that \( a_{\varnothing} < a_{\{0\}} \) for \( \tau(\bar{x}) \geq \tau(1) \). By induction we can define an \( \alpha \) sequence, \( \{a_\alpha\} \), with \( a_{\varnothing} < a_{\varnothing \{\alpha\}} \), a well known impossibility. Thus \( \lambda < \alpha \).

If \( \sum_{\lambda < \alpha} a_\lambda = \eta + \alpha \delta \), then for some \( \gamma \),

\[
\sum_{\lambda \geq \gamma} a_\lambda = \alpha \delta.
\]

Let \( \{b_\gamma\} \) be any permutation of the elements of \( \{a_\gamma\} \). As \( a_\gamma \) is regular, we are able to repeat the procedure given in [1] and find a \( \theta_\delta \) so that

\[
\sum_{\gamma \geq \theta_\delta} b_\gamma = \alpha \delta.
\]

Thus the value of \( \sum_{\lambda < \gamma} b_\lambda \) is determined by calculating the value of \( \sum_{\gamma < \delta} b_\gamma \). For each \( \theta_\delta \), there are \( \mathfrak{b}_\delta \) different subsets of \( \{a_\gamma\} \) of power \( \mathfrak{b}_\delta(2) \), and \( \mathfrak{b}_\delta \) permutations of the elements of each set. Hence

\[
\mathcal{N}(\sum a_\lambda) \leq \sum_{\lambda < \gamma} \mathfrak{b}_\lambda(2) = \mathfrak{b}_\gamma.
\]

Suppose \( \kappa_\gamma = \kappa_{\gamma + 1} \). Let \( a_\gamma = a_{\gamma + 1} \) for \( \xi < \alpha_{\gamma} \), and \( a_\gamma = 1 \) for \( \xi \geq \alpha_{\gamma} \). Then \( \mathcal{N}(\sum a_\lambda) = \kappa_{\gamma + 1} \) and the theorem is proved.

**Theorem 2.** If \( a_\gamma \) is a regular ordinal and \( \{a_\gamma\} \) is a non-decreasing transfinite sequence of ordinals, then \( \mathcal{N}(\sum a_\lambda) = 1 \).

Proof. Let \( a_\gamma \) be regular and \( \{a_\gamma\} \) non-decreasing. For \( \{b_\gamma\} \), a permutation of the elements of \( \{a_\gamma\} \), let \( b_{\lambda \{\gamma\}} = b_\lambda \alpha \) and suppose defined \( \{b_{\lambda \{\gamma\}}\} \). Let \( \lambda \) be the smallest ordinal \( \lambda \geq \tau(\bar{x}) \), \( \xi < \delta \). Since \( a_\gamma \) is regular, and since there are \( \mathfrak{b}_\delta \) elements \( a_\gamma \), \( a_\delta \geq a_\xi \), we can find a \( b_{\lambda \{\gamma\}} \geq a_\xi \), with \( \tau(\bar{x}) \geq \lambda \). Therefore

\[
\sum_{\lambda \leq \lambda} \leq \sum_{\lambda \geq \lambda} \leq \sum_{\lambda < \delta}.
\]

A similar procedure yields \( \sum_{\lambda \leq \lambda} \leq \sum_{\lambda \leq \lambda} \), so that the equality sign holds.

In conclusion we remark that if \( a \) is a non-regular limit number, then there exists an increasing sequence \( \{a_\gamma\} \) and a permutation \( \{b_\gamma\} \) such that \( \sum_{\lambda \leq \lambda} < \sum_{\lambda < \delta} \).

**Bibliography.**


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**On models of axiomatic systems.**

A. Mostowski (Warszawa).

This paper is devoted to a discussion of various notions of models which appear in the recent investigations of formal systems. The discussion will be applied to the study of the following problem: Given a formal system \( S \) based on an infinite number of axioms \( A_1, A_2, A_3, \ldots \), is it possible to prove in \( S \) the consistency of the system based on a finite number \( A_1, A_2, \ldots, A_n \) of these axioms?

**1. Notations and definitions.** We shall consider two systems \( S \) and a \( s \) based on the functional calculus of the first order)\(^1\). We shall not describe these systems in detail but give only some definitions which will be required later.

**System \( s \).** We assume that the following symbols occur among the primitive signs of \( s \):

1. **Variables:** \( x_1, x_2, \ldots, x, y, z, \ldots \)
2. **Individual constants:** \( f_1, \ldots, f_s \).
3. **Functions** (i.e. symbols for functions from individuals to individuals): \( g_1, \ldots, g_\eta \). We denote by \( q_\gamma \) the number of arguments of \( g_\gamma (\gamma = 1, \ldots, \eta) \).
4. **Predicates** (i.e. symbols for relations): \( r_1, \ldots, r_\zeta \). We denote by \( p_\eta \) the number of arguments of \( r_\eta (\eta = 1, \ldots, \zeta) \).
5. **Propositional connectives and quantifiers.** We use the symbol \( \overline{\cdot} \) for the “stroke function” and define other connectives in terms of the stroke. Quantifiers are denoted by symbols \( (\exists x_\gamma) \) and \( (\forall x) \).

Among expressions which can be constructed from these signs we distinguish the following:

6. **Terms.** Variables and individual constants are terms. If \( T_1, \ldots, T_s \) are terms, then so is \( g(T_1, \ldots, T_s), \gamma = 1, \ldots, \eta \).

Terms will be denoted by the letters \( T_1, T_2, T_3, \ldots \).
7. Elementary formulas are expressions of the form \( \rho(\Gamma_1, \ldots, \Gamma_n) \) where \( \Gamma_1, \ldots, \Gamma_n \) are terms. Elementary formulas will be denoted by the letters \( E, E_1, E_n, \ldots \).

8. Prime formulas are elementary formulas in which no variables occur. Prime formulas will be denoted by the letters \( P, P_1, P_n, \ldots \).

9. Matrices. Elementary formulas are matrices. If \( M_1 \) and \( M_2 \) are matrices, then so are \( M_1 \cup M_2 \) and \( (\exists x)M_1 \) for \( i = 1, 2, \ldots \). Matrices will be denoted by the letters \( M, M_1, M_2, \ldots \).

Note that matrices containing propositional connectives other than the stroke are easily definable by means of matrices containing only the stroke symbol. We shall occasionally use the following abbreviations:

- \( M^0 \) for \( \neg M \)
- \( M^1 \) for \( \sim M \)
- \( \bigwedge_{i=1}^n M_i \) for \( M_1 \lor \ldots \lor M_n \)
- \( \bigvee_{i=1}^n M_i \) for \( M_1 \land \ldots \land M_n \).

10. Free and bound variables. Substitution. The distinction between free and bound variables is assumed as known.

The formula which results from a formula \( A \) by the substitution of the terms \( \Gamma_1, \ldots, \Gamma_n \) for the variables \( x_1, \ldots, x_n \) will be denoted by \( \text{Sub} \ A(\Gamma_1, \ldots, \Gamma_n) \).

The operation of substitution is always performable when \( A \) is a term. If \( A \) is a matrix, it is sometimes necessary to re-name the bound variables occurring in \( A \) in order to make sure that the operation \( \text{Sub} \) can be performed. We shall always assume that the necessary changes in the bound variables of \( A \) have been performed before the operation \( \text{Sub} \) has been applied.

