}_\tau$ is a class of subsets of $\mathcal{Y}_\tau$, then $\boldsymbol{Y}_\tau^*$ will denote the class of all sets Y^* where $Y \in \boldsymbol{Y}_\tau$.

Let $\mathcal{Y}_\tau$ be the class of all prime ideals of L_τ, and let $\mathfrak{s}_\tau(a)$ (for $a \in L_\tau$) be the set of all prime ideals $\mathfrak{p} \in \mathcal{Y}_\tau$ such that a non $\in \mathfrak{p}$. Let $\boldsymbol{Y}_\tau$ be the class of all sets $\mathfrak{s}_\tau(a)$ $(a \in L_\tau)$. $\boldsymbol{Y}_\tau$ is a distributive lattice with respect to the set-theoretical union and intersection, and the mapping $Y = \mathfrak{s}_\tau(a)$ is the isomorphism of L_τ onto $\boldsymbol{Y}_\tau$[16]). Take the Cartesian product $\mathcal{Y}$ of all $\mathcal{Y}_\tau$ and apply the notations mentioned above. Let $\boldsymbol{Y}$ be the least class of subsets of $\mathcal{Y}$ such that

1⁰ if $U, V \in \boldsymbol{Y}$, then $U + V \in \boldsymbol{Y}$ and $UV \in \boldsymbol{Y}$;

2⁰ $\boldsymbol{Y}_\tau^* \subset \boldsymbol{Y}$ for every τ.

The class $\boldsymbol{Y}$ is a distributive lattice which is the $\mathfrak{L}_d$-product of all L_τ.

[15]) We assume here that, for $\tau \neq \tau'$, $K_\tau \cdot K_{\tau'}$ contains only the zero element common for all rings.

[16]) This representation of a distributive lattice is due to Stone [1]. See also Rieger [2]. A *prime ideal* is a set $\mathfrak{p} \subset L$ such that: (1) if $a, b \in \mathfrak{p}$, then $a \cup b \in \mathfrak{p}$; (2) if $a \in \mathfrak{p}$, $b \in L$, $b \subset a$, then $b \in \mathfrak{p}$; (3) if $a \cap b \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

In fact, L_τ is isomorphic to $\boldsymbol{Y}_\tau^*$. Let h_τ be a homomorphism of $\boldsymbol{Y}_\tau^*$ into a distributive lattice $\boldsymbol{C}$. We may assume that $\boldsymbol{C}$ is a distributive lattice of subsets of a set $\mathfrak{X}$ with the set-theoretical operations. The mapping $h_\tau(\mathfrak{s}_\tau(a)^*)$ is an homomorphisms of L_τ into $\boldsymbol{C}$. For every $x \in \mathfrak{X}$ let $\mathfrak{p} = \varphi_\tau(x)$ be the prime ideal of L_τ defined as follows

$$a \in \mathfrak{p} \quad \text{if and only if} \quad x \operatorname{non} \in h_\tau(\mathfrak{s}_\tau(a)^*).$$

φ_τ is a mapping of $\mathfrak{X}$ into $\mathcal{Y}_\tau$ and

$$\varphi_\tau^{-1}(Y) = h_\tau(Y^*) \quad \text{for} \quad Y \in \boldsymbol{Y}_\tau.$$

Let φ be the mapping $\varphi(x) = \{\varphi_\tau(x)\}$ of $\mathfrak{X}$ into $\mathcal{Y}$ and let $h(Y) = \varphi^{-1}(Y)$ for $Y \in \boldsymbol{Y}$. Clearly h is a homomorphism of $\boldsymbol{Y}$ into $\boldsymbol{C}$ which is the extension of all h_τ.

10. *Fields of sets.* Let $\mathfrak{F}$ be the class of all fields of sets. Algebraical operations in algebras $\in \mathfrak{F}$ are the set-theoretical union $X_1 + X_2$ and the complementation X'. The $\mathfrak{F}$-product exists for any family of fields $\{\boldsymbol{Y}_\tau\}$ of subsets of sets $\mathcal{Y}_\tau \neq 0$ respectively. Using the notation described in § 9, the $\mathfrak{F}$-product of all $\boldsymbol{Y}_\tau$ is the least field containing all fields $\boldsymbol{Y}_\tau^*$. For the proof see my paper [3].

