(α) If $\lambda=\beta+1$, then denote by p_λ the first element of (1) which is in the set

$$E-[\{\bigcup_{\xi\leqslant\eta_\beta} h_\xi^\beta\}\cup\{p_\xi|\xi<\lambda\}].$$

Since $\overline{\overline{\bigcup_{\xi\leqslant\eta_\beta} h_\xi^\beta}}\leqslant\overline{\eta}_\beta\aleph_\nu<\overline{\overline{E}}$, the element p_λ certainly exists.

(β) If λ is a limit number, then let p_λ be the first element of (1) which is in the set $E-\{p_\xi|\xi<\lambda\}$.

Let $M=\{p_\xi|\xi<\omega_\nu\}$. The set M is an element of H, and yet for no p_ξ is $p_\xi RM$. This is so since if $p_\xi RM$, then from the manner in which $p_{\xi+1}$ was selected, $p_{\xi+1}$ cannot be in M. As this is a contradiction, it follows that for some element y in E, there are at least $\aleph_\alpha$ sets h for which yRh, q. e. d.

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[1] P. Erdös, *Some remarks on set theory*, Proceedings of the American Mathematical Society **1** (1950), p. 127-141.

[2] G. Fodor and I. Ketskemety, *Some theorems on the theory of sets*, Fundamenta Mathematicae **37** (1950), p. 249-250.

[3] F. Hausdorff, *Mengenlehre*, p. 34.

A generalization of a theorem of Miss Anna Mullikin.

By

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Introduction. In 1923 Miss Mullikin proved the following theorem [1]):

Let $F_1, F_2, \dots$ be a sequence of mutually disjoint closed subsets of the euclidean plane E_2, such that for every i, E_2-F_i is connected, then $E_2-\Sigma F_i$ is connected.

A new and simpler proof of this theorem was given by S. Mazurkiewicz in Fundamenta Mathematicae **6** (1924), pp. 37-38.

The above mentioned theorem may in an obvious way be formulated as a property of the 2-sphere, and as such it appears essentially as a theorem of Phragmen-Brouwer type. Generalizations in this direction of Miss Mullikin's theorem seem to have received but little attention hitherto.

Recently the simplification and modernization of Miss Mullikin's proof of her theorem was proposed as a problem by Wiskundig Genootschap at Amsterdam [2]) (apparently in ignorance of Mazurkiewicz's proof). The present author succeeded in giving such a proof and at the same time generalized Miss Mullikin's theorem for n dimensions (equally ignorant of Mazurkiewicz's article). Owing to several useful hints of Prof. H. Freudenthal he realized that with the same methods the generalizations indicated here below (cf. Theorems I, II, III) could be proved. The method of the proof of theorem II strongly parallels Mazurkiewicz's arguments.

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[1]) A. M. Mullikin, *Certain Theorems Relating to Plane Connected Point Sets*, Trans. Am. Math. Soc. **24** (1923), pp. 144-162.

[2]) Programma van jaarlijkse prijsvragen, 1950.

Notations and Conventions.

$S_n = n$-dimensional sphere.
$E_n = n$-dimensional euclidean space. $\Big\}$ $(n \geqslant 2)$
$\emptyset =$ void set.
$\sim\; =$ "homologous to"-relation.
continuum = compact connected set.

Chaingroups, groups of cycles etc. will be taken over a field for convenience.

Throughout the following sections A will denote a fixed closed subset of S_n and $\mathfrak{z}$ a fixed cycle of A of dimension $n-k$.

If $k=n$, then $\mathfrak{z}$ is taken to be a 0-dimensional cycle ~ 0 in S_n.

1. A subset F of S_n is said to be an *Lk-set* (with respect to the pair $A, \mathfrak{z}$), or $\mathfrak{z}$ is said to be *linked with* F, if

(1) F is closed,

(2) $F \cdot A = \emptyset$,

(3) $\mathfrak{z}$ not ~ 0 in $S_n - F$, where ~ 0 is equivalent with "~ 0 on a compact subset $B \supset A$ of $S_n - F$" [3]).