11. Q-matrices. These are matrices in which no bound variables occur.

12. Axioms. We assume that the axioms of \( s \) are finite in number and have the form of \( Q \)-matrices in which no constants, functionals, or predicates occur besides those which were enumerated in 2, 3, and 4. The axioms will be denoted by \( a_i \) or \( a(x, y, \ldots, z) \), \( i = 1, \ldots, b \).

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On models of axiomatic systems

13. The rules of proof admitted in \( s \) are the usual ones. We adjoin to them the rule of explicit definitions and the \( \varepsilon \)-rule.

We shall add a few words to explain the \( \varepsilon \)-rule. To this end we define recursively the notions of \( \varepsilon \)-terms and \( \varepsilon \)-matrices.

The terms and matrices defined in 6 and 7 are \( \varepsilon \)-terms and \( \varepsilon \)-matrices. If \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \) are \( \varepsilon \)-terms, then

\( \text{Sub} \ v(\varepsilon\Gamma_1, \varepsilon\Gamma_2, \ldots, \varepsilon\Gamma_n) \)

is an \( \varepsilon \)-term provided that (i) \( x_1, \ldots, x_n \) are the free variables of \( \Gamma \) (ii) the bound variables of \( \Gamma \) are not free in \( \Gamma_1, \ldots, \Gamma_n \). If \( \Gamma_1, \ldots, \Gamma_n \) are \( \varepsilon \)-terms, then \( \varepsilon(\Gamma_1, \ldots, \Gamma_n) \) is an \( \varepsilon \)-matrix \( (i = 1, \ldots, n) \). If \( M_1 \) and \( M_2 \) are \( \varepsilon \)-matrices, then so are \( M_1 \cup M_2 \) and \( (\exists x)M_1 \). If \( M \) is an \( \varepsilon \)-matrix, then \( (\varepsilon x)M \) is an \( \varepsilon \)-term.

The \( \varepsilon \)-rule states that for every \( \varepsilon \)-matrix \( M \) the matrix

\( M \upharpoonright \text{Sub} \ M(x_1, \ldots, x_n) \)

can be assumed as a theorem of \( s \).

The question arises whether the assumptions concerning the form of axioms and rules of proof (cf. 12 and 13) are general enough to cover the cases of standard formal systems based on a finite number of axioms. The answer is affirmative. To see this we remark that the \( \varepsilon \)-rule enables us to get rid of quantifiers in the axioms provided that we introduce a sufficient number of \( \varepsilon \)-terms. Since the explicit definitions are allowed in \( s \), it follows that we can bring the axioms to the form of \( Q \)-matrices provided that we add a sufficient number of symbols to the symbols enumerated in 2 and 3. The resulting system then satisfies our assumptions and is equivalent to the given one provided that suitable definitions are introduced into the latter.

Example. Let one of the axioms have the form

(i) \( (\varepsilon x)M(x, y) \).

We introduce a new functor \( g(x) \) and an axiom

(ii) \( M(x, g(x)) \).

Clearly (i) is derivable from (ii). Conversely (ii) can be obtained from (i) by means of the \( \varepsilon \)-rule and the explicit definition \( g(x) = \varepsilon y M(x, y) \).

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1) An exact definition of the operation \( \text{Sub} \) is given in Church [1], pp. 56-58.

2) The ordinary rules of proof for the functional calculus are given e.g. in Church [1], p. 60. For the \( \varepsilon \)-rule see Hilbert-Bernays [5], pp. 9-18.

3) See Hilbert-Bernays [5], pp. 16-17.
The square-brackets notation explained in the foregoing paragraph will be used consistently in many other similar situations. So e.g. if $m_{ij}$ is the Gödel number of the matrix $M_{ij}$, then
$$[\prod_{i=1}^{m} \sum_{j=1}^{n} M_{ij}]$$
is the Gödel number of the matrix $\prod_{i=1}^{m} \sum_{j=1}^{n} M_{ij}$. If $as$ is the Gödel number of $M$, then $[sa]$ is the Gödel number of $M^s$ (i.e. of $\sim M$) and so on.

19. The following lemma is provable in $S$: In order that $e$ be an *elementary formula* it is necessary and sufficient that $e$ have the form $[i(t_1,\ldots,t_p)]$ where $i \leq y$ and $t_1,\ldots,t_p$ are terms in. The integer $i$ and terms $t_1,\ldots,t_p$ are determined by $e$. We put
$$i = \text{Ind}(e), \quad t_j = \text{Cont}(e), \quad j = 1,\ldots,p.$$

Functions $\text{Ind}$ and $\text{Cont}$ are definable in $S$.

20. The arithmetical counterpart of the function $\text{Subs}$ will be denoted by the symbol $\text{Sub}$. Thus if $a$ is the Gödel number of an expression $A$, and $t_1,\ldots,t_n$ are the Gödel numbers of terms $t_1,\ldots,t_n$, then $\text{Sub}(A, t_1,\ldots,t_n, a)$ is the Gödel number of the expression $\text{Subs}(A, t_1,\ldots,t_n, a)$.

21. If $\Gamma_1,\Gamma_2,\ldots,\Gamma_y$ are terms in, then the expressions
$$\text{Subs}(A, \Gamma_1,\ldots,\Gamma_y), \ldots, \text{Subs}(A, \Gamma_y)$$
and
$$g_j(\text{Subs}(\Gamma_1,\ldots,\Gamma_y), \ldots, \text{Subs}(\Gamma_y))$$
are identical.

22. The Gödel number of the $i$-th axiom of $S$ (see 12) will be denoted by $[g_j]$ or by $[a_i(r,y,\ldots)]$.

23. Arithmetical sentences expressing the consistency of $S$ and of $S'$ will be abbreviated as $\mathcal{N}(S)$ and $\mathcal{N}(S')$.

2. Models of the first kind. Let $R\beta(x,\gamma, \ldots, \gamma')$ be $\gamma+1$ matrices of $S$ with the indicated number of free variables and let $T_j(x_1,\ldots,x_m, y)$ be $\beta$ matrices of $S$ such that matrices
$$T_j(x_1,\ldots,x_m, y'), T_j(x_1,\ldots,x_m, y'') \vdash y' = y''$$
and
$$[g_j] T_j(x_1,\ldots,x_m, y)$$
are provable in $S$. We define in $S$ functors $\theta_j$ (where $j=1,2,...,\beta$) in the following way:

$$\theta_j(x_1,...,x_\gamma) = (\tau_j \theta_j(x_1,...,x_\gamma, y).$$

Finally let $F_1,...,F_\gamma$ be a constants definable in $S$.

The $\alpha + \beta + \gamma + 1$ tuple consisting of a constants $F_1$, of $\beta$ functors $\theta_j$, and of $\gamma + 1$ matrices $R_k$, and of $\gamma + 1$ matrices $B_1,...,B_\gamma$, is called a pseudo-model of the first kind of $s$ in $S$.

In order to define when a pseudo-model is a real model we shall introduce some auxiliary definitions.

To every term $\Gamma$ of $s$ we let correspond a term $T_{\Gamma}$ of $S$ in the following way. If $\Gamma$ is a variable, then $T_{\Gamma} = \Gamma$. If $\Gamma = F_j$, then $T_{\Gamma} = F_j$. Finally, if $\Gamma$ has the form $\theta_j(T_{\Gamma_1},...,T_{\Gamma_\gamma})$, then we put $T_{\Gamma} = \theta_j(T_{\Gamma_1},...,T_{\Gamma_\gamma})$.

