Proof. We have only to prove: *if $X \subset P$ is discrete and is not a Q-space, then P cannot be a Q-space.* Put $S = P - (\overline{X} - X)$. Then X is a closed subset of S, hence, by the above theorem, S is not a Q-space. Since P is metrisable, $S \in \boldsymbol{F}_\sigma(P)$, $S = \sum_{n=1}^{\infty} S_n$, $S_n \in \boldsymbol{F}(P)$. Lemma 7 implies that some S_n is not a Q-space; therefore, by Proposition 4, P is not a Q-space.

Remark. It is easy to show that a discrete space is a Q-space if and only if it does not admit of a non-reducible two-valued Borel measure, that is if its power has two-valued measure zero. Therefore, Theorem 3 may be given the following equivalent form: *A fully normal space P is a Q-space if and only if the power of any closed discrete subset of P has two-valued measure zero.*

References.

[1] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures et Appliquées (9) **23** (1944), pp. 65-76.

[2] E. Hewitt, *Rings of Continuous Functions, I*, Trans. Amer. Math. Soc. **64** (1948), pp. 45-99.

[2a] E. Hewitt, *Linear functionals on spaces of continuous functions*, Fund. Math. **37** (1950), pp. 161-189.

[3] C. Kuratowski, *Topologie I*, 2-me éd., Warszawa-Wrocław 1948.

[4] S. Lefschetz, *Algebraic Topology*, New York 1942.

[5] E. Marczewski and R. Sikorski, *Measures in non-separable metric spaces*, Coll. Math. **1** (1948), pp. 133-139.

[6] D. Montgomery, *Non-separable Metric Spaces*, Fund. Math. **25** (1935), pp. 527-533.

[7] A. H. Stone, *Paracompactness and Product Spaces*, Bull. Amer. Math. Soc. **54** (1948), pp. 977-982.

[8] J. Tukey, *Convergence and Uniformity in Topology*, Princeton 1940.

[9] N. Vedenisoff, *Sur les fonctions continues dans les espaces topologiques*, Fund. Math. **27** (1936), pp. 234-238.

Wrocław, May 1950.

On Real-Valued Functions in Topological Spaces.

By

M. Katětov (Praha).

The following theorem [1]) of H. Hahn is well-known: if g and h are real-valued functions in a metric space P, g is upper semicontinuous [2]), h is lower semicontinuous [2]), and $g(x) \leqslant h(x)$ for any $x \in P$, then there exists a continuous function f such that $g(x) \leqslant f(x) \leqslant h(x)$ for every $x \in P$. If $<$ is substituted for $\leqslant$, the theorem still holds. In his paper [4], J. Dieudonné has extended Hahn's theorem (with $\leqslant$ or $<$) to paracompact [3]) spaces. In the present note, it is shown that Hahn's theorem holds (i) with $\leqslant$, in arbitrary normal [4]) spaces (Theorem 1); (ii) with $<$, in a broad class (specified in Theorem 2) of normal spaces including paracompact, countably compact [5]) and perfectly normal [6]) ones (as a matter of fact, I do not know whether there exists any normal space not belonging to this class).

---

1) See e. g. [5], 36. 2. 6 (numbers in brackets refer to the list at the end of the present note).

2) A real-valued function g defined in a topological space P is called *upper semicontinuous* if, for any $a \in P$ and any $c > g(a)$, there exists a neighbourhood U of a such that $c > g(x)$ whenever $x \in U$. Substituting $<$ instead of $>$, we obtain the definition of the lower semicontinuity.

3) A topological space P is called *paracompact* if, for any open covering $\mathfrak{G}$ of P, there exists an open covering $\mathfrak{H}$ which refines $\mathfrak{G}$ (i. e. every $H \in \mathfrak{H}$ is contained in some $G \in \mathfrak{G}$) and is locally finite (i. e. such that every point has a neighbourhood intersecting only a finite number of sets $H \in \mathfrak{H}$). See J. Dieudonné's paper [4].

4) A topological space P is called *normal* if any two disjoint closed sets possess disjoint neighbourhoods.

