is a 6-dimensional Cantor-manifold. In fact by corollary 3 of Nr **13**, every one of the four summands is a 6-dimensional Cantor-manifold and the common part of two successive summands is the Cartesian product of the 4-dimensional (by (18)) set homeomorphic to $P_2 \times P_2$ or to $P_3 \times P_3$ and of a 2-dimensional element, hence [38]) it is also 6-dimensional.

15. ***Problems.*** Is the Cartesian product of an n-dimensional Cantor-manifold and a 1-dimensional continuum always an $(n+1)$-dimensional Cantor-manifold?

Is the Cartesian product of two locally contractible Cantor-manifolds always a Cantor-manifold?

If $A \times B$ is a locally contractible Cantor-manifold is it true that A and B are also Cantor-manifolds?

If $A \times B$ is an approximative pseudo-manifold is it true that A and B are also approximative pseudo-manifolds?

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[38]) See footnote [31]).

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Measures in Fully Normal Spaces.

By

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The present note contains two decomposition theorems concerning Borel measures in fully normal (i. e. paracompact) spaces. These theorems are closely related to the results of E. Marczewski and R. Sikorski [5] on Borel measures in metric spaces. The third theorem, proved by similar methods, asserts that every fully normal space is a Q-space, in the sense of E. Hewitt [2], unless some of its closed discrete subspaces are not so. It may be noticed that it is possible to deduce this result from the decomposition theorems of the present note and E. Hewitt's results [1]) concerning measures in Q-spaces.

All spaces considered are completely regular [2]) topological spaces.

The following notations are used: if P is a space, then $\boldsymbol{F}(P)$, $\boldsymbol{G}(P)$, $\boldsymbol{F}^*(P)$, $\boldsymbol{G}^*(P)$ denote, respectively, the family of all closed sets, the family of all open sets, the family of all sets of the form $f^{-1}(M)$, f continuous real-valued, M closed (or, equivalently, of the form $f^{-1}(0)$, f continuous real-valued), and the family of complements of sets from $\boldsymbol{F}^*(P)$. The meaning of $\boldsymbol{F}_\sigma(P)$, $\boldsymbol{F}^*_\sigma(P)$, $\boldsymbol{G}_\delta(P)$, $\boldsymbol{G}^*_\delta(P)$ is clear. $\boldsymbol{B}(P)$ or $\boldsymbol{B}^*(P)$ denotes the least σ-field containing $\boldsymbol{F}(P)$ or $\boldsymbol{F}^*(P)$ respectively. The sets belonging to $\boldsymbol{B}(P)$ will be called *Borel sets* (relative to P); those belonging to $\boldsymbol{B}^*(P)$ will be called *Baire sets* (relative to P).

Clearly, we always have $\boldsymbol{B}^*(P) \subset \boldsymbol{B}(P)$. If P is perfectly normal [3]), then $\boldsymbol{F}^*(P) = \boldsymbol{F}(P)$ (see e. g. [9]) and therefore $\boldsymbol{B}^*(P) = \boldsymbol{B}(P)$.

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[1]) See [2a], Theorem 16.

[2]) A Hausdorff space P is called *completely regular* if, for any closed set $A \subset P$ and any $x \in P - A$, there exists a real-valued continuous function f in P such that $f(x) = 1$, $f(A) = 0$.

[3]) A normal space P is called *perfectly normal* if $\boldsymbol{F}(P) \subset \boldsymbol{G}_\delta(P)$.

Lemma 1. *The class $F^*(P)$ is countably multiplicative, i.e. $\prod_{n=1}^{\infty} F_n \in F^*(P)$ whenever $F_n \in F^*(P)$, and finitely additive, i.e. $\sum_{n=1}^{p} F_n \in F^*(P)$ whenever $F_n \in F^*(P)$.*

Proof. Let $F_n = f^{-1}(0)$, f_n continuous real-valued, $0 \leqslant f_n(z) \leqslant 1$ for every $z \in P$. Put $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. Then f is continuous, $\prod_{n=1}^{\infty} F_n = f^{-1}(0)$. Put $g = \prod_{n=1}^{p} f_n$. Then g is continuous, $\sum_{n=1}^{p} F_n = g^{-1}(0)$.

Remarks. 1⁰ It can happen that $M \in F(P)$, $M \in B^*(P)$, but M non $\in F^*(P)$.

Example. P is obtained from the real line by declaring every irrational number to be isolated. It is easy to see that P is a normal space. Denoting by R the set of rational numbers, we have $\bar{R} = R$, $R \in F_\sigma^*(B) \subset B^*(P)$. If $G \subset P$ is open (in P), $G \supset R$, then there exists a set H, $G \supset H \supset R$, which is open in the usual topology of the line. Therefore, should R belong to $F^*(P) \subset G_\delta(P)$, then R would be a G_δ in the usual topology of the line, which it is not.