Analogously, let $\mathfrak{F}_\mathfrak{m}$ be the class of all $\mathfrak{m}$-additive fields[17]) of sets considered as algebras with the set-theoretical operations X' and $\sum_{\xi < \omega_\alpha} X_\xi$ where $\overline{\overline{\omega}}_\alpha = \mathfrak{m}$. The $\mathfrak{F}_\mathfrak{m}$-product exists for any family of $\mathfrak{m}$-additive fields $\boldsymbol{Y}_\tau$ of subsets of sets $\mathcal{Y}_\tau \neq 0$. The $\mathfrak{F}_\mathfrak{m}$-product is the least $\mathfrak{m}$-additive field of sets containing all the fields $\boldsymbol{Y}_\tau^*$. The proof in the case $\mathfrak{m} = \aleph_0$ is given in my paper [3]. If $\mathfrak{m} > \aleph_0$, the proof is analogous.

11. *Boolean algebras.* Let $\mathfrak{B}$ and $\mathfrak{B}_\mathfrak{m}$ denote respectively the class of all Boolean algebras and of all $\mathfrak{m}$-complete Boolean algebras[18]). Algebraical operations in the class $\mathfrak{B}$ are the Boolean union $A_1 + A_2$ and the complementation A'. Algebraical operations in $\mathfrak{B}_\mathfrak{m}$ are A' and the infinite Boolean union $\sum_{\xi < \omega_\alpha} A_\xi$ where $\overline{\overline{\omega}}_\alpha = \mathfrak{m}$.

[17]) In the sequel $\mathfrak{m}$ denotes a fixed infinite cardinal. A field of sets $\boldsymbol{X}$ is $\mathfrak{m}$-*additive* if the conditions $X_u \in \boldsymbol{X}$, $\overline{\overline{U}} \leqslant \mathfrak{m}$ imply that the set-theoretical union $\sum_{u \in U} X_u \in \boldsymbol{X}$. An $\aleph_0$-additive field is called also a *σ-field*.

[18]) Boolean algebras will be denoted by letters $A, B, C, \ldots$; their elements by $A, B, C, \ldots$ A Boolean algebra $\boldsymbol{A}$ is $\mathfrak{m}$-complete, if the Boolean union $\sum_{u \in U} A_u$ always exists whenever $\overline{\overline{U}} \leqslant \mathfrak{m}$ ($A_u \in \boldsymbol{A}$). Instead of "$\aleph_0$-complete" we shall also write "σ-complete".

The existence of the $\mathfrak{B}$-product of any family $\{\boldsymbol{A}_\tau\}$ of non-degenerate[19]) Boolean algebras follows from § 10 and Stone's representation theorem[20]).

The existence of the $\mathfrak{B}_\mathfrak{m}$-product of any family $\{\boldsymbol{A}_\tau\}$ of non-degenerate[19]) $\mathfrak{m}$-complete Boolean algebras follows from (viii). In fact, there is an $\mathfrak{m}$-complete Boolean algebra with $\mathfrak{n}$-free generators where $\mathfrak{n}$ is any cardinal[21]). The class of all $\mathfrak{m}$-complete Boolean algebras is closed, which implies (viii) (c). Given a family $\{\boldsymbol{A}_\tau\}$ of non-degenerate $\mathfrak{m}$-complete Boolean algebras, there is a Boolean algebra $\boldsymbol{D}$ which satisfies the condition (a) of (viii). It is sufficient to put $\boldsymbol{D}$ = MacNeille's minimal extension of the $\mathfrak{B}$-product $\boldsymbol{B}$ of all $\boldsymbol{A}_\tau$, since the isomorphism of $\boldsymbol{A}_\tau$ onto the subalgebra $\boldsymbol{B}_\tau \subset \boldsymbol{B} \subset \boldsymbol{D}$ preserves all infinite Boolean operations[22]).

In the case $\mathfrak{m} = \aleph_0$ the above-mentioned $\mathfrak{B}_{\aleph_0}$-product of σ-complete Boolean algebras is isomorphic to the σ-maximal product examined in my paper [5] (see theorem 6.4)[23]). If all $\boldsymbol{A}_\tau$ are σ-fields of sets, then the $\mathfrak{B}_{\aleph_0}$-product $\boldsymbol{B}$ of all $\boldsymbol{A}_\tau$ does not coincide, in general, with the $\mathfrak{F}_{\aleph_0}$-product $\boldsymbol{F}$ of all $\boldsymbol{A}_\tau$[24]). If $\boldsymbol{B} \neq \boldsymbol{F}$, then $\boldsymbol{B}$ is not isomorphic to a σ-field of sets (see the remark at the end of § 12 in my paper [5]).