5) A topological space is called *countably compact* if every countable open covering contains a finite subcovering.

6) A normal space it called *perfectly normal* if every closed set can be represented as the intersection of countably many open sets.

Theorem 3 of the present note concerns extensions of uniformly continuous functions defined in subsets of uniform spaces [7]. This theorem seems to be essentially known without having been explicitly stated as yet.

The proof of both Theorems 1 and 3 rests on a simple lemma concerning binary relations. Since Theorem 1 implies the classical Tietze-Urysohn Extension Theorem, we get, in this way, a direct proof of the Extension Theorem avoiding Urysohn's Lemma.

Notation. If A, B are propositions, then $A \to B$ stands for „A implies B". „Function" always means a real-valued function.

Definitions. Let R, T be sets, and let ϱ, τ be n-ary relations defined in R and, respectively, in T. Then R^T denotes the set of all transformations or the set T into R, and ϱ^τ denotes the n-ary relation in R^T defined as follows: $\varrho^\tau(f_1, \ldots, f_n)$ *if and only if* $\tau(t_1, \ldots, t_n) \to \varrho\big(f_1(t_1), \ldots, f_n(t_n)\big)$.

We shall say that a binary relation ϱ in R possesses the *Interpolation Property* (cf. Birkhoff [1], p. 52) if, given finite sets $A \subset R$, $B \subset R$ such that [8] $a\varrho b$ whenever $a \in A$, $b \in B$, there always exists $c \in R$ such that $a\varrho c$, $c\varrho b$ whenever $a \in A$, $b \in B$.

Lemma. *Let a binary relation ϱ in R possess the Interpolation Property. Let T be countable and let τ be a transitive irreflexive (i. e. such that $t\tau t$ never holds) relation in T. Then, for any $g \in R^T$ and $h \in R^T$ such that $h\varrho^\tau g$, there exists $f \in R^T$ such that $h\varrho^\tau f$, $f\varrho^\tau f$, $f\varrho^\tau g$.*

---

[7]) Let P be a set and let $\mathfrak{U}$ be a family of sets $U \subset P \times P$ such that (1) every set $U \in \mathfrak{U}$ contains all (x,x), $x \in P$; (2) if $U \in \mathfrak{U}$, $U \subset V \subset P \times P$, then $V \in \mathfrak{U}$; (3) if $U_1 \in \mathfrak{U}$, $U_2 \in \mathfrak{U}$, then $U_1 U_2 \in \mathfrak{U}$; (4) for any $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $(z,x) \in \mathfrak{U}$ whenever $(x,y) \in V$, $(y,z) \in V$. Then we shall say that $\mathfrak{U}$ is a *uniformity* in P; the set P together with the uniformity $\mathfrak{U}$, will be called a *uniform space* (see e. g. Bourbaki [2]). Example: a metric space P with $\mathfrak{U}$ consisting of all $U \subset P \times P$ containing, for some $\varepsilon > 0$, all (x,y) with $\varrho(x,y) < \varepsilon$.

A uniform space P is always given the topology defined as follows: $x \in \overline{M}$ if and only if every $U \in \mathfrak{U}$, where $\mathfrak{U}$ denotes the uniformity of P, contains some (x,y), $y \in M$.

A real-valued function f defined in a uniform space P (with the uniformity $\mathfrak{U}$) is called *uniformly continuous* if, for any $\varepsilon > 0$, there exists $U \in \mathfrak{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $(x,y) \in U$. This is, evidently a generalisation of the notion of uniform continuity in metric spaces.

[8]) If ϱ is a binary relation, then $a\varrho b$ means, of course, that a is in the relation ϱ to b.

Remark. Evidently, $f\varrho^\tau f$ if and only if f is a „homomorphism" with respect to τ, ϱ, i. e. if $t_1 \tau t_2$ implies $f(t_1)\varrho f(t_2)$.