2⁰ It can happen that $F(P) \subset G_\delta(P)$ in a non-normal space P.

Example. For any irrational $x \in P$, P denoting the set of all real numbers, choose a sequence $\{r_{x,n}\}$ of rational numbers converging (in the usual sense) to x. For any $M \subset P$, let the closure $\bar{M}$ consists of all $t \in M$ and all x such that M contains infinitely many points $r_{x,n}$. The space P is completely regular, but is not normal, since there are $\mathfrak{r}$ (= power of continuum) continuous real-valued functions in P (for the set R of rational numbers is dense in P), whereas there exist $2^{\mathfrak{r}}$ such functions in $P - R$ which is a closed discrete subspace of P. If $F \subset P$ is closed, then FR is open (every rational point being isolated), $F - R$ is open in $P - R$ and therefore belongs to $G_\delta(P)$, for $P - R \in G_\delta(P)$; thus $F \in G_\delta(P)$.

3⁰ I do not know whether there exists a completely regular non-normal space P with $B(P) = B^*(P)$ or a normal space P with $B(P) = B^*(P)$ which is not perfectly normal.

We shall call a *measure* in a space P every σ-additive non-negative real-valued (∞ excluded) set function μ in a σ-field $\mathfrak{B} \supset B^*(P)$. If $\mathfrak{B} = B(P)$, then μ will be called a *Borel measure*; if $\mathfrak{B} = B^*(P)$, then μ will be called a *Baire measure*. A measure is said to be *two-valued* if it assumes at most two values.

Remark. It is easy to see that every finite σ-additive function in a σ-field may be represented as the difference of two finite non-negative σ-additive functions. Therefore, many results concerning measures in topological spaces may be extended *mutatis mutandis* to finite σ-additive functions defined on Borel or Baire sets.

Given a space P and a measure μ defined in a σ-field $\mathfrak{B} \supset B^*(P)$ we shall say that a closed set $Q \subset P$ *semi-reduces* μ if (1) $\mu(G) > 0$ whenever G is open, $G \in \mathfrak{B}$, $GQ \neq 0$, (2) $\mu(F) = 0$ whenever F is closed, $F \in \mathfrak{B}$, $FQ = 0$. If, in addition, $Q \in \mathfrak{B}$ and $\mu(P - Q) = 0$ (this condition implies (2), of course) we shall say that Q *reduces* μ.

It is easy to see that, given a measure μ, there exists only one, if any, closed set Q reducing μ. This set, if it exists, consists of all $x \in P$ such that $\mu(U) > 0$ for any neighbourhood U of x belonging to $\mathfrak{B}$.

We shall say that a measure μ is *reducible* or, respectively, *semireducible* if there exists a set reducing or semi-reducing μ.

It is clear that, in a perfectly normal space, every semireducible measure is reducible (for, in such a space P, we have $P - Q = \sum_{n=1}^{\infty} F_n$, where F_n are closed).

We now give two examples of non-reducible measures.

Example 1. Let the ordered set P_1 of ordinals $\xi \leqslant \omega_1$ be given its usual topology. Let $\mathfrak{F}_1$ denote the collection of all uncountable $F \subset P_1$ such that $F + (\omega_1)$ is closed. It is easy to see that $\mathfrak{F}_1$ is countably multiplicative. Let $\mathfrak{B}_1$ denote the collection of all $B \in B(P_1)$ containing some $F \in \mathfrak{F}_1$; clearly, $\mathfrak{B}_1$ is countably multiplicative. Let $\mathfrak{B}$ denote the collection of all $M \subset P_1$ such that either M or $P_1 - M$ contains some $F \in \mathfrak{F}_1$. It is clear that $F(P_1) \subset \mathfrak{B}$, $\mathfrak{B}$ is a σ-field; hence $B(P_1) \subset \mathfrak{B}$. For $B \in B(P_1)$ put $\mu(B) = 1$ if $B \in \mathfrak{B}_1$; otherwise $\mu(B) = 0$. It is easy to verify that μ is a two-valued Borel measure. The one-point set (ω_1) semi-reduces μ, but μ is not reducible, for $\mu(P - (\omega_1)) = 1$.

Example 2. Put $P_2 = P_1 - (\omega_1)$. For $B \in B(P_2)$ put $\nu(B) = \mu(B)$, where μ is the measure from Example 1. Then ν is a two-valued Borel measure in P_2 and is not semi-reducible (for every point has a neighbourhood of measure zero whereas $\nu(P_2) = 1$).