To every elementary formula $R = \tau_j(T_{\Gamma_1},...,T_{\Gamma_\gamma})$ of $s$ we let correspond the matrix $R = R(T_{\Gamma_1},...,T_{\Gamma_\gamma})$ of $S$. We extend this definition to all $Q$-matrices of $s$ by putting $(M_1M_2)^{'} = M_1^{'}M_2^{'}$. In particular we put $A_i = A_i$, $(i=1,...,\gamma)$.

Definition. A pseudo-model

$$F_1,...,F_\gamma, \theta_1,...,\theta_\beta, R_1,...,R_\gamma$$

is a real model of the first kind of $s$ in $S$ if the formulas

$$R_\delta(x_1,...,x_\gamma) \equiv A_\delta(x_1,...,x_\gamma), \quad i=1,2,...,\delta,$$

are provable in $S$.

Models of the first kind are the ones with which one has to do in the usual proofs of consistency and of independence of axiomatic systems. For comparison with other notions of models to be defined later, we shall note the following general facts concerning models of the first kind:

1. The general notion of models of the first kind is defined not in $S$ but in the syntax of $S$.

2. Every particular pseudo-model is a finite set of matrices of $S$, and can therefore be defined in $S$. The problem whether it is or is not a real model can be formulated, and in particular cases also solved in $S$.

On models of axiomatic systems

3. If $s$ contains an infinite number of axioms (independent of whether their set is or is not definable in $S$), then the problem whether an explicitly given pseudo-model is or is not a real model of $s$ in $S$ would be expressible in the syntax of $S$ but not in $S$ itself.

The following theorems concerning models of the first kind are well known but are given here for the sake of comparison with other notions of models $^1$:

I. If (1) is a real model of the first kind of $s$ in $S$ and if a $Q$-matrix $A(x_1,...,x_\gamma, y)$ is provable in $S$, then the matrix

$$R_\delta(x_1,...,x_\gamma, y) \equiv A(x_1,...,x_\gamma, y)$$

is provable in $S$.

II. If (1) is a real model of the first kind of $s$ in $S$ and $S$ is consistent, then so is $s$.

Let us now assume that a real model of the first kind of $s$ has been explicitly defined in $S$.

Theorems I and II are provable in the syntax of $S$, hence they are translatable into arithmetic and therefore into $S$. Denoting by $X(s)$ and $X(s \Gamma)$ formulas of $S$ corresponding (via arithmetization) to the syntactic statement: $S$ (or $s \Gamma$) is self-consistent (cf. section 1, definition 23), we obtain from II:

III. The formula $X(s) \iff X(s)$ is provable in $S$.

In spite of this result models of the first kind are of no use when one is examining the problem whether the formula $X(s)$ itself is or is not provable in $S$. Models of the second and third kind which we shall discuss in the next sections will allow us to answer this question in many particular cases.

We note still the following theorem due to Wang [16]:

IV. If the formula $X(s)$ is provable in $S$, then a model of the first kind can be defined explicitly in $S$.

Indeed, the usual proofs of the completeness theorem of Gödel consist in exhibiting a model of the first kind of a (non-contradictory) first order system in the arithmetic of integers [15]. Taking $s$ as this system and repeating the argument of Gödel in $S$ (which is possible by our assumptions concerning $S$, cf. p. 138) we obtain the proof of theorem IV [15].

$^1$ Proofs of these theorems may be found e.g. in my book [17], Chapter XI.

$^2$ Gödel [2] and Hilbert-Bernays [5], p. 166.

$^3$ Wang [16], p. 287, gives a more detailed proof of this theorem.
3. Models of the second kind in the axiomatic theory of sets. We assume in this section that $S$ is an axiomatic system of set theory based, e.g., on Zermelo's axioms.

The following definitions are to be thought of as belonging to $S$.

Let $\mathcal{M}$ be an arbitrary set and $Z$ a finite set of positive integers. An $\mathcal{M}$-function with the set of arguments $Z$ is defined as a set $E$ of ordered pairs $\langle u, v \rangle$ such that $v \in \mathcal{M}$, $u$ runs over all finite sequences $^{12)}$ satisfying the conditions

$$D(u) = Z, \quad D^*(u) \subseteq \mathcal{M},$$

and the following condition of single-valuedness holds:

if $\langle u, v' \rangle \in E$ and $\langle u, v'' \rangle \in E$, then $v' = v''$.

The symbols $D(u)$ and $D^*(u)$ denote the domain and the counter-domain of $u$, i.e.

$$x \in D(u) = \{ y \in \mathcal{M} | \langle y, x \rangle \in u \},$$

$$x \in D^*(u) = \{ y \in \mathcal{M} | \langle y, x \rangle \in u \}.$$

If $u$ is a sequence satisfying the condition $D(u) = Z$ and $Y$ is an arbitrary set of positive integers, then we denote by $u|Y$ the sequence $u$ restricted to $Y$, i.e.

$$\langle i, u \rangle | Y = \langle i, u(i) \rangle (i \in Y).$$

An $\mathcal{M}$-relation with the set of arguments $Z$ is defined as a set $R$ of sequences $u$ such that $D(u) = Z$ and $D^*(u) \subseteq \mathcal{M}$.

If $Z$ consists of integers $i_1, i_2, \ldots, i_n$ and $u$ is a sequence with the domain $Z$ such that $\langle i_1, u \rangle, \langle i_2, u \rangle, \langle i_3, u \rangle, \ldots$ are elements of $u$, then instead of $u \in E$ we write $R(a, b, c, \ldots)$ and say that $R$ holds for the elements $a, b, c, \ldots$. Similarly, if $H$ is an $\mathcal{M}$-function, then instead of $\langle i, u \rangle \in H$ we write $H(u) = v$ or $H(a, b, c, \ldots) = v$.

Let $\mathcal{F}$ be elements of $\mathcal{M}$ ($i = 1, \ldots, n$), $G$ be functions with the sets of arguments $Z_j = \{ i_1, \ldots, i_n \}$, $j = 1, \ldots, \beta$, and let $R_k$ be relations with the sets of arguments $Z_k = \{ i_1, \ldots, i_n \}$, $k = 1, \ldots, \gamma$. The $a + \beta + \gamma + 1$ tuple

$$\mathcal{M}, \quad E, \ldots, F_a, \quad G_1, \ldots, G_\beta, \quad R_1, \ldots, R_\gamma,$$

will be called a pseudo-model of the second kind of $S$, in $S$.

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1) Results of this and the next section are due to Tarski [11] and [12].

2) Sequences are defined as functions (many-one relations) with domains contained in the set of positive integers. Cf. Tarski [11], p. 287.

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On models of axiomatic systems

We shall now explain when a pseudo-model is a real model. As in section 2 we need some auxiliary definitions.

We shall denote by $B(t)$ the set of free variables which occur in a term $t$ and by $B(m)$ the set of free variables which occur in a matrix $m$ of $n$.

With these definitions it is not difficult to prove the existence and uniqueness of a function $H_i(u)$ and a relation $Staf$ which are of fundamental importance in the investigations of the semantics of $S$.

The exact definitions of the function $H$ and the relation $Staf$ are given below in lemmas 1 and 2 together with the proofs of their existence and uniqueness. To facilitate our exposition we explain informally the intuitive meaning of these concepts.