Proof. Let all $t \in T$ be arranged in a sequence $\{t_n\}$, $t_m \neq t_n$ for $m \neq n$, and let T_n $(n = 1, 2, \ldots)$ denote the set of t_k, $k \leqslant n$. Suppose (which we evidently may for $n = 1$) that

(C_n) if $t\tau t'$, $t \in T_n$, $t' \in T_n$, then $h(t)\varrho f(t')$, $f(t)\varrho f(t')$, $f(t)\varrho g(t')$.

Let M denote the set of all $h(t)$ and $f(t)$ where $t \in T_n$, $t\tau t_{n+1}$, and let N denote the set of all $f(t)$ and $g(t)$ where $t \in T_n$, $t_{n+1}\tau t$. Since τ is transitive, we have $x\varrho y$ whenever $x \in M$, $y \in N$. Therefore, by the Interpolation Property, there exists $z \in R$ such that $x \in M \to x\varrho z$, $y \in N \to z\varrho y$. Putting $f(t_{n+1}) = z$ we see at once that (C_{n+1}) holds true. The proof is now completed by an obvious induction.

Theorem 1. *If P is a normal space, g and h are functions in P, g is upper semicontinuous, h is lower semicontinuous, and $g(x) \leqslant h(x)$ for any $x \in P$, then there exists a continuous function f in P such that, for any $x \in P$, $g(x) \leqslant f(x) \leqslant h(x)$.*

Proof. Let R denote the collection of all $X \subset P$; if $X \in R$, $Y \in R$, put $X\varrho Y$ if and only if $\overline{X} \subset \operatorname{Int} Y$. Let τ be the relation of (natural) order in the set T of rational numbers (that is, $t\tau t' \rightleftarrows t < t'$). For any rational t, let $H(t)$ denote the set of $x \in P$ such that $h(x) \leqslant t$, and let $G(t)$ denote the set of $x \in P$ such that $g(x) < t$. It is easy to see that every $H(t)$ is closed, every $G(t)$ is open, and $t_1 < t_2 \to H(t_1) \subset G(t_2)$. Thus we have $G \in R^T$, $H \in R^T$, $H\varrho^\tau G$. Since ϱ has the Interpolation Property (this follows at once from the normality of P) there exists, by the above lemma, $F \in R^T$ such that $H\varrho^\tau F$, $F\varrho^\tau F$, $F\varrho^\tau G$, hence $H(t_1) \subset \operatorname{Int} F(t_2)$, $\overline{F(t_1)} \subset \operatorname{Int} F(t_2)$, $\overline{F(t_1)} \subset \operatorname{Int} G(t_2)$ whenever $t_i \in T$, $t_1 < t_2$. For any $x \in P$, let $f(x)$ be equal to the g.l.b. of numbers $t \in T$ such that $x \in F(t)$. Then f is a real-valued function in P; for $\sum H(t) = P$, $\prod_t G(t) = 0$, hence every $x \in P$ lies in some $F(t)$ and in some $P - G(t)$, and therefore the values $f(x) = \pm\infty$ cannot occur. If $x \in F(t)$, then $x \in G(t')$ whenever $t' > t$, and therefore $g(x) \leqslant t$; if x non $\in F(t)$, then x non $\in H(t')$ whenever $t' < t$ and therefore $h(x) \geqslant t$. Hence $g(x) \leqslant f(x) \leqslant h(x)$ for every $x \in P$. If $t_1 < f(x) < t_2$, $t_i \in T$, then it is easy to see that $x \in \operatorname{Int} F(t_2) - \overline{F(t_1)}$, and $y \in \operatorname{Int} \overline{F(t_2)} - F(t_1) \to t_1 \leqslant f(y) \leqslant t_2$. Thus f is continuous.

From Theorem 1, it is easy to deduce [9]) the Tietze-Urysohn Extension Theorem: *If P is a normal space, $Q \subset P$ is closed, f is continuous mapping of P into an interval* [10]) *J of reals, then there exists a continuous mapping F of P into J coinciding with f in Q.*

Remark. If P is a non-normal completely regular [11]) space, then it may happen that, for some closed set $Q \subset P$, every bounded continuous function in Q has an extension [12]) over P whereas no unbounded continuous function in Q has such an extension.

