Some remarks on a relation between sets and elements.

By

S. Ginsburg (University of Miami).

Let E be any non-empty set and $H=\{h\}$ any non-empty family of subsets of E. Let R be a relation such that xRh holds for one and only one element x in each h. Let G_α be the subset of E consisting of those elements x in E for which xRh holds for at least $\aleph_\alpha$ sets h in H. The problem is to obtain an estimate of the power of G_α for different conditions on H and different $\aleph_\alpha$.

In [2] the following result was stated:

(*) *Let E be countably infinite, H the family of all finite subsets of E, and $a=0$. Then the power of G_α is $\aleph_0$.*

The theorem as stated above is incorrect as a simple example shows. Let E be the set of positive integers and xRh mean that x is the largest integer in the finite set h. For each integer x, the number of subsets of E in which x is the largest integer is finite. Obviously the set G_α is empty. (*) may be amended to get

Theorem 1. *Let $H=\{h\}$ be the family of all finite subsets of the uncountable set E and $\aleph_\alpha<\overline{\overline{E}}$. Then the power of G_α is $\overline{\overline{E}}$. For $\aleph_\alpha=\overline{\overline{E}}$, G_α may be empty.*

Proof. We first show that G_α is non-empty. In fact, assume the contrary, *i. e.* for each element x in E there are fewer than $\aleph_\alpha$ sets h for which xRh. Denote by $f(x)$ the set $\{y\,|\,y\in h, xRh\}$. Since $\overline{\overline{h}}<\aleph_0$ and the number of such sets h is less than $\aleph_\alpha$, it follows that $\overline{\overline{f(x)}}<\aleph_\alpha$. Well order the elements of E into an ω_η sequence, $\{y_\xi\}_{\xi<\omega_\eta}$, where $\aleph_\eta=\overline{\overline{E}}$. Let $x_0=y_0$ and $A_1=E-f(x_0)-\{x_0\}$. Either $x_0R(x_0,x)$ or $xR(x_0,x)$ for each x in A_1. As x is not in $f(x_0)$ we must have $xR(x_0,x)$, *i. e.* $(x_0,x)\subseteq f(x)$. Now assume defined x_ξ for each $\xi<\mu<\omega_\alpha$. We continue by transfinite induction. Let $B_\mu=\bigcup\limits_{\xi<\mu} f(x_\xi)\cup\{x_\xi|\xi<\mu\}$, and x_μ be the first element in $A_\mu=E-B_\mu$. Since the power of B_μ

is $\leqslant\aleph_\alpha$, the power of A_μ is $\overline{\overline{E}}$. Either $x_\xi R(x_\xi,x)$ or $xR(x_\xi,x)$ for each x in A_μ and each $\xi<\mu$. As x is not in $f(x_\xi)$ we must have $xR(x_\xi,x)$, *i. e.* $(x_\xi,x)\subseteq f(x)$.

Let $B_{\omega_\alpha}=\bigcup\limits_{\xi<\omega_\alpha} f(x_\xi)\cup\{x_\xi|\xi<\omega_\alpha\}$ and x_{ω_α} be the first element in $A_{\omega_\alpha}=E-B_{\omega_\alpha}$. Continuing as above we see that $x_{\omega_\alpha}R(x_\xi,x_{\omega_\alpha})$ for each $\xi<\omega_\alpha$. This implies that there are at least $\aleph_\alpha$ sets h for which $x_{\omega_\alpha}Rh$, a contradiction. Therefore G_α is not empty.

If we assume the generalized continuum hypothesis, *i. e.* $2^{\aleph_\xi}=\aleph_{\xi+1}$, then by a result due to Erdös [1], E contains a set $\{x_\xi|\xi<\overline{\overline{E}}\}$ such that $x_\xi\notin f(x_\nu)$ for $\xi\neq\nu$. In particular, $x_0\notin f(x_1)$ and $x_1\notin f(x_0)$. Consider the set (x_0,x_1). Either $(x_0,x_1)\subseteq f(x_0)$, or $(x_0,x_1)\subseteq f(x_1)$. But each of these possibilities is a contradiction. This yields a short proof that G_α is not empty.

Now let $E=\bigcup\limits_{\xi<\omega_\eta} E_\xi$, where $E_\xi\cap E_\nu=\Phi$ for $\xi\neq\nu$, and $\overline{\overline{E_\xi}}=\overline{\overline{E}}$. Since we have already proved that in each E_ξ there exists an element z_ξ for which $z_\xi Rh$ for at least $\aleph_\alpha$ sets h, we conclude that

$$\overline{\overline{\{z_\xi\,|\,\xi<\omega_\eta\}}}\leqslant\overline{\overline{G_\alpha}}\leqslant\overline{\overline{E}},$$

and thus, that the power of G_α is $\overline{\overline{E}}$.