12. $\mathfrak{m}$-*quotient algebras.* Let $\mathfrak{Q}_\mathfrak{m}$ be the class of all $\mathfrak{m}$-*quotient algebras*, *i. e.* Boolean algebras isomorphic to a quotient algebra $\boldsymbol{Y}/\boldsymbol{J}$ where $\boldsymbol{Y}$ is an $\mathfrak{m}$-additive field of subsets of a set $\mathcal{Y} \neq 0$, and $\boldsymbol{J}$ is an $\mathfrak{m}$-additive ideal of sets[25]). The element of $\boldsymbol{Y}/\boldsymbol{J}$ determined by a set $Y \in \boldsymbol{Y}$ will be denoted by $Y/\boldsymbol{J}$.

[19]) A Boolean algebra is *non-degenerate* if it contains at least two elements. It is *degenerate*, if it contains only one element.

If all A_τ are degenerate, then the $\mathfrak{B}$-product exists and it is degenerate. If some A_τ are degenerate and other A_τ are non-degenerate, the $\mathfrak{B}$-product of all A_τ does not exist.

[20]) Stone [2], pp. 98 and 106. See also Sikorski [5], Theorem 6.2.

[21]) Rieger [1], p. 37.

[22]) See Sikorski [5], Theorem 9.3.

[23]) It should be outlined that in my paper [5] all Boolean algebras under consideration are supposed to be non-degenerate.

[24]) A simple counter example is given in my paper [4], p. 17. Using the notation mentioned there, A is the $\mathfrak{F}_{\aleph_0}$-product of A' and A''. It follows from § 12 that C_σ/J is the $\mathfrak{B}_{\aleph_0}$-product of A' and A''. Clearly C_σ/J is not isomorphic to A since C_σ/J is isomorphic to no σ-field of sets (see my paper [1], Theorem 2.4).

[25]) Clearly the algebraical operations in $\mathfrak{Q}_\mathfrak{m}$ are the same as in $\mathfrak{B}_\mathfrak{m}$.

An ideal $\boldsymbol{I}$ is $\mathfrak{m}$-*additive* if $\sum_{u \in U} X_u \in \boldsymbol{I}$ whenever $\overline{\overline{U}} \leqslant \mathfrak{m}$ and $X_u \in \boldsymbol{I}$ for every $u \in U$.

The $\mathfrak{Q}_{\mathfrak{m}}$-product exists for every family $\{\boldsymbol{A}_\tau\}$ of non-degenerate $\mathfrak{m}$-quotient algebras. This can be proved immediately as follows.

We shall say that an $\mathfrak{m}$-*quotient algebra* $\boldsymbol{Y}/\boldsymbol{J}$ *has the property* $(H_{\mathfrak{m}})$ if, for any homomorphism h of $\boldsymbol{Y}/\boldsymbol{J}$ into any $\mathfrak{m}$-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ ($\boldsymbol{X}$ is an $\mathfrak{m}$-additive field of subsets of a set $\mathcal{X}\neq 0$, $\boldsymbol{I}$ is an $\mathfrak{m}$-additive ideal), there is a mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ which induces h, *i. e.*

$$h(Y/\boldsymbol{J})=\varphi^{-1}(Y)/\boldsymbol{I} \quad \text{for every} \quad Y\in\boldsymbol{Y}.$$

For instance, if $\boldsymbol{Y}=$ the least $\mathfrak{m}$-additive field generated by open-closed subsets of a generalized Cantor set, then $\boldsymbol{Y}/\boldsymbol{J}$ has the property $(H_{\mathfrak{m}})$ for each ideal $\boldsymbol{J}$[26]).

We may assume that $\boldsymbol{A}_\tau=\boldsymbol{Y}_\tau/\boldsymbol{J}_\tau$ where $\boldsymbol{Y}_\tau/\boldsymbol{J}_\tau$ has the property $(H_{\mathfrak{m}})$ (we may assume[27]) *e. g.* that $\boldsymbol{Y}_\tau$ is the least $\mathfrak{m}$-additive field generated by open-closed subsets of a generalized Cantor set). $\boldsymbol{Y}_\tau$ and $\boldsymbol{J}_\tau$ are respectively an $\mathfrak{m}$-additive field of subsets of a set $\mathcal{Y}_\tau$, and an $\mathfrak{m}$-additive ideal. Let $\mathcal{Y}$ be the Cartesian product of all spaces $\mathcal{Y}_\tau$. Employ the notations of § 9. Let $\boldsymbol{Y}$ be the least $\mathfrak{m}$-additive field containing all field $\boldsymbol{Y}_\tau^*$, and let $\boldsymbol{J}$ be the least $\mathfrak{m}$-additive ideal containing all ideals $\boldsymbol{J}_\tau^*$.