Definitions. A space P is called *fully normal* (cf. J. Tukey [8]) if there exists, for any open covering $\mathfrak{U}$, an open covering $\mathfrak{B}$ which has the following property: for any $x \in P$, there is a set $U \in \mathfrak{U}$ such that $V \subset U$ whenever $x \in V \in \mathfrak{B}$.

A collection[4]) $\mathfrak{M}$ of sets $M \subset P$ is called *locally finite* (relative to P) if every $x \in P$ has a neighbourhood which intersects only a finite number of sets $M \in \mathfrak{M}$.

A space P is called *paracompact* (J. Dieudonné [1]) if, for any open covering $\mathfrak{U}$, there exists a locally finite open covering $\mathfrak{B}$ such that every $V \in \mathfrak{B}$ is contained in some $U \in \mathfrak{U}$). It is known (cf. [8] and [1]) that every fully normal or paracompact space is normal. The following important theorem due to A. H. Stone [7] will be used throughout in the sequel:

A space is fully normal if and only if it is paracompact.

Lemma 2. *Let P be a normal space, let $\{G_\tau\}_{\tau \in T}$ be a locally finite covering of the space P, and let $F_\tau \in F^*(P)$, $F_\tau \subset G_\tau$, for any $\tau \in T$. Then $\sum_\tau F_\tau \in F^*(P)$.*

Remark. T is an arbitrary abstract set of indices, of course.

Proof. For any $\tau \in T$, there exists a continuous real-valued function f_τ in P such that $0 \leqslant f_\tau(x) \leqslant 1$, for any $x \in P$, $f_\tau^{-1}(0) = F_\tau$, $f_\tau^{-1}(1) \supset P - G_\tau$. For any $x \in P$, put $f(x) = \inf_{\tau \in T} f_\tau(x)$. Then $x \in F = \sum_{\tau \in T} F_\tau$ implies $f(x) = 0$. If x non $\in F$, then $f_\tau(x) > 0$ for all τ, and $f_\tau(x) = 1$ for all τ except a finite number (for $\{G_\tau\}$ is locally finite); therefore $f(x) > 0$. Thus $F = f^{-1}(0)$. Every $x \in P$ has a neighbourhood U such that $UG_\tau = 0$ except for $\tau = \tau_k$, $k = 1, \ldots, n$. Then $y \in U$ implies $f(y) = \inf_{k=1\ldots n} f_{\tau_k}(y)$. Therefore f is continuous which proves the lemma.

Lemma 3. *Let P be normal, let $\{G_\tau\}_{\tau \in T}$ be a locally finite open covering, and let $A_\tau \subset G_\tau$, $A_\tau \in F_\sigma^*(P)$ for any $\tau \in T$. Then $\sum_\tau A_\tau \in F_\sigma^*(P)$.*

Proof. We have $A_\tau = \sum_{n=1}^{\infty} F_{\tau,n}$, $F_{\tau,n} \in F^*(P)$. By Lemma 2, $\sum_\tau F_{\tau,n} \in F^*(P)$, $n = 1, 2, \ldots$ Hence $\sum_\tau A_\tau = \sum_{n=1}^{\infty} \sum_\tau F_{\tau,n} \in F_\sigma^*(P)$.

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[4]) Since indexed systems of sets (cf. [8] or [4]) will occur rather often in the sequel, we state the corresponding definition for such systems explicitly too: an indexed system $\{M_\tau\}_{\tau \in T}$ of sets $M_\tau \subset P$ will be called *locally finite* if every $x \in P$ has a neighbourhood U such that $UM_\tau = 0$ except for a finite number of indices τ.

Lemma 4. *If P is fully normal and $U_\varrho \subset P$, $\varrho \in R$, are such that $\sum_\varrho \operatorname{Int} U_\varrho = P$, then there exists a locally finite open covering $\{G_\tau\}_{\tau \in T}$ which refines $\{\operatorname{Int} U_\varrho\}$, and sets $H_\tau \in F^*(P)$, $\tau \in T$, such that $H_\tau \subset G_\tau$, $\sum_\tau H_\tau = P$.*

Proof. Since P is fully normal, $\{\operatorname{Int} U_\varrho\}$ has a locally finite refinement $\{G_\tau\}$. Now, the following theorem (see e. g. [4], p. 26) is well-known: if P is normal, then every point-finite (i. e. such that no $x \in P$ lies in infinitely many sets G_τ) covering $\{G_\tau\}$ is shrinkable, that is, there exist closed sets $F_\tau \subset G_\tau$ such that $\sum F_\tau = P$. Given such sets, there exist continuous real-valued functions h_τ equal to 0 in F_τ, to 1 in $P - G_\tau$. Then sets H_τ, each consisting of points $z \in P$ such that $f_\tau(z) \leqslant \frac{1}{2}$, have the properties required.

