(a) La droite L est une droite d'appui de Γ_1. En vertu de (2), on a $L\Gamma_1 = \Gamma(L\Pi E) = \Gamma(LE)$, et, puisque $L\Gamma_1 = \{x\}$, on voit que $LE = \{x\}$.

(b) La droite L divise le domaine Γ_1. Comme $L\Gamma_1 = \{x\}$, l'ensemble convexe Γ_1 est dans ce cas un segment rectiligne, dont les extrémités a et b se trouvent dans Π aux côtés différents de L. Une parallèle L_1 à L, menée par le point a, est alors droite d'appui de Γ_1, et, puisque $L_1\Gamma_1 = \{a\}$, on est ramené au cas précédent.

6. Toute direction singulière par rapport à E étant aussi, d'après **5**, singulière par rapport à $\Gamma(E)$, il suffit d'établir notre théorème pour les domaines convexes.

Soit d'abord Γ un domaine convexe contenu dans un plan Π. Toute direction λ singulière par rapport à Γ est nécessairement celle d'un faisceau de droites de Π. Par conséquent, il existe, dans le plan Π, une droite d'appui L de Γ ayant la direction singulière λ. L'ensemble $L\Gamma$, n'étant pas vide, contient plus d'un point; L est, par suite, droite d'appui singulière de Γ. L'ensemble des directions singulières par rapport à Γ a donc une puissance au plus égale à la puissance de l'ensemble des droites d'appui singulières de Γ; il est, par suite, tout au plus dénombrable.

Considérons enfin un domaine convexe Γ quelconque. Supposons que les droites L_1 et L_2 aient des directions différentes λ_1, λ_2 singulières par rapport à Γ. Il existe alors un plan d'appui Π de Γ parallèle à L_1 et L_2. Les directions λ_1, λ_2 sont aussi singulières par rapport au domaine convexe $\Pi\Gamma$, donc, celui-ci n'est pas situé sur une droite; Π est, par suite, plan d'appui singulier de Γ. Nous constatons ainsi, que toute direction singulière par rapport à Γ (s'il y en a au moins deux) est direction singulière par rapport à un ensemble $\Pi\Gamma$, où Π est un plan d'appui singulier de Γ.

Mais l'ensemble des plans d'appui singuliers de Γ est tout au plus dénombrable; d'autre part, l'ensemble des directions singulières par rapport à un domaine convexe plan l'est aussi. Donc, l'ensemble des directions singulières par rapport à Γ est tout au plus dénombrable.

La démonstration est ainsi achevée. Il est à remarquer que le théorème s'étend sans difficulté à l'espace euclidien à $n > 3$ dimensions.

On the distinct sums of λ-type transfinite series obtained by permuting the elements of a fixed λ-type series.

By

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Sierpiński [1] has demonstrated: (1) that the number of distinct sums of all ω-type transfinite series which are permutations of the elements of a fixed ω-type series, is finite; and (2) that the sum of any ω-type series which is a permutation of the elements of a fixed, non-decreasing ω-type sequence, is independent of the permutation. The purpose of this note is to generalize those results. In particular we show: α) that for any ω_ϱ-type series $\sum_{\xi<\omega_\varrho} a_\xi$, where $\omega_\varrho > \omega$ is a regular ordinal, the number of distinct sums of all the ω_ϱ series obtained by permuting the elements of Σa_ξ, is equal to or smaller than

$$\aleph_\varrho^{\aleph_\varrho} = \sum_{\xi<\varrho} \aleph_\varrho^{\aleph_\xi};$$

and β) that (2) is still true when ω is replaced by ω_ϱ.

Let $N(\sum_{\xi<\lambda} a_\xi)$ be the number of distinct sums of all the λ-type series obtained by permuting the elements of the series Σa_ξ. Then

Theorem 1. *If ω_ϱ, $\varrho>0$, is a regular ordinal, then $N(\sum_{\xi<\omega_\varrho} a_\xi) \leqslant \aleph_\varrho^{\aleph_\varrho}$; furthermore, if $\omega_\varrho = \omega_{\alpha+1}$, then there exists an ω_ϱ series, Σa_ξ, such that $N(\Sigma a_\xi) = \aleph_\varrho$.*

Proof. An element, a_{ξ_0}, will be said to *have property P* [1], whenever

$$\overline{\overline{\{a_\xi / a_\xi \geqslant a_{\xi_0}, \xi \geqslant \xi_0\}}} < \aleph_\varrho .$$

By generalizing Sierpiński's argument we will show that the elements $\{a_{\nu(\mu)}\}_{\mu<\lambda}$ which have property P, form a sequence of order type less than ω_ϱ. Assume the contrary, i. e., assume that $\lambda = \omega_\varrho$. To $a_{\nu(0)} = a_{\tau(0)}$ there corresponds a $\nu^*(0)$ such that $a_\xi < a_{\tau(0)}$ for $\xi \geqslant \nu^*(0)$. As $\{a_{\nu(\xi)}\}$ is an ω_ϱ sequence there exists a $\tau(1) \geqslant \nu^*(0)$ such that $a_{\nu(\xi)} < a_{\tau(0)}$ for $\nu(\xi) \geqslant \tau(1)$. By induction we can define an ω sequence, $\{a_{\tau(n)}\}$, with $a_{\tau(n)} < a_{\tau(n-1)}$, a well known impossibility. Thus $\lambda < \omega_\varrho$.