We shall prove that $\boldsymbol{Y}/\boldsymbol{J}$ is the $\mathfrak{Q}_{\mathfrak{m}}$-product of all $\boldsymbol{Y}_\tau/\boldsymbol{J}_\tau$.

The mapping $Y/\boldsymbol{J}_\tau\to Y^*/\boldsymbol{J}_\tau$ is a homomorphism of $\boldsymbol{Y}_\tau/\boldsymbol{J}_\tau$ onto $\boldsymbol{Y}_\tau^*/\boldsymbol{J}_\tau$. It is an isomorphism. In fact, if $Y\in\boldsymbol{Y}_{\tau_0}$ and $Y \operatorname{non}\in \boldsymbol{J}_{\tau_0}$, then

$$W=Y^*-(Z^*+\sum_{\tau\in U}Z_\tau^*)\neq 0$$

for arbitrary sets $Z\in\boldsymbol{J}_{\tau_0}$ and $Z_\tau\in\boldsymbol{J}_\tau$ where $U\subset T-(\tau_0)$ is a set of power $\leqslant\mathfrak{m}$. Let $y_{\tau_0}^0\in Y-Z$, and let $y_\tau^0\in\mathcal{Y}_\tau-Z_\tau$ $(\tau\in U)$. Each point $y=\{y_\tau\}\in\mathcal{Y}$ such that $y_\tau=y_\tau^0$ for $\tau\in U+(\tau_0)$ belongs to W.

Now let h_τ be any homomorphism of $\boldsymbol{Y}_\tau^*/\boldsymbol{J}$ into $\boldsymbol{C}=\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is an $\mathfrak{m}$-additive field of subsets of a set $\mathcal{X}\neq 0$ and $\boldsymbol{I}$ is an $\mathfrak{m}$-additive ideal. Since h_τ may be interpreted as a homomorphism of $\boldsymbol{Y}_\tau/\boldsymbol{J}_\tau$, there exists a mapping φ_τ of $\mathcal{X}$ into $\mathcal{Y}_\tau$ such that

$$h_\tau(Y^*/\boldsymbol{J})=\varphi^{-1}(Y)/\boldsymbol{I} \quad \text{for} \quad Y\in\boldsymbol{Y}_\tau.$$

[26]) This can easily be proved by means of the generalized characteristic function of a transfinite sequence of sets. For the case $\mathfrak{m}=\aleph_0$, see Sikorski [8], Theorem 6.2. The proof in the case $\mathfrak{m}>\aleph_0$ is similar.

[27]) Sikorski [7].

$\varphi(x)=\{\varphi_\tau^*(x)\}$ is a mapping of $\mathcal{X}$ into $\mathcal{Y}$. The formula

$$h(Y/\boldsymbol{J})=\varphi^{-1}(Y)/\boldsymbol{I} \quad \text{for} \quad Y\in\boldsymbol{Y}$$

defines a homomorphism of $\boldsymbol{Y}/\boldsymbol{J}$ into $\boldsymbol{X}/\boldsymbol{I}$ which is the common extension of all h_τ, Q. E. D.

A Boolean algebra is an $\aleph_0$-quotient algebra if and only if it is σ-complete[28]). The above-defined $\mathfrak{Q}_{\aleph_0}$-product of non-degenerate σ-complete Boolean algebras coincides with the $\mathfrak{B}_{\aleph_0}$-product and with the σ-maximal product defined in my paper [5]. The construction of the product described above is slightly different from that in my paper [5].

13. ***σ-complete Boolean algebras without neutral elements.*** An element A of a σ-complete Boolean algebra $\boldsymbol{A}$ is said to be *neutral* if $\mu(A)=0$ for every σ-measure μ defined on $\boldsymbol{A}$.