Definition. Let $\mathfrak{M}$ be a family of subsets of a space P. A set $A \subset P$ is said to *belong locally* to the family $\mathfrak{M}$ (or: to be *locally an $\mathfrak{M}$-set*) if every $x \in P$ has a neighbourhood U such that $UA \in \mathfrak{M}$.

Proposition 1. *Let P be fully normal. If $M \subset P$ belongs locally to $F^*(P)$, then $M \in F^*(P)$; if $M \subset P$ belongs locally to $F_\sigma^*(P)$, then $M \subset F_\sigma^*(P)$.*

Proof. Every $x \in P$ has a neighbourhood U_x such that $U_x M \in \mathfrak{F}$, where $\mathfrak{F}$ denotes $F^*(P)$ or, respectively, $F_\sigma^*(P)$. Since $\Sigma \operatorname{Int} U_x = P$, apply Lemma 4. Since every H_τ is contained in some U_x, we have $H_\tau M = U_x M H_\tau$. Since Lemma 1 implies that the family $\mathfrak{F}$ is (finitely) multiplicative, we have $H_\tau M \in \mathfrak{F}$. Now, since the system G_τ is locally finite, apply Lemma 2 or 3.

Remark. Theorems similar to the above Proposition 1 are ell-known in the theory of metric spaces (cf. e. g. C. Kuratowki [3], § 26, X). It is possible to show that many, if not all, such eorems (except, of course, those in which the assumption of sepability is essential) obtain in arbitrary fully normal spaces (of ich the metrisable ones constitute a special case).

Proposition 1a. *Let P be fully normal and let $\{A_\tau\}_{\tau \in T}$ be lly finite. If every A_τ belongs to $F^*(P)$, then $\sum_{\tau \in T} A_\tau$ belongs to ?); if every A_τ belongs to $F_\sigma^*(P)$, then $\sum_{\tau \in T} A_\tau \in F_\sigma^*(P)$.*

This follows at once from Proposition 1 and Lemma 1.

Proposition 2. *Let P be a fully normal space. Then, given an open covering $\mathfrak{G}$, there exists a completely additive field of sets $\mathfrak{A}$ such that* (1) $\mathfrak{A} \in \boldsymbol{F}_\sigma^*(P)$, (2) $P \in \mathfrak{A}$, (3) *every atom* [5]) *of $\mathfrak{A}$ is contained in some $G \in \mathfrak{G}$,* (4) *the collection of atoms of $\mathfrak{A}$ is locally finite.*

Proof. By Lemma 4, there exists a locally finite indexed system $\{H_\tau\}_{\tau \in T}$ such that $\sum_\tau H_\tau = P$, every H_τ belongs to $\boldsymbol{F}^*(P)$ and is contained in some $G \in \mathfrak{G}$. For any $x \in P$, let Γ_x denote the set of $\tau \in T$ such that $x \in H_\tau$, and let S_x denote the set of $z \in P$ such that $\Gamma_z = \Gamma_x$. If $x \in P$, $y \in P$, $S_x \neq S_y$, then $\Gamma_x \neq \Gamma_y$, hence there exists e. g. $\tau \in \Gamma_y - \Gamma_x$; then $z \in H_\tau$ whenever $z \in S_y$, $z \operatorname{non} \in H_\tau$ whenever $z \in S_x$, and therefore $S_x S_y = 0$.

Let $S(M)$ denote, for any $M \subset P$, the sum of all S_x, $x \in M$, and let $\mathfrak{A}$ denote the collection of all $S(M)$. Clearly, $\mathfrak{A}$ is a completely additive field of sets and has properties (2) and (3), for the sets S_x are precisely the atoms of $\mathfrak{A}$. We shall show that the collection of all S_x is locally finite. Consider a point $a \in P$. There exists a neighbourhood U of a such that $UH_\tau = 0$ for all τ, except for $\tau \in K \subset T$, K finite (non-void). For every $x \in U$, we have $\Gamma_x \subset K$. Since K is finite, we infer that there exist $x(k) \in U$, $k = 1, \ldots, n$, such that every Γ_x, $x \in U$, is equal to some $\Gamma_{x(k)}$. Then $\sum_{k=1}^{n} S_{x(k)} \supset U$ and therefore, two different S_x, S_y being always disjoint, the collection of atoms is locally finite.