If $\aleph_\alpha=\overline{\overline{E}}$, then the example stated previously can be generalized to show that G_α may be empty.

If we allow infinite sets to be elements of H, then

Theorem 2. *Let E be any set and $\aleph_\nu\leqslant\overline{\overline{E}}$. If $H=\{h\}$ is the family of those subsets of E which are of power $\aleph_\nu$, each, and if $\aleph_\alpha=\overline{\overline{E}}$, then $\overline{\overline{G_\alpha}}=\overline{\overline{E}}$.*

Proof. As in Theorem 1, it is sufficient to show that there is at least one element y in E for which yRh for $\aleph_\alpha$ sets h.

First suppose that $\aleph_\nu=\overline{\overline{E}}$. Thus the power of H is $\aleph_\nu^{\aleph_\nu}$. In other words, to each element of a set of power $\aleph_\nu^{\aleph_\nu}$ there corresponds an element of a set of power $\aleph_\nu$. By a well known theorem of König [3], it follows that for some element y in E, there are $2^{\aleph_\nu}$ sets h for which yRh. Now suppose that $\aleph_\nu<\overline{\overline{E}}$ and for no y in E is yRh for at least $\aleph_\alpha$ sets h. Well order the elements of E into the ω_α sequence

$$x_0,x_1,\dots,x_\xi,\dots \qquad (\xi<\omega_\alpha). \tag{1}$$

Let $p_0=x_0$ and denote by $h_0^0,\dots,h_{\eta_0}^0$ those sets h in H for which p_0Rh. Assume that $h_0^\xi,\dots,h_{\eta_\xi}^\xi$ and p_ξ have been defined for $\xi<\lambda<\omega_\nu$.

(α) If $\lambda=\beta+1$, then denote by p_λ the first element of (1) which is in the set

$$E-[\{\bigcup_{\xi\leqslant\eta_\beta} h_\xi^\beta\}\cup\{p_\xi|\xi<\lambda\}].$$

Since $\overline{\overline{\bigcup_{\xi\leqslant\eta_\beta} h_\xi^\beta}}\leqslant\overline{\eta}_\beta\aleph_\nu<\overline{\overline{E}}$, the element p_λ certainly exists.

(β) If λ is a limit number, then let p_λ be the first element of (1) which is in the set $E-\{p_\xi|\xi<\lambda\}$.

Let $M=\{p_\xi|\xi<\omega_\nu\}$. The set M is an element of H, and yet for no p_ξ is $p_\xi RM$. This is so since if $p_\xi RM$, then from the manner in which $p_{\xi+1}$ was selected, $p_{\xi+1}$ cannot be in M. As this is a contradiction, it follows that for some element y in E, there are at least $\aleph_\alpha$ sets h for which yRh, q. e. d.

Bibliography.

[1] P. Erdös, *Some remarks on set theory*, Proceedings of the American Mathematical Society **1** (1950), p. 127-141.

[2] G. Fodor and I. Ketskemety, *Some theorems on the theory of sets*, Fundamenta Mathematicae **37** (1950), p. 249-250.

[3] F. Hausdorff, *Mengenlehre*, p. 34.

A generalization of a theorem of Miss Anna Mullikin.

By

W. T. van Est (Utrecht).

Introduction. In 1923 Miss Mullikin proved the following theorem [1]):

Let $F_1, F_2, \dots$ be a sequence of mutually disjoint closed subsets of the euclidean plane E_2, such that for every i, E_2-F_i is connected, then $E_2-\Sigma F_i$ is connected.

A new and simpler proof of this theorem was given by S. Mazurkiewicz in Fundamenta Mathematicae **6** (1924), pp. 37-38.

The above mentioned theorem may in an obvious way be formulated as a property of the 2-sphere, and as such it appears essentially as a theorem of Phragmen-Brouwer type. Generalizations in this direction of Miss Mullikin's theorem seem to have received but little attention hitherto.

Recently the simplification and modernization of Miss Mullikin's proof of her theorem was proposed as a problem by Wiskundig Genootschap at Amsterdam [2]) (apparently in ignorance of Mazurkiewicz's proof). The present author succeeded in giving such a proof and at the same time generalized Miss Mullikin's theorem for n dimensions (equally ignorant of Mazurkiewicz's article). Owing to several useful hints of Prof. H. Freudenthal he realized that with the same methods the generalizations indicated here below (cf. Theorems I, II, III) could be proved. The method of the proof of theorem II strongly parallels Mazurkiewicz's arguments.

---

[1]) A. M. Mullikin, *Certain Theorems Relating to Plane Connected Point Sets*, Trans. Am. Math. Soc. **24** (1923), pp. 144-162.

[2]) Programma van jaarlijkse prijsvragen, 1950.