If $\sum\limits_{\xi<\omega_\varrho} a_\xi = k_1 + \omega^\delta$, then for some φ_0,

$$\sum_{\xi \geqslant \varphi \geqslant \varphi_0} a_\xi = \omega^\delta.$$

Let $\{b_\xi\}_{\xi<\omega_\varrho}$ be any permutation of the elements of $\{a_\xi\}$. As ω_ϱ is regular, we are able to repeat the procedure given in [1] and find a θ_0 so that

$$\sum_{\xi \geqslant \theta \geqslant \theta_0} b_\xi = \omega^\delta.$$

Thus the value of $\sum\limits_{\xi<\omega_\varrho} b_\xi$ is determined by calculating the value of $\sum\limits_{\xi<\theta_0} b_\xi$. For each θ_0 there are $\aleph_\varrho^{\overline{\overline{\theta_0}}}$ lifferent subsets of $\{a_\xi\}$ of power $\overline{\overline{\theta_0}}$ [2], and $\overline{\overline{\theta_0}}^{\overline{\overline{\theta_0}}}$ permutations of the elements of each set. Hence

$$N\Big(\sum_{\xi<\omega_\varrho} a_\xi\Big) \leqslant \sum_{\theta_0<\omega_\varrho} \aleph_\varrho^{\overline{\overline{\theta_0}}} \overline{\overline{\theta_0}}^{\overline{\overline{\theta_0}}} = \aleph_\varrho^{\aleph_\varrho}.$$

Suppose $\aleph_\varrho = \aleph_{\alpha+1}$. Let $a_\xi = \omega_{\alpha+1}$ for $\xi < \omega_\alpha$, and $a_\xi = 1$ for $\xi \geqslant \omega_\alpha$. Then $N(\Sigma a_\xi) = \aleph_{\alpha+1}$ and the theorem is proved.

Theorem 2. *If ω_ϱ is a regular ordinal and $\{a_\xi\}_{\xi<\omega_\varrho}$ is a non-decreasing transfinite sequence of ordinals, then $N(\Sigma a_\xi) = 1$.*

Proof. Let ω_ϱ be regular and $\{a_\xi\}$ non-decreasing. For $\{b_\xi\}_{\xi<\omega_\varrho}$, a permutation of the elements of $\{a_\xi\}$, let $b_{\tau(0)} = b_0 \geqslant a_0$ and suppose defined $\{b_{\tau(\xi)}\}_{\xi<\theta<\omega_\varrho}$. Let λ_θ be the smallest ordinal λ, $\lambda \geqslant \tau(\xi)$, $\xi < \theta$. Since ω_ϱ is regular, and since there are $\aleph_\varrho$ elements a_μ, $a_\mu \geqslant a_\theta$, we can find a $b_{\tau(\theta)} \geqslant a_\theta$, with $\tau(\theta) \geqslant \lambda_\theta$. Therefore

$$\Sigma a_\xi \leqslant \Sigma b_{\tau(\xi)} \leqslant \Sigma b_\xi.$$

A similar procedure yields $\Sigma b_\xi \leqslant \Sigma a_\xi$, so that the equality sign holds.

In conclusion we remark that if λ is a non-regular limit number, then there exists an increasing sequence $\{a_\xi\}_{\xi<\lambda}$ and a permutation $\{b_\xi\}_{\xi<\lambda}$ such that $\Sigma a_\xi < \Sigma b_\xi$.

Bibliography.

[1] W. Sierpiński, *Sur les séries infinies de nombres ordinaux*, Fund. Math. **36** (1949), p. 248-253.

[2] A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. **16** (1930), p. 181-304.

On models of axiomatic systems.

By

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This paper is devoted to a discussion of various notions of models which appear in the recent investigations of formal systems. The discussion will be applied to the study of the following problem: Given a formal system S based on an infinite number of axioms $A_1, A_2, A_3, \ldots$ is it possible to prove in S the consistency of the system based on a finite number $A_1, A_2, \ldots, A_n$ of these axioms?

1. Notations and definitions. We shall consider two systems S and s based on the functional calculus of the first order[1]. We shall not describe these systems in detail but give only some definitions which will be required later.

System s. We assume that the following symbols occur among the primitive signs of s:

1. Variables: $x_1, x_2, x_3, \ldots, x, y, z, \ldots$
2. Individual constants: $f_1, \ldots, f_\alpha$.
3. Functors (i. e. symbols for functions from individuals to individuals): $g_1, \ldots, g_\beta$. We denote by q_j the number of arguments of g_j $(j = 1, \ldots, \beta)$.
4. Predicates (i. e. symbols for relations): $r_1, \ldots, r_\gamma$. We denote by p_i the number of arguments of r_i $(i = 1, \ldots, \gamma)$.
5. Propositional connectives and quantifiers. We use the symbol $|$ for the "stroke function" and define other connectives in terms of the stroke. Quantifiers are denoted by symbols $(\exists x_i)$ and (x_i).

Among expressions which can be constructed from these signs we distinguish the following:

6. Terms. Variables and individual constants are terms. If $\Gamma_1, \ldots, \Gamma_{q_j}$ are terms, then so is $g_j(\Gamma_1, \ldots, \Gamma_{q_j})$, $j = 1, \ldots, \beta$.

Terms will be denoted by the letters $\Gamma, \Gamma_1, \Gamma_2, \ldots$

[1]) For the functional calculus of the first order see *e. g.* Church [1], Chapter II.