F is said to be a *0-Lk-set* if (1) and (2) still hold and if $\mathfrak{z} \sim 0$ in $S_n - F$, *i. e.* if there exists a compact subset $B \supset A$ of $S_n - F$ with $\mathfrak{z} \sim 0$ on B.

So *Lk*- and 0-*Lk*-sets are automatically understood to be closed.

If F is a 0-*Lk*-set and $F' \subset F$ a closed subset of F, then F' is a 0-*Lk*-set.

If F is an *Lk*-set, and $F' \supset F$ a closed set not meeting A, then F' is an *Lk*-set.

A closed subset of S_n not meeting A is either an *Lk*-set or a 0-*Lk*-set.

For 0-*Lk*-sets we have

Theorem of Phragmen-Brouwer-Alexandroff. *Let F_1 and F_2 be 0-Lk-sets, such that $F_1 \cdot F_2$ is $(k-2)$-acyclic, or if $k=2$ such that $F_1 \cdot F_2$ is connected, then $F_1 + F_2$ is a 0-Lk-set.*

In the following section we shall prove the following two theorems (stated separately though the first is a special case of the second):

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[3]) The Čech theory will be adopted as the homology theory for compact spaces.

Theorem I. *Let $F_1, F_2, \ldots$ be a sequence of mutually disjoint 0-Lk-sets and F a compact subset of ΣF_i, then F is a 0-Lk-set.*

Theorem II. *Let $F_1, F_2, \ldots$ be a sequence of 0-Lk-sets such that there is a fixed point $p \notin A$, with $F_i \cdot F_j \subset (p)$ for $i \neq j$. Let F be a compact subset of ΣF_i, then F is a 0-Lk-set.*

By omitting p, S_n becomes E_n, and theorem II may be rephrased as follows:

Theorem IIa. *Let A be a compact subset of E_n, and $\mathfrak{z}$ a cycle of A. Let $F_1, F_2, \ldots$ be a sequence of mutually disjoint closed subsets of E_n, each of which is not linked with $\mathfrak{z}$. Then, if F is a closed (with respect to E_n) subset of ΣF_i, F is also not linked with $\mathfrak{z}$.*

From theorem II Miss Mullikin's theorem may be derived directly. We shall however obtain it as simple consequence of theorem III (cf. section 5).

As may be expected beforehand the proofs of theorems I and II proceed on parallel lines, with the only difference that the proof of theorem II involves a certain property (σ) of irreducible *Lk*-sets which is not needed in the proof of theorem I.

2. ***Proposition 1.*** *Let $F_1, F_2, \ldots$ be a sequence of Lk-sets such that* $\limsup F_i = F$ [4]) *does not meet A. Then F is an Lk-set.*

Proof. Suppose F to be a 0-*Lk*-set. Then there exists a compact subset $B \supset A$ of $S_n - F$ such that $\mathfrak{z} \sim 0$ on B. Since $B \cdot F = \emptyset$, there exists an open neighbourhood U of F with $U \cdot B = \emptyset$. F being $\limsup F_i$, U contains almost all F_i, and hence almost all F_i would be 0-*Lk*-sets, which contradicts our hypotheses.

Corollary 1.1. *If F_i is a convergent sequence with limit F* [4]) *under the same hypotheses as above, then F is an Lk-set.*

Corollary 1.2. *The intersection F of a non-increasing sequence $F_1 \supset F_2 \supset \ldots$ of Lk-sets is an Lk-set.*

Proof. In this case we have $F = \lim F_i$. Furthermore since $F_i A = \emptyset$ we have $F \cdot A = \emptyset$. Hence we may apply corollary 1.1.

Corollary 1.3. *Any Lk-set F contains an irreducible Lk-set, i. e. an Lk-set F_0 such that no closed proper subset of F_0 is an Lk-set.*

Proof. The assertion is immediately obtained from corollary 1.2 by application of Brouwer's reduction theorem.

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[4]) The concept of lim sup is taken in the sense of topological *limes superior* (cf. F. Hausdorff, *Mengenlehre*, 3-rd ed., pp. 147, 149, 150, 163).