Let $t$ be a term, $m$ a matrix of $n$, and let $u$ be a sequence $\{ \langle i_1, u_1 \rangle, \langle i_2, u_2 \rangle, \langle i_3, u_3 \rangle, \ldots \}$ where $i_1, i_2, \ldots$ are the free variables of $t$ or of $m$. Then $H_i(u)$ is what is usually called the value of $t$ for the values $a, b, c, \ldots$, respectively, given to the free variables of $t$. The relation $Staf$ of $m$ holds if and only if the elements $a, b, c, \ldots$ satisfy the matrix $m$ in the domain $\mathcal{M}$ of individuals.

Lemma 1. There exists exactly one function $H_i(u) = H_i(u)$ such that

1. $t$ runs over terms of $S$;

2. $H_i(u)$ considered as a function of $u$ alone is an $\mathcal{M}$-function with the set of arguments $B(t)$;

3. if $t$ is a variable, then $H_i(\langle i, x \rangle) = x$;

4. if $t = t_i$, then $H_i(u) = F_i$, $i = 1, \ldots, n$;

5. if $t = \{ \langle g, y_1, \ldots, y_\beta \rangle \}$, then $H_i(u) = G_i(H_{i_1}(u_1), \ldots, H_{i_\beta}(u_\beta))$ where $G_i = \{ u | B(u) \}$, $u = 1, \ldots, q_i$, $j = 1, \ldots, \beta$.

Lemma 2. There exists exactly one binary relation $Staf$ such that

1. the codomain of $Staf$ consists of matrices of $S$;

2. for a fixed matrix $m$ the set $E_{m}[u | Staf \ m]$ is an $\mathcal{M}$-relation with the set of arguments $B(m)$;

3. if $m$ is the elementary formula $[t_1, \ldots, t_p]$, then

$$u \ Staf \ m \iff R_i(F_{i_1}(u_1), \ldots, F_{i_p}(u_p)),$$

where $u_n = u_i(B_{i_1}(u_1))$ for $n = 1, \ldots, p_i$, $i = 1, \ldots, \gamma$.

4. if $u \ Staf \ m$, then $u \ non-Staf \ m$, or $u \ non-Staf \ m$.
A. Mostowski:

If \( u = \beta [\beta j \circ m] \) and the variable \( \gamma \) is not free in \( m \), then \( u \text{Stef} m = u \text{Stef} m \). If however \( \gamma \) is free in \( m \), then

\[ u \text{Stef} m = \text{there is an element } a \in \mathcal{M} \text{ such that } \]

\[ u \vdash \langle \beta j, a \rangle \text{Stef} m \]

Note that both lemmas are provable in \( S \).

Definition 14. A pseudo-model (2) is a real model of the second kind of \( s \) in \( S \) if for every axiom \([ \alpha j \circ m ]\) of \( s \) and for every sequence \( s \) satisfying the conditions \( D(\alpha) = B(\alpha j) \) and \( D(\alpha) \subseteq \mathcal{M} \), the following formula holds

\[ u \text{Stef} [m], \quad i = 1, \ldots, \delta. \]

Using this definition one can prove the following theorems:

\[ \text{VI 14}. \]

If (2) is a real model of the second kind of \( s \) in \( S \) and if \( m \) is a matrix provable in \( s \), then \( u \text{Stef} m \) for an arbitrary sequence \( u \) satisfying the conditions \( D(\alpha) = B(\alpha j) \) and \( D(\alpha) \subseteq \mathcal{M} \), then \( \mathcal{N}(\alpha) \).

Theorems V and VI as well as all the previous definitions and theorems belong to the system \( S \).

We now abandon \( S \) and pass to its syntax. We can then formulate the following statements concerning models of the second kind. These statements should be compared with statements 1-3 of section 2, pp. 138-139:

1. The general notion of models of the second kind is definable within \( S \).
2. Every individual model of the second kind is an element of the universe of discourse of \( S \).
3. Models of the second kind can also be defined in cases where the number of axioms of \( s \) is infinite and statements 1 and 2 above also remain valid.

From the circumstance that theorem VI has been proved in \( S \) we infer that the following theorem holds:

\[ \text{VII}. \]

If the existence of a real model of the second kind of \( s \) is provable in \( S \), then so is the formula \( \mathcal{N}(\alpha) \) expressing the consistency of \( s \).

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On models of axiomatic systems

Hence models of the second kind enable us to obtain absolute consistency proofs whereas models of the first kind yield merely relative consistency proofs.

We note finally that just as in section 2 we can derive from the completeness theorem of Gödel the following theorem which is the converse of VII:

\[ \text{VIII}. \]

If the sentence \( \mathcal{N}(\alpha) \) is provable in \( S \), then so is the sentence stating the existence of at least one model of the second kind of \( s \) in \( S \).

4. Impossibility of a finite axiomatization of set-theory. Let us assume as in section 3 that \( S \) is an axiomatic system of set-theory and let \( s \) be a system based on a finite number of axioms of \( S \). Since we assume the \( r \)-rule both in \( S \) and in \( s \), we can assume that the axioms of \( s \) contain no quantifiers and that \( S \) contains all functors occurring in the axioms of \( s \).

From now on until the formulation of theorem IX we again assume that our discussion takes place in system \( S \).

Let \( \mathcal{M} \) be an arbitrary non-void set such that

\[ f_1, \ldots, f_\beta \in \mathcal{M}, \]

if \( m_1, \ldots, m_\beta \in \mathcal{M} \), then \( g_j(m_1, \ldots, m_\beta) \in \mathcal{M}, \quad j = 1, \ldots, \beta. \)

Put \( F_1 = f_1, \ldots, F_\beta = f_\beta \) and define the \( \mathcal{M} \)-functions \( G_j (j = 1, \ldots, \beta) \) as sets of pairs

\[ \langle \langle 1, m_1 \rangle, \ldots, \langle G_j, m_\beta \rangle, g_j(m_1, \ldots, m_\beta) \rangle, \]

where \( m_1, \ldots, m_\beta \) vary independently in \( \mathcal{M} \). Further let \( R_k (k = 1, \ldots, \gamma) \) be \( \mathcal{M} \)-relations defined by the equivalence

\[ \langle \langle 1, m_k \rangle, \ldots, \langle p_k, m_p \rangle \rangle \in R_k \iff r_k(m_1, \ldots, m_\beta). \]

The \( a \vdash \beta + \gamma + 1 \) tuple

\[ \mathcal{M}, F_1, \ldots, F_\beta, G_1, \ldots, G_\beta, R_1, \ldots, R_\gamma \]

constitutes a pseudo-model of the second kind of \( s \) in \( S \).

To show that this pseudo-model is a real model we remark that if \( s_\alpha \) is an axiom of \( s \) with the free variables \( z_0, \ldots, z_\delta \) and if \( u \) is any sequence \( \langle z_0, u_0 \rangle, \ldots, \langle z_\delta, u_\delta \rangle \) with \( u_0, \ldots, u_\delta \in \mathcal{M} \), then

\[ u \text{Stef} [s_\alpha] = s_\alpha(u_0, \ldots, u_\delta) \]
(cf. "convention W" in Tarski [11], p. 306). Since the right side of (4) is an axiom of $S$, we obtain $u \text{Strf}[a]$. This formula is provable in $S$, we infer that (3) is a real model of $s$ in $S$.