Example. Let E be the space of real numbers and let $Q \subset E$ be the set of all integers. Let βE denote the Čech (bi)compactification [13]) of E and put $P = \beta E - (\bar{Q} - Q)$, where $\bar{Q}$ denotes, of course, the closure of Q in βE. Since E is normal, every bounded continuous function in Q admits of an extension over E, hence over P. Now let f be an unbounded continuous function in Q and suppose that there exists a continuous function F in P such that $x \in Q \to F(x) = f(x)$. It is easy to see that there exists a closed (in E) set $A \subset E - Q$ such that $F(A)$ is not bounded. Since $\bar{Q}$ and $\bar{A}$ (closures in βE) are disjoint, the closure of A in P is equal to $\bar{A}$, hence compact. Thus $F(A)$ is bounded and we have a contradiction.

Theorem 2. *If P is normal, then the following conditions are equivalent*:

(a) *if g, h are functions in P, g is upper semicontinuous, h is lower semicontinuous and $g(x) < h(x)$ for every $x \in P$, then there exists a continuous function f in P such that, for any $x \in P$, $g(x) < f(x) < h(x)$;*

(b) *every countable open covering of P has a locally finite refinement;*

(c) *every countable open covering of P has a point-finite* [14]) *refinement;*

(d) *every countable open covering is shrinkable* [15]);

(e) *if $F_n \subset P$ are closed, $F_n \subset F_{n+1}$ $(n = 1, 2, \ldots)$, $\prod_{n=1}^{\infty} F_n = 0$, then there exist open sets $G_n \supset F_n$ such that $\prod_{n=1}^{\infty} G_n = 0$.*

Proof. I. If (a) holds, let G_n be open, $\sum_{n=1}^{\infty} G_n = P$. Put

$$U_n = G_1 + \ldots + G_n \qquad (n = 1, 2, \ldots)$$

and put

$$h(x) = \begin{cases} 1 & \text{if } x \in U_1, \\ n^{-1} & \text{if } x \in U_n - U_{n-1} \end{cases} \qquad (n = 2, 3, \ldots).$$

Since h is clearly a lower semicontinuous function, there exists (for we can put $g(x) = 0$, for any x, and make use of the property (a)) a positive continuous function f in P such that $x \in U_1 \to f(x) < 1$,

$$x \in U_n - U_{n-1} \to f(x) < n^{-1} \qquad (n = 2, 3, \ldots).$$

Let I_k $(k = 1, 2, \ldots)$ denote the open interval with endpoints $(k+2)^{-1}$, k^{-1} and put $H_k = f^{-1}(I_k)$. Clearly, $H_k \subset U_{k+1}$ $(k = 1, 2, \ldots)$, $\sum_{k=1}^{\infty} H_k = P$, and every $x \in P$ has a neighbourhood intersecting two sets H_k at most. It is easy to show that the collection of all non-void sets $H_k G_l$, $l \leqslant k+1$, is a locally finite open covering of P which refines $\{G_n\}$. Thus (a) implies (b).

II. Evidently, (b) implies (c).

III. If (c) holds, let $\{G_n\}$ $(n = 1, 2, \ldots)$ be an open covering of P. Let $\{H_\nu\}$, ν running over an arbitrary given set of indices, be a point-finite refinement of $\{G_n\}$. For any ν, choose $m = m(\nu)$ such that $H_\nu \subset G_m$. Let U_n $(n = 1, 2, \ldots)$ denote the sum of all H_ν such that $m(\nu) = n$. Then $U_n \subset G_n$ $(n = 1, 2, \ldots)$, $\{U_n\}$ is point-finite. Now apply the following well-known (see e. g. Lefschetz [5], p. 26) theorem: *every point-finite covering of a normal space is shrinkable.*

---

[9]) In a well-known way: if $I = [\alpha, \beta]$ is closed, put $\varphi(x) = \psi(x) = f(x)$ if $x \in Q$ and $\varphi(x) = \alpha$, $\psi(x) = \beta$ if $x \in P - Q$. Then $\varphi(x) \leqslant \psi(x)$, φ is upper semicontinuous, ψ is lower semicontinuous. Hence there exists a continuous function F in P with $\varphi(x) \leqslant F(x) \leqslant \psi(x)$, for any $x \in P$. Clearly, $x \in Q \to F(x) = f(x)$. For the case of a non-closed interval I see e. g. Bourbaki [3], p. 65.