Let $\mathfrak{B}_n$ be the class of all σ-complete Boolean algebras which contain no neutral element different from the zero element[29]). The $\mathfrak{B}_n$-product $\boldsymbol{A}$ of a family $\boldsymbol{A}_\tau\in\mathfrak{B}_n$ $(\tau\in T)$ always exists. In fact $\boldsymbol{A}=\boldsymbol{B}/\boldsymbol{I}$ where $\boldsymbol{B}$ is the $\mathfrak{B}_{\aleph_0}$-product of all $\boldsymbol{A}_\tau$, and $\boldsymbol{I}$ is the σ-ideal of all neutral elements of $\boldsymbol{B}$. The easy proof is left to the reader.

Ryll-Nardzewski gave an example[30]) of two σ-fields of sets $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ such that their $\mathfrak{B}_n$-product $\boldsymbol{A}$ is different from their $\mathfrak{F}_{\aleph_0}$-product. Consequently $\boldsymbol{A}$ is not isomorphic to a σ-field of sets.

14. ***Topological spaces.*** Each topological space[31]) $\mathcal{X}$ may be interpreted as an abstract algebra denoted by $\boldsymbol{S}(\mathcal{X})$. Elements of $\boldsymbol{S}(\mathcal{X})$ are all subsets of $\mathcal{X}$. The algebraical operations in $\boldsymbol{S}(\mathcal{X})$ are

(1) complementation $X'=\mathcal{X}-X$ $(X\subset\mathcal{X})$;

(2) addition $\sum_\xi X_\xi$ of an arbitrary transfinite sequence of subsets $X_\xi\subset\mathcal{X}$;

(3) the closure operation $\overline{X}$ $(X\subset\mathcal{X})$.

The class of all such algebras will be denoted by $\mathfrak{T}$.

Since the type of the sequence $\{X_\xi\}$ in the operation (2) is not fixed and may be arbitrary, the algebras $\boldsymbol{S}(\mathcal{X})$ are not abstract algebras in the sense of the definition in § 1. However, we can easily generalize the definition in § 1 so that some operations,

[28]) Loomis [1], p. 757, and Sikorski [1], p. 256.

[29]) The algebraical operations in $\mathfrak{B}_n$ are assumed to be the same as in $\mathfrak{B}_{\aleph_0}$.

[30]) Not published.

[31]) By a topological space we shall mean a set $\mathcal{X}$ with a closure operation satisfying axioms I-IV in § 15, p. 226.

feasible on every transfinite sequence of elements, will be admitted; and it is easy to see that the definition and the fundamental properties of the product also hold after this generalization. Therefore we can apply the results of §§ 1-5 to the case of the class $\mathfrak{T}$.

On the other hand, if we are concerned with a given family $S(\mathcal{X}_\tau)\in\mathfrak{T}$ $(\tau\in T)$, instead of the unrestricted unions (2) we can consider only the union of sequences $\{X_\xi\}$ of a fixed type ω_α, where ω_α is a sufficiently great initial ordinal such that $\aleph_\alpha\geqslant$ the cardinal of any class composed of subsets of the spaces $\mathcal{X}_\tau$ and of their product [32]). In fact, each union of sets under consideration can then be reduced to the union of a sequence of sets of the type ω_α. This is the second reason which permits the application of the notions and theorems of §§ 1-5 to the case of the class $\mathfrak{T}$.

According to § 1, a subalgebra of $S(\mathcal{X})\in\mathfrak{T}$ is a completely additive subfield [33]) of $S(\mathcal{X})$ closed with respect to the closure operation. A homomorphism of $S(\mathcal{Y})$ into $S(\mathcal{X})$ ($\mathcal{X},\mathcal{Y}$ — topological spaces) is a mapping h such that

(1) $$h(\mathcal{Y}-Y)=\mathcal{X}-h(Y)\quad\text{for}\quad Y\subset\mathcal{Y};$$

(2) $$h(\sum_\xi Y_\xi)=\sum_\xi h(Y_\xi)\text{ for each transfinite sequence } Y_\xi\subset\mathcal{Y};$$

(3) $$h(\overline{Y})=\overline{h(Y)}\quad\text{for}\quad Y\subset\mathcal{Y}.$$

It follows from (1) and (2) that there is a point mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ such that

(4) $$h(Y)=\varphi^{-1}(Y)\quad\text{for}\quad Y\subset\mathcal{Y}.$$

h is an isomorphism of $S(\mathcal{Y})$ onto $S(\mathcal{X})$ if and only if φ is one-to-one and $\varphi(\mathcal{X})=\mathcal{Y}$. Condition (3) implies that φ is then a homeomorphism. Conversely, if φ is a homeomorphism of $\mathcal{X}$ onto $\mathcal{Y}$, then equation (4) defines an isomorphism h of $S(\mathcal{Y})$ onto $S(\mathcal{X})$.