The construction carried out above is expressible in $S$, therefore on using theorem VII we obtain:

IX. If $s$ is a finitely axiomatizable sub-system of $S$, then the sentence $N(s)$ is provable in $S$.

Remarks. 1. The assumption made above that $s$ and $S$ both contain the $\varepsilon$-rule is not essential for the validity of theorem IX. Indeed, let $s'$ and $S'$ be systems without the $\varepsilon$-rule which become equivalent to $s$ and $S$ after adjunction of that rule and explicit definitions. It is evident that $N(s')$ is not stronger than $N(s)$ and hence provable in $S$. Since $N(s')$ is an arithmetical sentence in which the $\varepsilon$-symbol does not occur, it follows from the second $\varepsilon$-theorem of Hilbert and Bernays that $N(s')$ is provable without the $\varepsilon$-rule, i.e. in $S'$.

2. One might ask why our construction breaks down when $s$ contains infinitely many axioms, e.g. when $s\cong S$. To answer this question we recall that equivalences of the form (4) are provable in $S$ for each $a_i$ separately. There are no means by which to express in $S$ anything which could serve as a logical product of infinitely many such equivalences.

The following theorems are easy corollaries of IX:

X. If $S$ is self-consistent, it is not finitely axiomatizable 1).

Proof. First of all we remark that there exists a finitely axiomatizable sub-system $s_2$ of $S$ which is at least as strong as the arithmetic of integers based on Peano’s axioms with the axiom of mathematical induction (conceived as an axiom-schema).

To prove this we remark that this system of arithmetic is equivalent to the system (Z) of Hilbert-Bernays [2], p. 384. It has been shown by Novak and Wang [8], p. 90, that upon extending (Z) by the introduction of a new primitive notion and suitable axioms we obtain a system which is finitely axiomatizable. The new primitive notion is that of a predicative class of integers. The resulting system $s_2$ is therefore certainly weaker than $S$ since in $S$ we have at our disposal the general notion of classes which satisfies all the absolute axioms formulated by Novak and Wang in their construction. Hence $s_2$ is a sub-system of $S$.

The existence of $s_2$ being proved, we proceed as follows. According to a theorem of Gödel 2) the sentence $N(s)$ is provable in no self-consistent system $s$ which contains (Z). Hence if $S$ is self-consistent and $s$ is an axiomatizable sub-system of $S$ which contains $s_2$ then $N(s)$ is not provable in $s$. Since $N(s)$ is provable in $S$ according to IX, we infer that systems $s$ and $S$ are not equivalent.

Theorem X is thus proved.

XI. If $S$ is self-consistent, it is $\omega$-incomplete 3).

Proof. Let $S'$ be a system equivalent to $S$ but without the $\varepsilon$-rule and with a fixed set of primitive functions and predicates (i.e. explicit definitions are not allowed in $S'$). Let $a_1, a_2, \ldots$ be axioms of $S'$. We can assume that no $a_i$ contains free variables. Put $m(i) = [a_1, a_2, \ldots, a_i, \neg a_{i+1}, \ldots, a_n]$ and let $\Theta(a)$ be the matrix $m(a)$ is unprovable in the first order functional calculus.

It is easy to see that this matrix can be written in purely arithmetical terms and is therefore a matrix of $S$.

According to IX, sentences $\Theta(1), \Theta(2), \ldots$ are all provable in $S$ whereas the general statement $\forall x \Theta(x)$ is equivalent to $N(S)$ and hence unprovable in $S$ unless $S$ is inconsistent.

XII 4). If $S$ is self-consistent, then there exist consistent but $\varepsilon$-inconsistent sets of arithmetical sentences.

Indeed, sentences $\neg N(S), \Theta(1), \Theta(2), \Theta(3), \ldots$ form such a set.

5. Models of the second kind in the axiomatic theory of real numbers. Almost all we have said in sections 3 and 4 can be repeated when $S$ is an axiomatic theory of real numbers. When speaking of the arithmetic of real numbers, we have in mind systems in which the class of integers as well as the development of any real number into decimal (or other) fractions is definable and can be proved to exist.

---

1) Gödel [3], theorem XI, p. 196.
2) For the notion of $\omega$-completeness see Tarski [13]. The result obtained in theorem XI is of course not new.
3) Of course this result is not new either. See Gödel [3], p. 190 and Tarski [13], p. 108.
4) Fundamenta Mathematicae, T. XXXX.
Note that the arithmetic of real numbers in its usual formulation is based on an infinite number of axioms because the axiom of continuity cannot be expressed otherwise in a schema. The notions of functions and relations do not occur explicitly among the primitive notions of arithmetic. Some particular cases of these notions, however, are definable in arithmetic and these particular cases are general enough to enable us to carry over the proofs given in sections 4 and 5 from set-theory to arithmetic.

The procedure is as follows:

First, we define one-to-one correspondences between integers and finite sequences of integers, $k=1, 2, \ldots$. If an integer $n$ is made to correspond with a $k$-tuple $(n_1, \ldots, n_k)$ and $p_k$ is the $k$-th prime, then we shall identify the integer $p_k^n$ with the $k$-tuple $(n_1, \ldots, n_k)$. In this way we obtain an arithmetical substitute for the notion of a finite sequence of integers.

It is well known that we can effectively establish a one-to-one correspondence between real numbers and sets of integers. In other words, we can find a matrix $\Phi(s, a)$ such that the following formulas are provable in $S$:

$$\Phi(s, a) \models \text{an integer},$$

$$a' = a'^n \equiv (\exists x) \Phi(x', a) \equiv \Phi(x', a).$$

A real number $a$ will be called a $k$-termed relation if $(\forall x) (\exists y) \Phi(x, a)$ and $(\exists x) \Phi(x, a) = p_k^n$. Integers $n_1, \ldots, n_k$ are said to be in relation $x$ if the integer $n$ corresponding to the sequence $(n_1, \ldots, n_k)$ satisfies the condition $\Phi(x, a^n)$. In this case we write $x(n_1, \ldots, n_k)$.

A real number $a$ is called a function with $k$ arguments if it satisfies the following conditions:

$x$ is a binary relation,

$$a(x, y) \equiv \exists z (\exists w) \Phi(x, a) \equiv \Phi(x, a),$$

$$a(x, y) \equiv \exists z (\exists w) \Phi(x, a) \equiv \Phi(x, a).$$

The value of the function $a$ for the arguments $n_1, \ldots, n_k$ is defined as $(n_1, \ldots, n_k)^n$ where $m$ is the integer corresponding to the sequence $n_1, \ldots, n_k$.

Having defined the notions of functions and relations, we can reconstruct without difficulty all the definitions and proofs which were given in sections 3 and 4. In this way we arrive at the following results:

**XIII.** If the system $S$ of the arithmetic of real numbers is self-consistent and $a$ is a finitely axiomatizable sub-system of $S$, then the sentence $\forall a(s)$ is provable in $S$.

**XIV.** The arithmetic of real numbers is not finitely axiomatizable.

**6. Models of the third kind.** The method described in sections 3-5 does not apply in the case in which $S$ is the system of the arithmetic of positive integers based on Peano's postulates.

The failure of the method is caused by the fact that no model of the second kind is definable in the arithmetic of positive integers for a system $s$ in which the existence of infinitely many individuals is provable.

Models of the third kind which we shall discuss presently will enable us to prove theorems similar to IX-XII for the case in which $S$ is the system of the arithmetic of integers. These models were first defined by Hilbert and Bernays who stated their definitions in a non-formal language $\varepsilon$. We shall present here an arithmetical counterpart of the Hilbert-Bernays definition in order to discuss the possibility of its use in a formal system $S$.