We have now only to prove that every S_x belongs to $\boldsymbol{F}_\sigma^*(P)$. This will imply, the collection of all S_x being locally finite, that $\mathfrak{A}$ has property (1). Consider S_a, $a \in P$. Every $x \in P$ has a neighbourhood $V \in \boldsymbol{F}^*(P)$ which intersects H_τ only for $\tau \in K$, $K \subset T$ finite. It is easy to see that VS_a consists of points $x \in V$ which belong to each H_τ, $\tau \in \Gamma_a$, and to no H_τ, $\tau \in K - \Gamma_a$. Therefore, VS_a is equal to the intersection of a finite number of sets belonging either to $\boldsymbol{F}^*(P)$ or to $\boldsymbol{G}^*(P)$, hence to $\boldsymbol{F}_\sigma^*(P)$. Hence, by Lemma 1, $VS_a \in \boldsymbol{F}_\sigma^*(P)$. Thus S_a belongs locally to $\boldsymbol{F}_\sigma^*(P)$ whence, by Proposition 1, $S_a \in \boldsymbol{F}_\sigma^*(P)$ which completes the proof.

Lemma 5. *In order that a Borel measure μ in a normal space P be semi-reducible, it is necessary and sufficient that its Baire restriction* [6]) *be semi-reducible.*

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[5]) If $\mathfrak{A}$ is a field of sets, then $A \in \mathfrak{A}$ is called an *atom* (of the field $\mathfrak{A}$) if $A \neq 0$ and $A \supset B \neq 0$, $B \in \mathfrak{A}$ implies $B = A$.

[6]) That is, the Baire measure μ' defined by $\mu'(B) = \mu(B)$ for $B \in \boldsymbol{B}^*(P)$.

Proof. It is clear that if Q semi-reduces μ, then it semi-reduces its Baire restriction μ' too. If a closed set Q semi-reduces μ', let F be closed, $FQ = 0$. Then clearly there exists $F_1 \in \boldsymbol{F}^*(P)$, $F_1 \supset F$, $F_1 Q = \Phi$, and we have $\mu(F_1) = \mu'(F_1) = 0$ and therefore $\mu(F) = 0$.

The proof of the following lemma may be omitted.

Lemma 6. *Let P be a space, $S \subset P$. If $B \subset P$ is a Borel or Baire set (in P), then $BS \in \boldsymbol{B}(S)$ or $BS \in \boldsymbol{B}^*(S)$, respectively. If S is a Borel or Baire set, then $\boldsymbol{B}(S) \subset \boldsymbol{B}(P)$ or $\boldsymbol{B}^*(S) \subset \boldsymbol{B}^*(P)$ respectively.*

Proposition 3a. *If every (two-valued) Borel measure in a space P is semi-reducible, and $S \subset P$ is closed, then every (two-valued) Borel measure in S is semi-reducible.*

Proof. Let μ be a Borel measure in S. For $B \in \boldsymbol{B}(P)$, put $\nu(B) = \mu(BS)$; this is possible, by Lemma 6. Let $Q \subset P$ be a closed set which semi-reduces ν. Put $Q_1 = QS$. It is easy to see that Q_1 semi-reduces μ.

Proposition 3b. *If every (two-valued) Baire measure in a normal space P is semi-reducible, and $S \subset P$ is closed, then every (two-valued) Baire measure in S is semi-reducible.*

Proof. If μ is a Baire measure in S, put $\nu(B) = \mu(BS)$, for any $B \in \boldsymbol{B}^*(P)$, which is possible by Lemma 6. If $Q \subset P$ is closed and semi-reduces ν, put $Q_1 = QS$. If $H \subset S$ is open (in S), $H \in \boldsymbol{B}^*(S)$, $HQ_1 \neq 0$, then choose a point $a \in HQ_1$; it is easy to see that there exists a set $G \in \boldsymbol{G}^*(P)$ such that $a \in G$, $(S - H)G = 0$. Then $\mu(GS) = \nu(G) > 0$; hence $\mu(H) > 0$, for $H \supset GS$. If $F \subset S$ is closed (in S), $F \in \boldsymbol{B}^*(S)$, $FQ_1 = 0$, then $FQ = 0$ and therefore, for some $F_1 \in \boldsymbol{F}^*(P)$ we have $F \subset F_1$, $F_1 Q = 0$, $\nu(F_1) = 0$, $\mu(F_1 S) = 0$, $\mu(F) = 0$.

Proposition 3c. *If every (two-valued) Borel measure in a space P is reducible, and $S \subset P$, then every (two-valued) Borel measure in S is reducible.*

Proof. If μ is a Borel measure in S, put $\nu(B) = \mu(BS)$ for $B \in \boldsymbol{B}(P)$. If $Q \subset P$ reduces ν, put $Q_1 = QS$. Then $\mu(S - Q_1) = \mu((P - Q)S) = \nu(P - Q) = 0$. If $H \subset S$ is open in S, let $H = GS$, G open in P. If $HQ_1 \neq 0$, then $GQ \neq 0$, $\mu(H) = \nu(G) > 0$.