Proposition 2. *Let F and F' be closed subsets of S_n. Let F' be a $(k-2)$-acyclic set, or, if $k=2$, a connected set. Then $F+F'$ is an Lk-set if and only if for at least one component Z of F, $Z+F'$ is an Lk-set.*

Proof. The sufficiency of the condition is clear. Now let us conversely suppose that $F+F'$ is an Lk-set. F being a compact set, there exists a dyadic system $F_{i_1,i_2,\dots,i_k}$ ($k=1,2,\dots$; $i_j=0,1$) of closed subsets $F_{i_1,i_2,\dots,i_k}$ of F with the following properties:

(α) For a fixed k the system consisting of the 2^k sets $F_{i_1,\dots,i_k}$ is a system of mutually disjoint closed subsets of F with $\Sigma F_{i_1,\dots,i_k}=F$.

(β) Given $F_{i_1,\dots,i_k}$ then for any choice of i_{k+1},

$$F_{i_1,\dots,i_k,i_{k+1}}\subset F_{i_1,\dots,i_k}.$$

(γ) Given any sequence $i_1,i_2,\dots$, with $i_k=0$, 1 for $k=1,2,\dots$, then the intersection $F_{i_1}\cdot F_{i_1i_2}\cdot F_{i_1i_2i_3}\cdot\dots$ is a component Z of F.

Now we have

$$(F_0+F')+(F_1+F')=F+F',$$
$$(F_0+F')\cdot(F_1+F')=F'.$$

Since $F+F'$ is an Lk-set, it follows from Phragmen-Brouwer-Alexandroff's theorem that either F_0+F' or F_1+F' is an Lk-set. Suppose that F_0+F' is an Lk-set. Then repeating the same argument we find: either $F_{00}+F'$ or $F_{01}+F'$ is an Lk-set, etc. Thus we construct a non-increasing sequence $F_{i_1}+F'\supset F_{i_1i_2}+F'\supset\dots$ of Lk-sets. According to corollary 1.2, $(F_{i_1}+F')\cdot(F_{i_1i_2}+F')\cdot\dots=Z+F'$ is an Lk-set, where Z is a component of F according to (γ).

By setting $F'=\emptyset$ we find

Corollary 2.1. *A closed subset F of S_n is an Lk-set if and only if at least one component Z of F is an Lk-set.*

Corollary 2.2. *An irreducible Lk-set is connected.*

Proof. Corollary 1.3 and corollary 2.1.

Now the proof of theorem I is immediate.

Proof of theorem I. Let F be a compact Lk-set contained in ΣF_i. Then according to corollary 2.1, F contains an irreducible and hence connected Lk-set F_0. Since $F_0\subset F\subset\Sigma F_i$ we have $F_0=\Sigma F_i\cdot F_0$. The sets F_i being mutually disjoint, so are the sets $F_i\cdot F_0$, and thus we have obtained a representation of a continuum F_0 as the sum of a countable system of mutually disjoint closed subsets. According to a wellknown theorem of Sierpiński[5]) this is only possible if for at least one index i, $F_0=F_i\cdot F_0$, and hence for the other indices j, $F_j\cdot F_0=\emptyset$. So F_0 is already contained in some F_i. Since F_0 is an Lk-set, F_i must be an Lk-set, which is a contradiction.

For the proof of theorem II we need the following property (σ) of irreducible Lk-sets:

(σ) Let F be an irreducible Lk-set and p an arbitrary point of F. Then there is a non-decreasing sequence of continua $K_1\subset K_2\subset\dots\subset F-(p)$ with $F=\overline{\Sigma K_j}$, where the bar denotes the closure operation with respect to S_n.

The proof of (σ) will be given in section 3.

From (σ) there may be derived

Proposition 3. *Let F be an irreducible Lk-set, p an arbitrary point of F, and $F_1,F_2,\dots$ a sequence of compact subsets of F with $F_i\cdot F_j\subset(p)$ for every pair i,j, $i\neq j$, and $F=\Sigma F_i$. Then there is an index i such that $F=F_i$, and hence for $j\neq i$, $F_j\subset(p)$.*

Proof. Let K denote a subcontinuum of $F-(p)$. Since $F_i\cdot F_j\subset(p)$, and $\Sigma F_i=F$, we have

$$(K\cdot F_i)\cdot(K\cdot F_j)=\emptyset,\quad i\neq j,$$
$$K=\Sigma K\cdot F_i.$$

Hence according to Sierpiński's theorem for at least one index i, $K\cdot F_i=K$, i. e. $K\subset F_i$. Let $K'\supset K$ also be a subcontinuum of $F-(p)$. Then there is an index j for which $K'\subset F_j$. Let us suppose that $K\neq\emptyset$, then on account of $p\notin K,K'$, we have

$$F_j\cdot F_i-(p)\supset F_i\cdot K\cdot F_j\cdot K'\supset K\neq\emptyset.$$

This is only possible if $i=j$. Using (σ) we find: There is an index i such that $F_i\supset F$.