If h is any homomorphism, then it can be proved that the transformation φ in (4) is open (*i. e.* φ is continuous, and $\varphi(U)$ is open in $\mathcal{Y}$ for every set U open in $\mathcal{X}$). Conversely if φ is an open mapping of $\mathcal{X}$ into $\mathcal{Y}$, then equation (4) defines a homomorphism of $S(\mathcal{Y})$ into $S(\mathcal{X})$.

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[32]) In the construction of the $(\mathfrak{T},\Phi_t)$-product on p. 225 it is sufficient to assume that $\aleph_\alpha\geqslant\exp\,(\prod_{\tau\in T}\overline{\overline{\mathcal{Y}}}_\tau)$.

[33]) A field F is *completely additive* if the set-theoretical union of any family of sets $Y\in F$ also belongs to F.

I do not know whether the $\mathfrak{T}$-product of a family $S(\mathcal{Y}_\tau)\in\mathfrak{T}$ exists. The $\mathfrak{T}$-product is defined by an extension property of homomorphisms, *i. e.* of open mappings between topological spaces. No such extension property of open mappings is known. However, open mappings form a very special class of mappings between topological spaces. The natural class which should be considered here is the class of all continuous mappings. Therefore, instead of homomorphisms we shall consider the class Φ_t of transformations induced by continuous mappings.

More exactly we define the class Φ_t as follows: $f\in\Phi_t$ if and only if simultaneously

(a) f is defined on a subalgebra $A\subset S(\mathcal{Y})\in\mathfrak{T}$; the values of f are in an algebra $S(\mathcal{X})\in\mathfrak{T}$;

(b) there is a continuous mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ such that $f(Y)=\varphi^{-1}(Y)$ for every $Y\in A$.

It is easy to verify that the class Φ_t satisfies conditions (a-e) of § 5.

If $S(\mathcal{Y}_\tau)\in\mathfrak{T}$, $\mathcal{Y}_\tau\neq 0$ for every $\tau\in T$, then the $(\mathfrak{T},\Phi_t)$-product of the family $\{S(\mathcal{Y}_\tau)\}_{\tau\in T}$ exists. More exactly, if $\mathcal{Y}$ is the Cartesian product of the spaces $\mathcal{Y}_\tau$ with the usual topology, then $S(\mathcal{Y})$ is the $(\mathfrak{T},\Phi_t)$-product of all $S(\mathcal{Y}_\tau)$. In other words, the $(\mathfrak{T},\Phi_t)$-product of topological spaces $\mathcal{Y}_\tau$ coincides with the Cartesian product of these spaces.

To prove this statement, use the notations of § 9. Clearly $S(\mathcal{Y}_\tau)^*$ is isomorphic to $S(\mathcal{Y}_\tau)$ and all $S(\mathcal{Y}_\tau)^*$ generate $S(\mathcal{Y})$. Let $f_\tau\in\Phi_t$ be a transformation of $S(\mathcal{Y}_\tau)^*$ into an $S(\mathcal{X})\in\mathfrak{T}$ $(\tau\in T)$. Then there is a continuous mapping ψ_τ of $\mathcal{X}$ into $\mathcal{Y}$ such that

$$f_\tau(Y^*)=\psi_\tau^{-1}(Y^*)\quad\text{for every}\quad Y\subset\mathcal{Y}_\tau.$$

Let $\varphi_\tau(x)$ be the projection of $\psi_\tau(x)$ on the axis $\mathcal{Y}_\tau$. The mapping $\varphi_\tau(x)$ of $\mathcal{X}$ into $\mathcal{Y}_\tau$ is continuous and

$$f_\tau(Y^*)=\varphi^{-1}(Y)\quad\text{for}\quad Y\subset\mathcal{Y}_\tau.$$

Consequently $\varphi(x)=\{\varphi_\tau(x)\}$ is a continuous mapping of $\mathcal{X}$ into $\mathcal{Y}$ and the formula

$$f(Y)=\varphi^{-1}(Y)\quad\text{for}\quad Y\subset\mathcal{Y}$$

defines a transformation $f\in\Phi_t$ which is a common extension of all f_τ.