We shall make the same assumptions concerning the systems $S$ and $s$ as in section 1. The system $S$ will however be slightly enlarged by the addition of the symbols $\Lambda$ and $\nu$, denoting the Boolean zero and the Boolean unit. Boolean addition and multiplication will be denoted by $+$ and $\cdot$ and, when many summands or factors are present, by the $\Sigma$- and $\Pi$-symbols. Boolean complementation will be denoted by an upper index 1. For symmetry we put $\Lambda^1 = \Lambda$ and $\nu^1 = \nu$.

A term $T$ of $s$ will be called a constant term if no variable occurs in it. It is easy to construct in arithmetic (and hence in $S$) a functor $T$ with one free variable such that the following formulas are provable in $S$:

- If $y$ is an integer, then $T(y)$ is a constant term;
- If $x$ is a constant term, then $x = T(y)$ for some $y$.

We shall now construct in $S$ a function $S(s, m)$ which enumerates all matrices that can be obtained from a $Q$-matrix $m$ by all possible substitutions of constant terms for the free variables of $m$.

We introduce first the auxiliary functors $s, S$, and $\mu$.

\[\dag\] See Hilbert-Bernays [9], pp. 32-36.
Let \( \sigma \) be a functor of \( S \) with two free variables such that the formula
\[
x = 2^\sigma_1 \cdot 3^\sigma_2 \cdot \cdots \cdot \alpha_{\sigma} \cdots
\]
is provable in \( S \). In other words, the definition of \( \sigma(y, x) \) is obtained by expressing \( S \) the following definition:

\[
\sigma(y, x) = 1 + (\text{the exponent of the } y\text{-th prime in the development of } x \text{ into the product of primes}).
\]

For any term \( t \) we put
\[
S(t, x, 0) = SB(\sigma_0, T, (t, x)),
\]
\[
S(t, x, y + 1) = SB(S(t, x, y), (y + 1, T, (t, y + 1, x))).
\]

This is clearly an inductive definition of the type which can be represented in \( S \) by a single functor. Hence \( S(t, x, y) \) is a functor of \( S \).

Let \( \mu(t) \) be the functor of \( S \) defined as the largest integer \( y \) such that \( \sigma_y \) occurs in \( t \). We put
\[
S(t, x) = S(t, x, \mu(t))
\]
and call \( S(t, x) \) the \( \alpha \)-th substitution of \( t \). \( S(t, x) \) is clearly a functor of \( S \).

It can be proved in \( S \) that
\[
(\alpha, 1) \cdot S(t, x) \text{ is a constant term}.
\]

If \( e = [\sigma_1, \ldots, \sigma_n] \) is an elementary formula, then we put
\[
S(\sigma, e) = [\sigma_1, S(t_1, x), \ldots, S(t_n, x)]
\]
and call \( S(\sigma, e) \) the \( \alpha \)-th substitution of \( e \). The following statement is provable in \( S \):

\[
(\alpha, e) \cdot S(\sigma, e) \text{ is a prime formula}
\]
(cf. section 1, definition 8).

We define now the \( \alpha \)-th substitution of an arbitrary \( Q \)-matrix \( m \). The definition proceeds by induction. If \( m = e \), then \( S(e, x) \) has been defined above. If \( m = [m_1, m_2] \), then we put \( S([m_1, m_2], x) = S([m_1, x], S([m_2, x])) \).

By a standard procedure we transform this inductive definition into an explicit one which can be expressed in \( S \).

We shall call a pseudo-model of the third kind or briefly a valuation, a functor \( \Phi \) or \( S \) with one free variable such that the following formula is provable in \( S \):

\[
\text{if } p \text{ is a prime formula, then } \Phi(p) = \top \text{ or } \Phi(p) = \bot.
\]

Let \( \Phi \) be an arbitrary valuation. We consider a functor \( \text{Val}_0(m, x) \) of \( S \) with two free variables satisfying the following conditions: a) the first argument of \( \text{Val}_0 \) runs over \( Q \)-matrices and the second over arbitrary positive integers; b) the following statements (5) and (6) are provable in \( S \):

\[
\text{Val}_0(e, x) = \Phi(S(e, x)),
\]
\[
\text{Val}_0([m_1, m_2], x) = (\text{Val}_0(m_1, x) \cdot \text{Val}_0(m_2, x))
\]
(the upper index 1 denotes here the Boolean complementation).

It is easy to construct effectively a functor \( \text{Val}_0 \) which satisfies the above conditions. All we have to do is to remark that conditions (5) and (6) can be considered as an inductive definition of \( \text{Val}_0 \) and that inductive definitions of this kind can be transformed into ordinary definitions which are expressible in \( S \).

Definition. A pseudo-model \( \Phi \) is called a real model of the third kind of \( s \) in \( S \) if the following formulas are provable in \( S \):

\[
\forall \sigma \left( \text{Val}_0([s_\sigma], x) = \top \right), \quad i = 1, \ldots, \sigma.
\]

Models of the third kind are in some respects similar to models of the first kind. Indeed, it follows from the definition that

1. The general notion of models of the third kind is defined not in \( S \) but in its syntax (since \( \Phi \) was defined as a functor of \( S \));
2. Every particular pseudo-model of the third kind can be defined within \( S \) (since each particular \( \Phi \) can be written down by means of symbols allowed in \( S \));
3. The notion of models of the third kind retains its meaning also for cases in which \( s \) contains an infinite number of axioms.

Indeed, \( \Phi \) is a real model if the formula

\[
(m) \left[ (m \text{ is an axiom of } s) \right] \left( \forall \sigma \left( \text{Val}_0([s_\sigma], x) = \top \right) \right)
\]
is provable in \( S \). This definition is meaningful not only when the axioms of \( s \) are finite in number and more generally when their set is definable in \( S \).

In this respect there is an analogy between models of the third and second kinds.

We shall now investigate the problem whether models of the third kind can be used to obtain absolute proofs of consistency.
Following Hilbert-Bernays [20], we shall call a Q-matrix m
verifiable if \( \exists x \forall \lambda \forall y (m, x, y) = \gamma \). The following theorem can then be proved:

**XV. The formula**

\( \langle m \rangle \langle w \rangle \langle \text{is a Q-matrix provable in } s \rangle, \exists \langle m \rangle \langle w \rangle \langle \text{is verifiable} \rangle \rangle 

is provable in S.

The proof of this theorem has been given by Hilbert-Bernays [5], pp. 33-36. We note that this proof is straightforward for the case in which m can be obtained from the axioms by the elementary calculus with free variables [20]. Hence the essential step in the proof of XV consists in proving in S the implication:

\( \langle m \rangle \langle w \rangle \langle \text{is provable in } s \rangle, \exists \langle m \rangle \langle w \rangle \langle \text{is verifiable} \rangle \rangle 

(\text{exponents } k \text{ as well as the symbols } \Pi \text{ and } \Sigma \text{ have here the Boolean meaning}).

**Proof.** Immediate from (5) and (6).

**Lemma 3.** If \( \Phi \) is an arbitrary valuation, then the following formula is provable in S:

\[
\forall x \forall \lambda \forall y (m, x, y) = \gamma
\]

(\text{exponents } k \text{ as well as the symbols } \Pi \text{ and } \Sigma \text{ have here the Boolean meaning}).