Proof of theorem II. Let F be a compact Lk-set$\subset\Sigma F_i$, and F_0 an irreducible Lk-subset of F. Then the rest of the proof proceeds in the same way as the proof of theorem I, using Sierpiński's theorem if $p\notin F_0$, and using proposition 3 if $p\in F_0$.

3. We still have to prove (σ). In fact we shall prove a somewhat more general property which includes (σ) as a special case.

Proposition 4. *Let F be an Lk-set, and p an arbitrary point of F. Then there exists a non-decreasing sequence $K_1\subset K_2\subset\dots$ of subcontinua of $F-(p)$ such that $\overline{\Sigma K_i}$ is an Lk-set.*

[5]) cf. F. Hausdorff, loc. cit., p. 162.

Proof. We suppose S_n to be metrized as geometrical sphere in E_{n+1}. Then a closed ε-neighbourhood of a point is a closed spherical region. Furthermore if ε is sufficiently small, which will be supposed henceforth, the closed ε-neighbourhood U_ε of p does not meet A. Since F is an Lk-set, $F+U_\varepsilon=\overline{(F-U_\varepsilon)}+U_\varepsilon$ is an Lk-set. Then according to proposition 2 there is at least one component Z_ε of $\overline{(F-U_\varepsilon)}$ such that $Z_\varepsilon+U_\varepsilon$ is an Lk-set. For every $\varepsilon>0$ we choose a fixed Z_ε with this property. Suppose ε_i to be a decreasing sequence of positive numbers converging to zero. Now we define a double sequence of continua $Z_i^k \subset F-(p)$ as follows:

We set $Z_j^k=Z_{\varepsilon_k}$ for $j\geqslant k$.

Let for a fixed k, Z_i^k with $i\leqslant k$ already be constructed as a subcontinuum of Z_{i+1}^k such that

$$Z_i^k\subset\overline{S_n-U_{\varepsilon_i}},$$
$$Z_i^k+U_{\varepsilon_i}\ \text{ is an } Lk\text{-set.}$$

Then $Z_i^k+U_{\varepsilon_i}+U_{\varepsilon_{i-1}}$ is also an Lk-set. Since $U_{\varepsilon_{i-1}}\supset U_{\varepsilon_i}$ we have $Z_i^k+U_{\varepsilon_{i-1}}$ is an Lk-set. Now take Z_{i-1}^k as a component of $\overline{(Z_i^k-U_{\varepsilon_{i-1}})}$ such that $Z_{i-1}^k+U_{\varepsilon_{i-1}}$ is an Lk-set. The construction of the sequence Z_j^k for $j=k, k-1, \ldots$ will be finished as soon as Z_1^k is constructed. Thus we have obtained a double sequence with the following properties:

(1) $\quad$ if $\ i\leqslant k\ $ then $\ Z_i^k\subset\overline{(S_n-U_{\varepsilon_i})}$,

(2) $\quad Z_i^k+U_{\varepsilon_i}$ is an Lk-set, $i\leqslant k$,

(3) $\quad$ if $i\geqslant k$ then $Z_i^k=Z_k^k=Z_{\varepsilon_k}$,

(4) $\quad Z_j^k\subset Z_{j+1}^k$ for $k,j\geqslant 1$,

(5) $\quad Z_j^k$ is a subcontinuum of F for $k,j\geqslant 1$.