15. *Borel C-algebras.* A *closure algebra* [34]) is a σ-complete Boolean algebra $\boldsymbol{A}$ in which there is defined a closure operation $\bar{A}$ determined for all $A \in \boldsymbol{A}$ and such that

$$\text{I. } \overline{A+B}=\bar{A}+\bar{B}, \qquad \text{II. } \bar{0}=0,$$
$$\text{III. } A\subset\bar{A}, \qquad \text{IV. } \overline{(\bar{A})}=\bar{A}.$$

A closure algebra $\boldsymbol{A}$ is said to be a *C-algebra* [35]) if

V. there is an enumerable sequence of open elements $R_n \in \boldsymbol{A}$ such that every open $G \in \boldsymbol{A}$ is the sum of all R_n such that $R_n \subset G$.

A C-algebra is said to be a *Borel C-algebra* if the least σ-subalgebra containing all open elements of $\boldsymbol{A}$ is the algebra $\boldsymbol{A}$ itself. For instance, the class $\boldsymbol{B}(\mathfrak{X})$ of all Borel subsets of a separable metric space is a Borel C-algebra.

Let $\mathfrak{C}$ be the class of all Borel C-algebras [36]), and let Φ_c be the class of all *continuous Boolean homomorphisms* [37]), *i. e.* the class of all transformations h of a subalgebra of an $\boldsymbol{A} \in \mathfrak{C}$ into another $\boldsymbol{B} \in \mathfrak{C}$ such

$$\sum_{n=1}^{\infty} h(A_n)=\sum_{n=1}^{\infty} h(A_n),$$
$$h(A')=h(A)',$$

and

$$\overline{h(A)}\subset h(\bar{A}).$$

The $(\mathfrak{C}, \Phi_c)$-product of any finite or enumerable family of non-degenerate algebras $\boldsymbol{A}_\tau \in \mathfrak{C}$ $(\tau \in T,\ \overline{\overline{T}} \leqslant \aleph_0)$ always exists.

In fact, we may suppose [38]) that $\boldsymbol{A}_\tau=\boldsymbol{B}(\mathcal{Y}_\tau)/\boldsymbol{J}_\tau$ where $\mathcal{Y}_\tau$ is a separable complete metric space and $\boldsymbol{J}_\tau$ is a suitable σ-ideal. Let $\mathcal{Y}$ be the Cartesian product of all $\mathcal{Y}_\tau$. Assume the notations of §§ 9 and 12. Let $\boldsymbol{J}$ be the least σ-ideal generated by all σ-ideals $\boldsymbol{J}_\tau^*$.

The Borel C-algebra $\boldsymbol{B}(\mathcal{Y})/\boldsymbol{J}$ is the $(\mathfrak{C}, \Phi_c)$-product of all $\boldsymbol{A}_\tau$.

In fact, the subalgebra $\boldsymbol{B}(\mathcal{Y}_\tau)^*/\boldsymbol{J}$ is isomorphic to $\boldsymbol{A}_\tau$ since the mapping

$$Y/\boldsymbol{J}_\tau \to Y^*/\boldsymbol{J} \quad \text{for} \quad Y \in \boldsymbol{B}(\mathcal{Y}_\tau)$$

[34]) See Sikorski [6], p. 170.

[35]) See Sikorski [6], p. 182.

[36]) Clearly the algebraical operations in $\mathfrak{C}$ are: the enumerable Boolean union $\sum_{n=1}^{\infty} A_n$, the complementation A', and the closure operation $\bar{A}$.

[37]) See Sikorski [6], p. 175.

[38]) Sikorski [6], Theorem 15.2.

is an isomorphism of $\boldsymbol{A}_\tau$ onto $\boldsymbol{B}(\mathcal{Y}_\tau)^*/\boldsymbol{J}$ (see § 12). Clearly all subalgebras $\boldsymbol{B}(\mathcal{Y}_\tau)^*/\boldsymbol{J}$ generate $\boldsymbol{B}(\mathcal{Y})/\boldsymbol{J}$.