**Proof.** Immediate from (5) and (6).

**Lemma 4.** Under the assumptions of lemma 3 the following equivalence is provable in S:

\[
\forall x \forall \lambda \forall y (m, x, y) = \gamma
\]

(\text{exponents } k \text{ as well as the symbols } \Pi \text{ and } \Sigma \text{ have here the Boolean meaning}).

**Proof.** Immediate from lemma 3.

Let \( t \) run over constant terms of \( s \). We consider a functor \( \Theta \) of S such that the following equations be provable in S:

\[
\Theta(t_1) = t_1, \quad i = 1, \ldots, \alpha,
\]

\[
\Theta((t_1, \ldots, t_n)) = g(\Theta(t_1), \ldots, \Theta(t_n)), \quad j = 1, \ldots, f.
\]

It is easy to construct explicitly a functor satisfying these conditions. Indeed, (7) and (8) contain an inductive definition which can be transformed into an explicit one and the definitions of the explicit definition thus obtained is the required \( \Theta \).

The intuitive meaning of the functor \( \Theta \) can be explained as follows: Consider an arbitrary constant term \( t \) of \( s \); of course it denotes an integer. Let \( t \) be the Gödel number of \( t \). Then \( \Theta(t) \) is the integer denoted by \( t \).

We put \( \Theta(T(x)) = \Theta(x) \). Note that \( \theta \) is a term of \( S \) with one free variable.

\( \theta(x) \) is of course the integer denoted by the \( x \)-th constant term (in the enumeration of constant terms given by the function \( T \)).

**Lemma 5.** Let \( \Gamma \) be a term of \( s, t \) its Gödel number and \( k \) the largest integer such that \( t_k \) occurs in \( t \); Then the following equation is provable in S:

\[
\Theta(T(t_1)) = \begin{cases} 
\Theta(t_1), & \text{if } k = 1, \\
\Theta(t_2), & \text{if } k = 2, \\
\vdots \\
\Theta(t_k), & \text{if } k = k, \\
\vdots \\
\end{cases}
\]

\[
\Theta(T(t_{k+1})) = \begin{cases} 
\Theta(t_{k+1}), & \text{if } k + 1 \not\in t, \\
\vdots \\
\end{cases}
\]

(\text{free variables } i_1, \ldots, i_k \text{ and } x, y, z).
Proof. If \( \Gamma = \pi_t \), then \( t = 3_t \) \( S(t, x) = T(c(t, x)) \), and hence \( \Theta(S(t, x)) = \Theta(\sigma(t, x)) \). On the other hand the right side of (9) is \( \Theta(\sigma(t, x)) \). Hence the lemma is true in this case.

If \( \Gamma = \pi_f \), then \( t = 3_f \) and the left and right sides of (9) can easily be shown to be equal to \( f \).

Let us assume that the lemma is proved for terms \( \Gamma_1, \ldots, \Gamma'_q \) with the Gödel numbers \( t_1, \ldots, t'_q \). Let \( \Gamma \) be the term \( g_l(\Gamma_1, \ldots, \Gamma'_q) \); the Gödel number of \( \Gamma \) is \( l = [g_l(\Gamma_1, \ldots, \Gamma'_q)] \). It follows from lemma 21 in section 1 and from (8) that equations

\[
\begin{align*}
S(t, x) &= [g_l(S(t_1, x), \ldots, S(t'_q, x))], \\
\Theta(S(t, x)) &= g_l(\Theta(S(t_1, x)), \ldots, \Theta(S(t'_q, x)))
\end{align*}
\]

are provable in \( S \). On the other hand if we put

\[
\Gamma^{(q)} = \text{Subst} \Gamma^{(q)}[\alpha(1, x), \ldots, \alpha_q(\sigma(\Gamma) - \sigma(t))],
\]

we obtain from lemma 21 in section 1 the equation

\[
\text{Subst} \Gamma^{(q)}[\sigma(1, x), \ldots, \sigma_q(\sigma(\Gamma) - \sigma(t))]) = g_l(\Gamma^{(q)}, \ldots, \Gamma'_q^{(q)})
\]

are provable in \( S \). Since by the inductive assumption equations

\[
\Theta(S(t, x)) = \Gamma^{(q)}
\]

are provable in \( S \), we obtain the desired result by comparing formulas (10) and (11).

Lemma 5 is thus proved. Observe that this lemma is not a theorem of \( S \) but a theorem-schema. The statement of this lemma must be proved separately for every \( \Gamma \).

Definition. If \( \Gamma \) is a term and \( M \) a matrix of \( S \) and \( h \) is the largest integer such that \( x_h \) occurs in \( \Gamma \) or in \( M \), then we put

\[
\Gamma^{(0)} = \text{Subst} \Gamma^{(0)}[\alpha(1, x), \ldots, \alpha_h(\sigma(\Gamma) - \sigma(t))],
\]

\[
M^{(0)} = \text{Subst} M^{(0)}[\alpha(1, x), \ldots, \alpha_h(\sigma(\Gamma) - \sigma(t))].
\]

Note that \( \Gamma^{(0)} \) is a term of \( S \) with one free variable \( x \). Similarly \( M^{(0)} \) is a matrix of \( S \) with one free variable \( x \).

Lemma 6. If \( M = \sum_{i=1}^{m} E_{i, k}^{(0)} \) where the \( E_{i, k}^{(0)} \) are elementary formulas and the \( i, k \) are equal to 0 or 1, then

\[
M^{(0)} = \sum_{i=1}^{m} E_{i, k}^{(0)} M^{(0)}
\]

The proof is obvious.

We shall now define a valuation \( \Phi \) of which we shall show later that it is a real model of \( S \) in \( B \).

\[
\{ \Phi(p) = \epsilon \} = \sum_{j=1}^{\aleph_0} \{ \text{Ind } (p) = j \} \cdot \text{S} \cdot \text{S}(\text{Comp}_1(p), \ldots, \text{S}(\text{Comp}_p(p)))(\epsilon = \forall)
\]

\[
\{ \text{S} \cdot \text{S}(\text{Comp}_1(p), \ldots, \text{S}(\text{Comp}_p(p)))(\epsilon = \Lambda) \}.
\]

(Cf. section 1, def. 19, p. 137, for the definition of the functions \( \text{Ind } \) and \( \text{Comp} \)).

It is evident that \( \Phi \) is a functor of \( S \). The formula

\[
\Phi(p) = \forall \quad \text{or} \quad \Phi(p) = \Lambda
\]

is provable in \( S \). Indeed, it can be proved in \( S \) that if \( p \) is a prime formula, then \( \text{Comp}_p(p) \) is a constant term and hence \( \Theta(\text{Comp}_p(p)) \) is a perfectly defined term of \( S \).