Theorem 1. *Let P be fully normal. In order that every (two-valued) Borel or Baire measure in P be semi-reducible, it is necessary and sufficient that every (two-valued) Borel measure in any closed discrete subspace of P be reducible.*

Remark. The above necessary and sufficient condition may be stated in terms of „power of measure zero", see [5], as follows: *the power of any closed discrete subspace of P has (two-valued) measure zero.*

Proof. Necessity follows from Proposition 3a and 3b.

Sufficiency. By Lemma 5, we may restrict ourselves to Baire measures. Let μ be a Baire measure in P which is not semi-reducible. Let Q denote the set of points $x \in P$ such that $\mu(U)>0$ for every neighbourhood $U \in \boldsymbol{B}^*(P)$ of x. Clearly, Q is closed and $\mu(G)>0$ whenever G is open, $G \in \boldsymbol{B}^*(P)$, $GQ \neq 0$. Hence there exists a closed set $S \in \boldsymbol{B}^*(P)$ such that $SQ=0$, $\mu(S)>0$; for otherwise μ would be semi-reducible. We may assume that $S \in \boldsymbol{F}^*(P)$; for otherwise there exists $S_1 \in \boldsymbol{F}^*(P)$, $S_1 \supset S$, $S_1 Q=0$, and we may consider S_1 instead of S. Every $x \in S$ has an open neighbourhood $U_x \in \boldsymbol{B}^*(P)$ such that $\mu(U_x)=0$. Since $\sum_x U_x \supset S$, and S is fully normal (for every closed subspace of a fully normal space is fully normal), there exists, by Proposition 2, a completely additive field of sets $\mathfrak{A} \subset \boldsymbol{F}^*_\sigma(S)$ such that

(1) the collection $\mathfrak{A}_0$ of all atoms of $\mathfrak{A}$ is locally finite;

(2) $A \in \mathfrak{A}_0$ implies $\mu(A)=0$; for, by Lemma 6, $A \in \boldsymbol{F}^*(P)$, and, by Proposition 2, every $A \in \mathfrak{A}_0$ is contained in some U_x;

(3) $S \in \mathfrak{A}$.

Now choose, for any $A \in \mathfrak{A}_0$, a point $x=x(A) \in A$, denote by X the set of all points $x(A)$, let, for any $M \subset X$, $\varphi(M)$ denote the sum of all $A \in \mathfrak{A}_0$ intersecting M, and put, for $M \subset X$, $\nu(M)=\mu(\varphi(M))$. Since $\mathfrak{A}_0$ is a locally finite collection of disjoint sets, X is closed discrete. Clearly, ν is a (two-valued) Borel measure [7]) in X. The measure ν is not reducible; for otherwise, X being discrete, some one-point subset of X would be of positive measure and therefore we should have $\mu(A)>0$ for some $A \in \mathfrak{A}_0$, which contradicts the above property (2).

Theorem 2. *Let P be hereditarily fully normal. In order that every (two-valued) Borel measure in P be reducible, it is necessary and sufficient that every (two-valued) Borel measure in any discrete subspace of P be reducible.*

[7]) It is clear that a Borel measure in a discrete space Z is simply a finite non-negative σ-additive function in the (completely additive) field of all subsets of Z.

Remark. It is again possible to state this condition in terms of „power of measure zero" as follows: *the power of any discrete subspace of P has (two-valued) measure zero.*

Proof. Necessity follows from Proposition 3a.

Sufficiency. Suppose that a Borel measure μ in P is not reducible. Denoting by Q the set of $x \in P$ such $\mu(U)>0$ for every open neighbourhood U of x, we have $\mu(P-Q)>0$, for otherwise Q would reduce μ. Consider the restriction μ' of μ to $\boldsymbol{B}(P-Q)$. Clearly, μ' is not semi-reducible. Hence, by Theorem 1, there exists a discrete set $X \subset P-Q$ and a (two-valued) Borel measure in X which is not reducible.

Remarks. 1° It is well-known that every metrisable space is fully normal; see e. g. Tukey [8]. Therefore, Theorems III, IV, V, VI of [5] follow at once from the above Theorem 2.

2° The class of fully normal spaces is much broader than that of metrisable ones. On the other hand, the proofs of decomposition theorems (for metrisable spaces) given in [5] are much simpler than the corresponding proofs in the present note. In [5], Montgomery's theorem (see [6], Lemma 2, p. 528) is essential. It may be noticed that the proof of A. H. Stone's theorem (asserting that every fully normal, e. g. every metrisable space is paracompact) rests on an idea closely related to that of Montgomery, and that A. H. Stone's theorem is vital for the proofs of the theorems of the present note.

We shall now consider Q-spaces.

Definition. A completely regular space P is called a Q-space if it is impossible to imbed P into a completely regular space $S \supset P$ in such a way that $S=\overline{P}$, $S \neq P$, and every (bounded or not) continuous real-valued function f in P is extensible [8]) over S.