Let $f_\tau \in \Phi_c$ be a transformation of $\boldsymbol{B}(\mathcal{Y}_\tau)^*/\boldsymbol{J}$ into any Borel C-algebra $\boldsymbol{C}$. We may suppose [38]) that $\boldsymbol{C}=\boldsymbol{B}(\mathfrak{X})/\boldsymbol{I}$ where $\mathfrak{X}$ is a separable metric space, and $\boldsymbol{I}$ is a σ-ideal. Since f_τ may be interpreted as a Boolean σ-homomorphism [39]) of $\boldsymbol{A}_\tau$ into $\boldsymbol{C}$, and since $\boldsymbol{A}_\tau$ has the property $(H_{\aleph_0})$ [40]) (see § 12), there is a mapping φ_τ of $\mathfrak{X}$ into $\mathcal{Y}_\tau$ such that

$$f_\tau(Y^*/\boldsymbol{J})=\varphi^{-1}(Y)/\boldsymbol{I} \quad \text{for every} \quad Y \in \boldsymbol{B}(\mathcal{Y}_\tau).$$

There exists a set $X_\tau \in \boldsymbol{I}$ such that $\varphi_\tau | \mathfrak{X}-X_\tau$ is continuous [41]).

The transformation $\varphi(x)=\{\varphi_\tau(x)\}$ of $\mathfrak{X}$ into $\mathcal{Y}$ induces the Boolean σ-homomorphism f of $\boldsymbol{B}(\mathcal{Y})/\boldsymbol{J}$ into $\boldsymbol{C}$:

$$f(Y/\boldsymbol{J})=\varphi^{-1}(Y)/\boldsymbol{I} \quad \text{for} \quad Y \in \boldsymbol{B}(\mathcal{Y})$$

which is a common extension of all f_τ. Since φ is continuous on the set $\mathfrak{X}-\sum_\tau X_\tau$ and $\sum_\tau X_\tau \in \boldsymbol{I}$, the Boolean σ-homomorphism f is continuous [41]), *i. e.* $f \in \Phi_c$, Q. E. D.

It is possible that the $(\mathfrak{C}, \Phi_c)$-product of Borel C-algebras $\boldsymbol{B}(\mathcal{Y}_\tau)$ (where $\mathcal{Y}_\tau$ — separable metric spaces) is not isomorphic to a σ-fields of sets. However, if all $\mathcal{Y}_\tau$ are absolute Borel sets [42]), then the $(\mathfrak{C}, \Phi_c)$-product of an at most enumerable family $\boldsymbol{B}(\mathcal{Y}_\tau)$ is the Borel C-algebra $\boldsymbol{B}(\mathcal{Y})$ where $\mathcal{Y}$ is the Cartesian product of all $\mathcal{Y}_\tau$. This follows from the fact that in the above construction the assumption that $\mathcal{Y}_\tau$ are complete can be replaced by the hypothesis that $\mathcal{Y}_\tau$ are absolute Borel sets [40]).

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[39]) That is, a homomorphism with respect to the operations admitted in $\mathfrak{B}_{\aleph_0}$.

[40]) Sikorski [2], Theorem 5.1 c).

[41]) Sikorski [6], Theorem 21.1.

[42]) Y is an *absolute Borel set* if it is homeomorphic to a Borel subset of the Hilbert cube.

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On Continuous Mappings on Cartesian Products.

By

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In every Hausdorff space we can distinguish two different topologies: the original *neighbourhood topology* and the *sequential topology*. The sequential topology is determined by the concept of a convergent sequence defined in the neighbourhood topology [1]). These two topologies are not, in general, equivalent (under sequential topology the space is only a Fréchet $\mathcal{L}^*$-space). The equivalence holds if the space satisfies the first axiom of countability [2]).

Let A and B be two Hausdorff spaces. By a *continuous mapping* of A into B we shall always understand a mapping Φ continuous in the neighbourhood topology, that is: for every neighbourhood V of $\Phi(a)$ there is a neighbourhood U of $a \in A$ with $\Phi(U) \subset V$. We shall say that a mapping Φ of A into B is *sequentially continuous* if it is continuous in the sequential topology of A and B, *i. e.* if $a = \lim a_n$ in A implies $\Phi(a) = \lim \Phi(a_n)$ in B.

The two above notions of continuity, corresponding to the classical definitions of Cauchy and Heine respectively, are not, in general, equivalent. Continuity always implies sequential continuity; the converse is true only under certain additional hypotheses, *e. g.* if A satisfies the first axiom of countability, in particular if A satisfies the second axiom of countability [3]) or if A is metrizable.

In this paper it will be shown that the equivalence of neighbourhood and sequential continuity holds also if the space B has the property

[1]) We write $a = \lim a_n$ if every neighbourhood of a contains all elements a_n except a finite number.

[2]) That is, for each point a there is a sequence of its neighbourhoods $\{U_n\}$ such that if U is any neighbourhood of a, then $U_n \subset U$ for an integer n.

[3]) That is, the space possesses an enumerable open basis.