[10]) Any interval, closed or not, bounded or unbounded.

[11]) A topological space P is called *completely regular*, if, for any closed $M \subset P$ and any $x \in P - M$, there exists a continuous function f in P such that $f(x) = 1$ and $y \in M \to f(y) = 0$.

[12]) This means: there exists a continuous function F in P such that $x \in Q \to F(x) = f(x)$.

[13]) If S is a completely regular space, then there exists an essentially unique compact (=bicompact) space, denoted by βS and called the *Čech compactification of S*, such that $S \subset \beta S$, $\bar{S} = \beta S$ and every bounded continuous function in S has an extension over βS.

[14]) A covering $\mathfrak{A}$ of a space P is called *point-finite* if every $x \in P$ belongs to a finite number of sets $A \in \mathfrak{A}$.

[15]) If $\{G_\nu\}$, ν running over an arbitrary given set of indices, is an open covering of a space P, then we shall say that $\{G_\nu\}$ is *shrinkable* if there exist closed sets $F_\nu \subset G_\nu$ such that $\sum_\nu F_\nu = P$. Cf. Lefschetz [5], p. 26.

IV. If (d) holds, let F_n be closed, $F_n \supset F_{n+1}$, $\prod_{n=1}^{\infty} F_n = 0$. Put $H_n = P - F_n$. Then $\sum_{n=1}^{\infty} H_n = P$ and therefore there exist closed sets $A_n \subset H_n$ such that $\sum_{n=1}^{\infty} A_n = P$. Put $G_n = P - A_n$. Then $G_n \supset F_n$, $\prod_{n=1}^{\infty} G_n = 0$.

V. If (e) holds, let g and h be functions in P; suppose that g is upper semicontinuous, h is lower semicontinuous, $g(x) < h(x)$ for all $x \in P$. Let F_n $(n=1,2,\ldots)$ denote the set of $x \in P$ such that $h(x) - g(x) \leqslant 3^{-n+1}$. Clearly, $F_n \supset F_{n+1}$, $\prod_{n=1}^{\infty} F_n = 0$. Therefore, there exist open sets $G_n \supset F_n$ such that $\prod_{n=1}^{\infty} G_n = 0$. Since P is normal, there exist continuous functions φ_n in P $(n=1,2,\ldots)$ such that we always have $0 \leqslant \varphi_n(x) \leqslant 1$, $\varphi_n(x) = 0$ if $x \in F_n$, $\varphi_n(x) = 1$ if $x \in P - G_n$. For any $x \in P$, put $\varphi(x) = \sum_{n=1}^{\infty} 3^{-n} \varphi_n(x)$. Then φ is a continuous function in P. Clearly we have $0 < \varphi(x) \leqslant \frac{1}{2}$, for any $x \in P$, and $\varphi(x) \leqslant \frac{1}{2} 3^{-n}$ for $x \in F_n$. Since every $x \in P$ lies either in $P - F_1$ or in some $F_n - F_{n+1}$, we get at once $2\varphi(x) < h(x) - g(x)$, for any $x \in P$. Putting $g_1 = g + \varphi$, $h_1 = h - \varphi$ and applying Theorem 1 to g_1 and h_1, we see that (e) implies (a). This completes the proof.

Remarks. 1° I do not know whether there exists any normal space which does not possess properties (a)—(e).

2° It is clear that every paracompact or countably compact space has property (b) and every perfectly normal space has property (e). Hence the class of normal spaces possessing properties (a)—(e) includes paracompact, countably compact and perfectly normal ones.

We shall now consider uniformly continuous functions in uniform spaces.