Q-spaces are known (see [2]) to possess many important properties. Therefore, it is interesting to find properties sufficient for a completely regular space to be a Q-space. Thus E. Hewitt [2] has posed the problem whether every metrisable space is a Q-space. We shall prove that, under certain restrictions, every fully normal space is a Q-space.

[8]) This means: there exists a continuous real-valued (∞ and $-\infty$ excluded) function F in S such that $F(x)=f(x)$ whenever $x \in P$.

Proposition 4. *Every closed subspace of a Q-space is a Q-space.*

Proof. Suppose that $P \subset R$ is closed and is not a Q-space. Then there exists a completely regular space $S \supset P$, $S = \overline{P}$, $S \neq P$, such that every continuous real-valued function φ in P is extendible over S (this extension is unique, of course). Evidently, we may suppose that $S - P$ contains a single point ξ. For any $f \in \Phi$ where Φ denotes the set of all continuous real-valued functions f in R, let f' denote the restriction of f to P, and let f^* denote the extension of f' over $P + (\xi)$.

Now let us define a topology in $T = R + (\xi)$ as follows: $\xi \in \overline{M}$, where $M \subset R$, if and only if $f^*(\xi) \in \overline{f(M)}$, for any $f \in \Phi$; R is imbedded in T as an open set. It is easy to see that T is completely regular and every $f \in \Phi$ is extendible (continuously) over T. Therefore, R is not a Q-space.

Lemma 7. *If S is a completely regular space, $P \subset S$ is not closed, and every continuous real-valued function f in P is extendible to S, then P non $\in \boldsymbol{F}_\sigma(S)$.*

Proof. Suppose, on the contrary, that $P = \sum_{n=1}^{\infty} A_n$, A_n closed in S. Choose a point $\xi \in \overline{P} - P$. There exist continuous functions f_n in S such that (1) $0 \leqslant f_n(x) \leqslant 1$, for any $x \in S$, (2) $f_n(\xi) = 0$, (3) $f_n(x) = 1$ if $x \in A_n$. Put $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. Then f is continuous, $f(\xi) = 0$, $f(x) > 0$ if $x \in P$. Now put for $x \in P$, $g(x) = \log f(x)$. Then g is a continuous function in P not extendible over S.

Lemma 8. *If $P \subset S$, P is normal, and every bounded continuous real-valued function in P is extendible over S, then $\overline{A}\overline{B} = 0$ whenever A, B are disjoint closed (in P) subsets of P.*

Proof. Suppose that $\xi \in \overline{A}\overline{B}$; since P is normal, there exists a bounded continuous real-valued function f in P such that $f(A) = 0$, $f(B) = 1$. Clearly, f is not extendible over ξ.

Theorem 3. *In order that a fully normal space P be a Q-space, it is necessary and sufficient that every closed discrete subspace of P be a Q-space.*

Proof. Necessity follows from Proposition 4.

Sufficiency. Suppose now that P is not a Q-space. Then there exists a completely regular space $S \supset P$, $S \neq P$, such that $S = \overline{P}$ and every continuous real-valued function in P is extendible to S; we may, of course, suppose that $S - P$ contains a single point ξ. Since P is covered by open sets $G \subset P$ such that ξ non $\in \overline{G}$, there exists, by Proposition 2, a completely additive field of sets $\mathfrak{A} \in \boldsymbol{F}_\sigma^*(P)$ such that the collection $\mathfrak{A}_0$ of all atoms of $\mathfrak{A}$ is locally finite and ξ non $\in \overline{A}$ whenever $A \in \mathfrak{A}_0$. For any $A \in \mathfrak{A}_0$, there exist sets $B(A,n) \in \boldsymbol{F}^*(P)$ such that $\sum_{n=1}^{\infty} B(A,n) = A$. Since every collection $\{B(A,n)\}$, n fixed, is locally finite, we have, by Proposition 1a, $B_n \in \boldsymbol{F}^*(P)$, where B_n denotes the sum of all $B(A,n)$, n fixed. Since $\sum_{n=1}^{\infty} B_n = P$, we have, by Lemma 7, $\xi \in \overline{B}_p$, for some p. Put $B = B_p$ and let $\mathfrak{B}$ denote the collection of all BA, $A \in \mathfrak{A}$. Clearly, $\mathfrak{B}$ is a completely additive σ-field, the collection $\mathfrak{B}_0$ of atoms of $\mathfrak{B}$ consists of sets $BA = B(A,p)$, $A \in \mathfrak{A}_0$, hence $\mathfrak{B}_0 \subset \boldsymbol{F}^*(P)$, and therefore, by Proposition 1a, every set from $\mathfrak{B}$ belongs to $\boldsymbol{F}^*(P)$. It is clear that

(a) ξ non $\in \overline{C}$ whenever $C \in \mathfrak{B}_0$.