By the diagonal procedure we can select a subsequence $Z_{\varepsilon_{k_i}}$ from the sequence Z_{ε_k} such that $Z_j^{k_i}$, for any fixed j, is a converging sequence of continua [4]). As a consequence, for any fixed k_j, the sequence $Z_{k_j}^{k_i}$ converges also. Hence we may suppose that the continua Z_j^k are already constructed in such a way, that for any fixed j, the sequence Z_j^k converges. We set

$$\lim_k Z_j^k=Z_j.$$

According to (4) we have $Z_j^k\subset Z_{j+1}^k$, and hence

(6) $$Z_j\subset Z_{j+1}.$$

According to (1) we have

(7) $$Z_i\subset\overline{(S_n-U_{\varepsilon_i})}.$$

Furthermore $Z_j^k\subset F$ according to (5), and F being closed in S_n we have

(8) $$Z_j\subset F.$$

Finally applying corollary 1.1 we have

(9) $$\lim_k\,(Z_i^k+U_{\varepsilon_i})=Z_i+U_{\varepsilon_i}\quad\text{is an } Lk\text{-set.}$$

(For any fixed i, $Z_i^k+U_{\varepsilon_i}$ is an Lk-set for almost all k.)

From (6), (7), (8) we may conclude that $\{Z_i\}$ is a non-decreasing sequence of subcontinua of $F-(p)$. (Every Z_i is a continuum since it is the limit of a sequence of continua [4])). Furthermore we get from (9) by applying corollary 1.1:

$$\lim_i\,(Z_i+U_{\varepsilon_i})=\overline{\Sigma Z_i}+(p)$$

is an Lk-set. According to proposition 2, either $\overline{\Sigma Z_i}$ or (p) is an Lk-set. Since (p) is not an Lk-set, $\overline{\Sigma Z_i}$ must be an Lk-set. So proposition 4 is completely proved.

It will be seen that (σ) is an immediate corollary of proposition 4.

4. It will be seen that theorems I and II are of Phragmen-Brouwer type. To render the analogy more complete we shall establish the implication (A)→(B) of the following properties (A) and (B):

(A) $\mathfrak{z}$ is not linked with any compact subset of ΣF_i.

(B) $\mathfrak{z}\sim 0$ in $S_n-\Sigma F_i$, where the Čech-homology theory [6]) is taken as the homology theory for $S_n-\Sigma F_i$.

Let F_σ be a countable sum of closed subsets of S_n, and let G_δ denote its complement. The Čech-homology theory will be taken as the basic homology theory for G_δ. Let A denote a fixed compact subset of G_δ and let $\mathfrak{z}$ denote a fixed cycle in A. We shall prove

Proposition 5. *If $\mathfrak{z}$ is not linked with any compact subset B of F_σ, in other words if any compact subset B of F_σ is a 0-Lk-set with respect to the pair $A,\mathfrak{z}$, then $\mathfrak{z}\sim 0$ in G_δ.*

The proof depends on two lemmas.

[6]) cf. Lefschetz, *Algebraic Topology*, New York 1948, Chapter VII.

Lemma 1. *Let S be a metric space and R a subspace of S. Let $\{U_1,...,U_n\}$ be a covering of R with sets $U_i \subset R$, which are open relative R. Then there exists a system $\{O_1,...,O_n\}$ of open subsets $O_i \subset S$, with $O_i \cap R = U_i$, and such that the two systems $\{U_1,...,U_n\}$, $\{O_1,...,O_n\}$ have isomorphic nerves.*

Proof. The distance function in S will be denoted by ϱ. Let $p \in R$, then the (ε, R)-neighbourhood $(\varepsilon > 0)$ of p is the set of all points $q \in R$ with $\varrho(p,q) < \varepsilon$. Let $p \in S$, then the ε-neighbourhood of p is the set of all points $q \in S$ with $\varrho(p,q) < \varepsilon$ $(\varepsilon > 0)$.

Let $p \in R$, and $p \in U_i$ for a certain i. Then $\varrho_i(p)$ is defined as the largest ε such that the (ε, R)-neighbourhood of p is $\subset U_i$. Now we define O_i as the sum of all $\varrho_i(p)/3$-neighbourhoods of the points $p \in U_i$. It will be evident that $U_{i_1} \cdot U_{i_2} \cdot ... \cdot U_{i_k} \neq 0$ implies $O_{i_1},...,O_{i_k} \neq 0$.