Theorem 3. *Let P be a uniform space and let f be a bounded uniformly continuous function in a subspace $Q \subset P$. Then there exists a bounded uniformly continuous function F in P which coincides with f in Q.*

Proof. Let R denote the collection of all $X \subset P$. If $X \in R$, $Y \in R$, put $X \varrho Y$ if and only if there exists $U \in \mathfrak{U}$ (where $\mathfrak{U}$ denotes, of course, the uniformity of the space P) such that $x \in X$, $(x,y) \in U \rightarrow y \in Y$. It is easy to see (cf. footnote 7, property (4)) that the relation ϱ has the Interpolation Property. Let T denote the set of rational numbers; if $t_1 \in T$, $t_2 \in T$ put $t_1 \tau t_2 \rightleftarrows t_1 < t_2$. Let a and β denote respectively the g. l. b. and the l. u. b. of numbers $f(x)$, $x \in Q$. If $t \in T$, $a \leqslant t \leqslant \beta$, let $A(t)$ denote the set of points $x \in Q$ such that $f(x) \leqslant t$ and put $B(t) = A(t) + (P - Q)$. If $t \in T$, $t < a$, put $A(t) = B(t) = 0$; if $t \in T$, $t > \beta$, put $A(t) = B(t) = P$. It is easy to see that, for $t_1 \in T$, $t_2 \in T$, $t_1 < t_2$ implies $A(t_1) \varrho B(t_2)$. Thus we have $A \varrho^\tau B$ and therefore, by the lemma on binary relations, there exists $C \in R^T$ such that $A \varrho^\tau C$, $C \varrho^\tau C$, $C \varrho^\tau B$, that is $A(t_1) \varrho C(t_2)$, $C(t_1) \varrho C(t_2)$, $C(t_1) \varrho B(t_2)$ whenever $t_i \in T$, $t_1 < t_2$.

For any $x \in P$, let $F(x)$ be equal to the g. l. b. of numbers $t \in T$ such that $x \in C(t)$. If $x \in Q$, $t_1 < f(x) < t_2$, where $t_i \in T$, then clearly $x \in A(t_2) - B(t_1)$ and therefore $x \in C(t_2') - C(t_1')$ whenever $t_i' \in T$, $t_1' < t_1$, $t_2 < t_2'$; hence $t_1 \leqslant F(x) \leqslant t_2$. Therefore, $x \in Q \rightarrow F(x) = f(x)$. Clearly, $x \in P \rightarrow a \leqslant F(x) \leqslant \beta$.

It remains to prove that F is uniformly continuous. Given $\varepsilon > 0$, choose numberr $t_k \in T$ $(k = 0,1,\ldots,n+1)$ such that $0 < t_{k+1} - t_k < \frac{1}{2}\varepsilon$, $t_0 \leqslant a$, $\beta \leqslant t_{n+1}$. Since $C(t_k) \varrho C(t_{k+1})$, there exist sets $U_k \in \mathfrak{U}$ $(k=0,1,\ldots,n)$ such that $y \in C(t_{k+1})$ whenever $x \in C(t_k)$, $(x,y) \in U_k$. Put $U = \prod_{k=0}^{n} U_k$. Then $U \in \mathfrak{U}$ (cf. footnote 7, property (3)). Clearly we have $F(y) \leqslant t_{k+1}$ whenever $F(x) < t_k$, $(x,y) \in U$; therefore, $|F(x) - F(y)| \leqslant \varepsilon$ whenever $(x,y) \in U$. This completes the proof.

References.

[1] G. Birkhoff, *Lattice Theory*, revised edition. New York 1948.

[2] N. Bourbaki, *Eléments de mathématiques, Les structures fondamentales de l'analyse*, Livre III, Topologie générale, Chap. I et II, Paris 1940.

[3] —, *Eléments de mathématique, Les structures fondamentales de l'analyse*, Livre III, Topologie générale, Chap. IX, Paris 1948.

[4] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures et Appliquées (9) **23**, (1944), pp. 65-76.

[5] H. Hahn, *Reelle Funktionen I*, Wien 1932.

[6] S. Lefschetz, *Algebraic Topology*, New York 1942.