Lemmas 7 and 8 imply that

(b) given disjoint $C_k \in \mathfrak{B}$ $(k = 1, 2, \ldots)$ such that $\sum_{k=1}^{\infty} C_k = B$, there exists exactly one k_0 such that $\xi \in \overline{C}_{k_0}$.

Let $X \subset B$ intersect every set from $\mathfrak{B}_0$ in exactly one point. Since $\mathfrak{B}_0$ is locally finite, X is a closed discrete subspace of P. For any $M \subset X$, let $\varphi(M)$ denote the sum of all sets from $\mathfrak{B}$ which intersect M; let $\mathfrak{M}$ denote the collection of $M \subset X$ such that $\xi \in \overline{\varphi(M)}$. Then the above property (a) implies that

(a′) $\mathfrak{M}$ contains no finite set,

and property (b) implies

(b′) if $M_k \subset X$ $(k = 1, 2, \ldots)$ are disjoint, $\sum_{n=1}^{\infty} M_k = X$, then $M_k \in \mathfrak{M}$ for exactly one k.

Let η be an arbitrary element not belonging to X. Put $T = X + (\eta)$; for $M \subset T$, put $\overline{M} = M + (\eta)$ if $MX \in \mathfrak{M}$, otherwise $\overline{M} = M$. Then $\overline{X} = T$, T is completely regular (we make use of (a′) here). It is easy to conclude from (b′) that every real-valued function in X is extensible to ξ. Therefore, X is not a Q-space.

Corollary. *A metrisable space P is always a Q-space unless it contains a discrete set which is not a Q-space.*

Proof. We have only to prove: *if $X \subset P$ is discrete and is not a Q-space, then P cannot be a Q-space.* Put $S = P - (\overline{X} - X)$. Then X is a closed subset of S, hence, by the above theorem, S is not a Q-space. Since P is metrisable, $S \epsilon \boldsymbol{F}_\sigma(P)$, $S = \sum_{n=1}^{\infty} S_n$, $S_n \epsilon \boldsymbol{F}(P)$. Lemma 7 implies that some S_n is not a Q-space; therefore, by Proposition 4, P is not a Q-space.

Remark. It is easy to show that a discrete space is a Q-space if and only if it does not admit of a non-reducible two-valued Borel measure, that is if its power has two-valued measure zero. Therefore, Theorem 3 may be given the following equivalent form: *A fully normal space P is a Q-space if and only if the power of any closed discrete subset of P has two-valued measure zero.*

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On Real-Valued Functions in Topological Spaces.

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The following theorem [1]) of H. Hahn is well-known: if g and h are real-valued functions in a metric space P, g is upper semicontinuous [2]), h is lower semicontinuous [2]), and $g(x) \leqslant h(x)$ for any $x \epsilon P$, then there exists a continuous function f such that $g(x) \leqslant f(x) \leqslant h(x)$ for every $x \epsilon P$. If $<$ is substituted for $\leqslant$, the theorem still holds. In his paper [4], J. Dieudonné has extended Hahn's theorem (with $\leqslant$ or $<$) to paracompact [3]) spaces. In the present note, it is shown that Hahn's theorem holds (i) with $\leqslant$, in arbitrary normal [4]) spaces (Theorem 1); (ii) with $<$, in a broad class (specified in Theorem 2) of normal spaces including paracompact, countably compact [5]) and perfectly normal [6]) ones (as a matter of fact, I do not know whether there exists any normal space not belonging to this class).

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[1]) See e. g. [5], 36. 2. 6 (numbers in brackets refer to the list at the end of the present note).

[2]) A real-valued function g defined in a topological space P is called *upper semicontinuous* if, for any $a \epsilon P$ and any $c > g(a)$, there exists a neighbourhood U of a such that $c > g(x)$ whenever $x \epsilon U$. Substituting $<$ instead of $>$, we obtain the definition of the lower semicontinuity.

[3]) A topological space P is called *paracompact* if, for any open covering $\mathfrak{G}$ of P, there exists an open covering $\mathfrak{H}$ which refines $\mathfrak{G}$ (i. e. every $H \epsilon \mathfrak{H}$ is contained in some $G \epsilon \mathfrak{G}$) and is locally finite (i. e. such that every point has a neighbourhood intersecting only a finite number of sets $H \epsilon \mathfrak{H}$). See J. Dieudonné's paper [4].

[4]) A topological space P is called *normal* if any two disjoint closed sets possess disjoint neighbourhoods.

[5]) A topological space is called *countably compact* if every countable open covering contains a finite subcovering.

[6]) A normal space it called *perfectly normal* if every closed set can be represented as the intersection of countably many open sets.