Suppose now that $O_{i_1} \cdot O_{i_2} \cdot ... \cdot O_{i_k} \neq 0$, *e. g.* $O_1 \cdot ... \cdot O_k \neq 0$. Let $q \in O_1 \cdot O_2 \cdot ... \cdot O_k$. Then according to the definitions of the sets $O_1,...,O_k$ there exist points $p_i \in U_i$ $(i=1,...,k)$ with $\varrho(q,p_i) < \varrho_i(p_i)/3$. There is no loss in generality in supposing $\varrho_1(p_1) \leqslant \varrho_i(p_i)$, $i \geqslant 1$. We have then

$$\varrho(p_1,p_i) \leqslant \varrho(p_1,q) + \varrho(q,p_i) < (\varrho_1(p_1) + \varrho_i(p_i))/3$$
$$\leqslant 2/3\ \varrho_i(p_i) < \varrho_i(p_i).$$

Hence according to the definition of $\varrho_i(p_i)$, we have $p_1 \in U_i$ for each i $(i=1,...,k)$, *i. e.* $U_1 \cdot ... \cdot U_k \neq 0$. From

$$O_{i_1} \cdot ... \cdot O_{i_k} \neq 0 \leftrightarrow U_{i_1} \cdot ... \cdot U_{i_k} \neq 0$$

it follows that the nerves of the systems $\{U_1,...,U_n\}$, and $\{O_1,...,O_n\}$ are isomorphic.

Lemma 2. *Let S be a topological space. Let $\mathfrak{z}$ be a cycle of S, and let $R \subset S$ be a carrier of $\mathfrak{z}$, with $\mathfrak{z} \sim 0$ on R. Then $\mathfrak{z} \sim 0$ on S.*

Proof. We have to prove: Given a finite open covering $\{O_1,...,O_n\}$ of S with nerve Φ. Then if $\mathfrak{z}\Phi$ is a representative of $\mathfrak{z}$ in Φ, $\mathfrak{z}\Phi \sim 0$ in Φ.

The system $\{U_1,...,U_n\}$ with $U_i = O_i \cap R$ is a relative open covering of R with nerve Φ_R. Then in a wellknown way Φ_R may be considered as a simplicial subcomplex of Φ, and $\mathfrak{z}\Phi$ may be considered as a cycle of Φ_R. Since $\mathfrak{z} \sim 0$ on R, we have $\mathfrak{z}\Phi \sim 0$ in Φ_R, and Φ_R being a simplicial subcomplex of Φ, $\mathfrak{z}\Phi \sim 0$ on Φ too.

Proof of proposition 5. Let there be given a finite relative open covering $\{U_1,...,U_n\}$ of G_δ with nerve Φ, and let $\mathfrak{z}\Phi$ denote a representative of $\mathfrak{z}$. According to lemma 1 each U_i may be enlarged to an open set O_i of S_n such that the system $\{O_1,...,O_n\}$ has also the "same" nerve Φ. Let ΣO_i be denoted by G. Then $B = S_n - G \subset S_n - \Sigma U_i = F_\sigma$. Since B is a compact subset of F_σ, $\mathfrak{z}$ is not linked with B, *i. e.* $\mathfrak{z} \sim 0$ on a compact subset of G. According to lemma 2, $\mathfrak{z} \sim 0$ on G. Hence any representative of $\mathfrak{z}$ in the nerve Φ of the covering $\{O_1,...,O_n\}$ is ~ 0. Since $\mathfrak{z}\Phi$ may be considered as such a representative, $\mathfrak{z}\Phi \sim 0$ in Φ. Thus we have found:

Let $\{U_1,...,U_n\}$ be a finite relative open covering of G_δ with nerve Φ. Then if $\mathfrak{z}\Phi$ is a representative of $\mathfrak{z}$ in Φ we have $\mathfrak{z}\Phi \sim 0$ in Φ.

This is precisely the significance of: $\mathfrak{z} \sim 0$ in G_δ.

5. From the foregoing there follows

Theorem III. *Let $F_1, F_2, ...$ be a sequence of 0-Lk-sets such that there is a fixed point $p \notin A$ with $F_i \cdot F_j \subset (p)$, $i \neq j$. Then $\mathfrak{z} \sim 0$ in $S_n - \Sigma F_i$.*

Proof. Theorem II together with proposition 5.