Lemma 7. Let \( \Gamma_1, \ldots, \Gamma_p \) be an elementary formula and \( \epsilon = [\epsilon(t_1, \ldots, t_p)] \) its Gödel number. The following equivalences are then provable in \( S \) for \( \lambda = 0, 1, 2, \ldots, \)

\[
\Phi(S(t, x) = \forall \lambda = \Gamma_1^{(0)}, \ldots, \Gamma_p^{(0)}).
\]

Proof. By definition \( S(t, x) = [\epsilon(S(t_1, x), \ldots, S(t_p, x))] \) whence it follows that equations

\[
\text{Ind } (S(t, x)) = \epsilon, \\
\text{Comp}_1(S(t, x)) = S(t_1, x), \\
\cdots \\
\text{Comp}_p(S(t, x)) = S(t_p, x)
\]

are provable in \( S \). Now observe that the formula

\[
S(t, x) \text{ is a prime formula}
\]

is provable in \( S \). Upon using the definition of \( \Phi \) we obtain therefore the formula provable in \( S \)

\[
\Phi(S(t, x)) = \forall \lambda = \Gamma_1^{(0)}, \ldots, \text{S}(\text{S}(t_p, x)).
\]

Since equations \( \Theta(S(t_1, x)) = \Gamma_1^{(0)} \) are provable in \( S \) (cf. lemma 5), we get the desired result directly from the last formula.

Observe that lemma 7 is not a single theorem but a theorem-schema of \( S \).
To prove this theorem we need the following

Lemma 8. There exists a finitely axiomatizable sub-system $s$ of $S$ such that if $s$ is a finitely axiomatizable sub-system of $S$ which contains $s_0$, then the sentence $X(s)$ is not provable in $s$.

We shall content ourselves with a sketch of the proof.

We shall take for granted the arithmetization of the system $S$ along the lines indicated by Gödel [5]. The arithmetical counterparts of the metamathematical notions will be denoted by symbols used by Gödel although strictly speaking the symbols should be modified because the system arithmetized by Gödel is different from $S$.

Let $F$ be a primitive recursive function such that $F(w, n, p) = 0$ if and only if $w$, $p$ are sentences of $S$ and $w$ is a proof of the implication $p \implies n$ in the functional calculus of the first order.

Let $F'$ be defined as follows

\[ F'(w, n, p) = F\left(w, \text{Sh} n, \left\{ 19 \right\}, p \right). \]

Let $f$ be the Gödel number of the equation $F'(w, n, p) = 0$. Hilbert and Bernays [5], pp. 310-325, have shown that the following implication is provable in $S$:

\[ \forall \exists \left( 17 \vee 19 \vee 23 \right) \left( Z(m) \wedge \exists n, p \right). \]

Analyzing their proof we find that $\exists x$ whose existence is stated in (13) is a primitive recursive function $Q$ of $m, n, p$; the axioms of $S$ which occur in the proof with the Gödel number $Q(m, n, p)$ are finite in number and independent of $m, n, p$. If we denote by $A_1, \ldots, A_n$ these axioms and by $ax$ the Gödel number of their conjunction, we obtain instead of (13) the following implication provable in $S$:

\[ \forall \exists \left( 17 \vee 19 \vee 23 \right) \left( Z(m) \wedge \exists n, p \right) \wedge ax = 0. \]

Hilbert and Bernays have further shown (cf. [5], pp. 307-308) that the implication

\[ (\exists x) F'(x, 17 \text{ Gen } n, p) = 0 \implies \left( Z(q) \right) F'(x, \text{ Sh} n, \left\{ 19 \right\}, p) = 0 \]

is provable in $S$. 

---

This theorem is due to Ryll-Nardzewski who obtained it in [10] by a different method.
Now let $s_a$ be a sub-system of $S$ based on the axioms $A_1, \ldots, A_n$ and on those axioms of $S$ which are necessary to prove implications (14) and (15) as well as the recursion-equations for the functions $F, Sb, Imp, Neg, Z,$ and $Q$.

We shall show that $s_a$ has the property stated in the lemma. Indeed, let $s$ be a finitely axiomatizable sub-system of $S$ containing $s_a$ and denote by $A \vee \neg A$ the Gödel number of the conjunction of the axioms of $s$. It follows that (14) and (15) (when expressed in the symbols of $s$) are theorems of $s$.

From (14) we infer that the following implication is provable in $s$:

\[(16) \quad F'(m, n, p) \vdash F'(m, n, p), Sb(5, \{Z_n(m) \wedge Z_p(n, Z_p(m)) \neg A_x\}) = 0.\]

We put

\[\beta = Sb(5, \{Z(M)\}) \quad \gamma = 17 \text{ Gen Neg } \beta \quad \delta = Sb(5, \{Z(\gamma)\}).\]

It is easy to see that

\[\delta = 17 \text{ Gen Neg Sb } \beta(5, \{Z(\gamma)\}).\]

Repeating the argument of Gödel [3], pp. 187-188, we can prove that $\delta$ is unprovable in $s_a$, provided that $s$ is consistent, i.e.,

\[(17) \quad \neg F(s, \delta, A_x) \vdash 0.\]

This proof can be repeated word by word in the system $s$ owing to the circumstances that (15), (16), and recursion-equations for the functions $F, Sb, Imp, Neg, Z,$ and $Q$ are available in $s$. Hence if we denote by $\omega$ the Gödel number of the sentence $F(\omega)$ and observe that the Gödel number of the sentence $(s) = F(\omega, \delta, A_x) = 0$ is $\delta$, we obtain as an arithmetical counterpart of (17) in $s$ the equation

\[(18) \quad F'(\omega, \delta, A_x) = 0,\]

where $A$ is the Gödel number of the proof of (17) in $s$.

It follows from (18) that if $F(\omega)$ were provable in $s$ (i.e., if $\omega$ were provable in $s$), then $\delta$ would be provable in $s$ and hence by (17) $s$ would be inconsistent.

Lemma 8 is thus proved.

Theorem XIX results from lemma 8 by the same argument which was used in the proof of theorem X.
On Labil and Stabil Points.

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1. The concept of the (homotopically) labil point is due to H. Hopf and E. Pannwitz¹). Its definition can be formulated as follows:

**Definition 1.** A point \( a \) of a space \( M \) is **homotopically labil** whenever for every neighbourhood \( U \) of \( a \) there exists a continuous mapping \( f(x,t) \) which is defined in the Cartesian product \( M \times I \) of \( M \) and of the interval \( I: 0 \leq t \leq 1 \) and which satisfies the following conditions:

\[
\begin{align*}
(1) & \quad f(x,t) \in M \quad \text{for every} \quad (x,t) \in M \times I, \\
(2) & \quad f(a,0) = a \quad \text{for every} \quad x \in M, \\
(3) & \quad f(x,t) \rightarrow a \quad \text{for every} \quad (x,t) \in (M - U) \times I, \\
(4) & \quad f(x,t) \in U \quad \text{for every} \quad (x,t) \in U \times I, \\
(5) & \quad f(x,1) = a \quad \text{for every} \quad x \in M.
\end{align*}
\]

A point \( a \in M \) will be called **homotopically stabil** if it is not homotopically labil.

**Remark.** If \( a \) is a homotopically labil point of a space \( M \) and \( b \) a point of another space \( N \) and if there exists a homeomorphic mapping \( h \) of a neighbourhood \( U_b \) of \( a \) in \( M \) onto a neighbourhood \( V_b \) of \( b \) in \( N \) such that \( h(a) = b \), then \( b \) is a homotopically labil point.

¹) H. Hopf and E. Pannwitz, *Über die Deformationen von Komplexen in sich*, Math. Ann. **108** (1933), pp. 433-465. See also P. Alexandroff and H. Hopf, *Topologie I*, Berlin 1933, p. 323. In the present paper we slightly modify the terminology. Namely we shall refer to the points called by H. Hopf and E. Pannwitz labil, as homotopically labil. The term "labil" will be used here in the other sense.

²) H. Hopf and E. Pannwitz use the term "locally stabil point".