From the above theorem there follows

Theorem of Miss Mullikin. *Let $F_1, F_2, ...$ be a sequence of compact subsets of S_n with the following properties:*

(α) there is a fixed point $p \in S_n$, such that $F_i \cdot F_j \subset (p)$, $i \neq j$,

(β) for every i, $S_n - F_i$ is connected.

Under these conditions $S_n - \Sigma F_i$ is connected.

Proof. Let q_1 and q_2 denote two arbitrary points of $S_n - \Sigma F_i$. The zero cycle $q_1 - q_2$ is not linked with any F_i, since $S_n - F_i$ is connected for every i. Hence according to theorem III, $q_1 - q_2 \sim 0$ in $S_n - \Sigma F_i$, *i. e.* q_1 and q_2 belong to the same quasi-component of $S_n - \Sigma F_i$ [7]). Since q_1 and q_2 are arbitrary points of $S_n - \Sigma F_i$, $S_n - \Sigma F_i$ can only have one quasi-component and hence $S_n - \Sigma F_i$ is connected [7]).

Added in proof: After submitting the above paper for publication, the paper of S. Kaplan: *Homology properties of arbitrary subsets of Euclidean spaces*, Trans. Am. Math. Soc. 62 (1947) pp. 248—271, came to my attention.

In this paper Kaplan develops a homology theory for separable metric spaces based upon the family of countable starfinite open coverings, which family derives its importance from the fact that any open covering of a separable metric space has a countable star-finite refinement. The homology theory thus obtained

[7]) Lefschetz, loc. cit., pp. 257-261.

seems to be an appropriate tool for treating duality theorems of the kind studied above. The same homology theory has been developed also by P. Alexandroff in: *A general law of duality for non-closed sets in n-dimensional space*, Doklady Akad. Nauk SSSR (N. S.) 57 (1947) pp. 107—110 (Russian).

Using the Kaplan-Alexandroff homology, theorem III may be formulated as follows

Theorem IIIa. *Let $F_1, F_2, \ldots$ be a sequence of closed sets not meeting A and such that there is a point $p \in A$ with $F_i \cdot F_j \subset (p)$, $i \neq j$. Then $\mathfrak{z} \sim 0$ in $S_n - \sum F_i$ if and only if each F_i is a 0-Lk-set.*

The proof of Miss Mullikin's theorem utilizing this homology concept remains verbally the same.

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Enden und Primenden *).

Von

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Ein nichtkompakter Raum kann immer auf vielerlei Weisen kompaktifiziert werden; eine Gerade z. B. durch einen oder durch zwei „unendlich ferne" Punkte, die Ebene durch eine uneigentliche Gerade zur projektiven Ebene oder durch einen uneigentlichen Punkt zur funktionentheoretischen Ebene (zweidimensionalen Sphäre). Sucht man nun eine „natürliche" Kompaktifizierung, so kann bei diesen Beispielen die Wahl nicht schwierig sein: Die Gerade besitzt zwei deutlich unterschiedene Unendlichkeiten — warum sollte man die in einen Punkt zusammenfallen lassen? Die Ebene dagegen steht nicht in einer topologisch eindeutig bestimmten Beziehung zu einer etwa hinzugefügten Geraden; die Einführung *eines* uneigentlichen Punktes befriedigt hier das Kompaktifizierungsbedürfnis.

Man wird die „ideale" Kompaktifizierung [2-5] durch zwei Forderungen erzwingen:

1. Die hinzugefügte Punktmenge soll möglichst „dünn" sein (z. B. kein Kontinuum, wenn ein Punkt ausreicht).

2. Die Menge der neuen Limesrelationen soll möglichst klein sein (nicht jede divergente Folge soll zu einer konvergenten ernannt werden, wenn man sich mit bescheideneren Festsetzungen begnügen kann).

De Groot [5] hat diese Forderungen folgendermaßen präzisiert:

Das Kompaktum $\tilde{R}$ heißt *ideale Kompaktifikation* des separabelen metrisierbaren Raumes R, wenn

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