For every $\Lambda' \in \mathfrak{I}_{m,m+k}$ let $E_{\Lambda'}$ be the set of all $x \in X_0^{m+k}$ such that $(x,a) \in E_\Lambda$, $\Lambda = \Lambda' + (m+1+k)$. By (vii), X_0^{m+k} is the union of all sets $E_{\Lambda'}$, $\Lambda' \in \mathfrak{I}_{m,m+k}$. We shall prove the property (iv).

Let $\Lambda' \in \mathfrak{I}_{m,m+k}$, $\Lambda = \Lambda' + (m+1+k) \in \mathfrak{I}_{m+1,m+1+k}$. Let $P' = Y_1' \times \ldots \times Y_{m+k}'$, where $Y_j' = (a_j)$ for $j \in \Lambda'$ be any Λ'-subset of X_0^{m+k}, and let P be the Λ-set of all points $(x_1, \ldots, x_{m+k}, a) \in X^{m+1+k}$ where $x_j = a_j$ for $j \in \Lambda$. We have $\overline{\overline{PE_\Lambda}} < \aleph_\tau$. Since $P'E_{\Lambda'}$ is the set of all $x \in X_0^{m+k}$ such that $(x,a) \in PE_\Lambda$, we infer that $\overline{\overline{P'E_{\Lambda'}}} < \aleph_\tau$.

Corollary 1. *Let k be any positive integer. The continuum hypothesis is equivalent to the assertion that the $(k+2)$-dimensional Euclidean space is the sum of $\binom{k+2}{2}$ sets $E_{(i,j)}$[8]) such that the set $PE_{(i,j)}$ is finite for every k-dimensional hyperplane P perpendicular to the i-th and j-th axes of coordinates.*

Corollary 2. *Let k be any positive integer. The continuum hypothesis is equivalent to the assertion that the $(k+1)$-dimensional Euclidean space is the sum of $k+1$ sets E_i $(i=1,\ldots,k+1)$ such that, for every k-dimensional hyperplane P perpendicular to the i-th axis of coordinates, the set PE_i is at most denumerable.*

In order to prove the above corollaries it is sufficient to put in the Theorem

$$\tau = 0 \quad \text{and} \quad m = 2,$$

or:

$$\tau = 1 \quad \text{and} \quad m = 1.$$

[8]) Here $\Lambda = (i,j) \in \mathfrak{I}_{2,k+2}$, i. e. (i,j) is a two-element subset of I_{k+2}.

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The Space of Measures on a Given Set[1]).

By

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This paper is an attempt at a systematic discussion of the concept of weak convergence of measures. We shall introduce a neighborhood topology in the set M_R of all measures on a given set (or space) R, and discuss the relations between the properties of R and the topology of M_R. This topology specializes to weak convergence under certain conditions.

The space of measures. Let R be an abstract set with a class of subsets called „open", satisfying, for the present, only

Axiom I: *R is an open set.*

A *measure* is a set function defined for all sets, satisfying:

(1) $$\varphi(A) \geqslant 0, \quad \varphi(0) = 0, \quad \Phi(R) \text{ finite.}$$

(2) $$A \subset B \Rightarrow \varphi(A) \leqslant \varphi(B)$$

(3) $$\varphi\Big(\sum_{i=1}^{\infty} A_i\Big) \leqslant \sum_{i=1}^{\infty} \varphi(A_i)$$

(4) $$\varphi(A) = LB\, \varphi(O) \text{ for all open sets } O \supset A \text{ (Regularity).}$$

(5) *Open sets are (Carathéodory) measurable.*

Definition: A unitary neighborhood $\mathcal{O}(\varphi_0, O, a)$ of a measure φ_0 is the set of all measures φ for which $\varphi_0(O) < \varphi(O) + a$ and $|\varphi(R) - \varphi_0(R)| < a$, where O is open and $a > 0$.

Any finite product of unitary neighborhoods of φ_0 is called a neighborhood of φ_0.

The measures on R thus constitute a topological space M_R. Neighborhoods are open sets, but we shall not prove this.

[1]) Presented to the American Mathematical Society April 30, 1949. The author is indebted to Professor Witold Hurewicz for advice given during the preparation of this paper.

We shall state a rather superficial theorem showing that all results concerning the space of measures can be derived from corresponding results concerning the space of normalized measures (i. e. measures for which $\varphi(R)=1$).

Let M denote the subspace of all normalized measures, and M' the subspace of all measures except the zero measure. Let I be the space of positive real numbers.

Theorem 1: *M' is topologically equivalent to $M\times I$.*

The one-to-one mapping $(\varphi,x)\to x\varphi$ of $M\times I$ on to M' is a homeomorphism. We omit the proof, which is straight forward.

Henceforth, let all measures be normalized.

Theorem 2: *M is a T_1-space.*

Proof: Let $\varphi_0\neq\varphi_1$. There is an open set O for which $\varphi_0(O)>\varphi_1(O)$. For suppose not; then $\varphi_0\leqslant\varphi_1$ for all open sets, and hence for *all* sets. But φ_0 and φ_1 are not identical, and therefore there is an open set O' for which $\varphi_0(O')<\varphi_1(O')$. Then $\varphi_0(R-O')>\varphi_1(R-O')$, which is a contradiction. Hence the open set O described above exists. Let $a=\varphi_0(O)-\varphi_1(O)$. Then $\mathcal{O}(\varphi_0,O,a)$ is a neighborhood of φ_0 which does not contain φ_1.

Definition: $\varphi_n\xrightarrow{\circ}\varphi$ if $\varphi(O)\leqslant\underline{\mathrm{Lim}}\,\varphi_n(O)$ for each open set O.

This was first stated by A. D. Alexandroff [2]) as a necessary and sufficient condition for weak convergence. It is easy to see that a sequence φ_n converges to φ if and only if $\varphi_n\xrightarrow{\circ}\varphi$. A priori, little is known about the properties of this convergence; for example, limits need not be unique.

We shall use the term *separable* to mean that the open sets have a finitely additive countable basis. (If R is a topological space, then this is ordinary separability).

Theorem 3: *If R is separable, then M has the first countability property.*

Proof: Let ω be a finitely additive countable basis for R. We assert that the neighborhoods $\mathcal{O}(\varphi_0,A,r)$ ($A\in\omega$ and $r>0$ and rational) and their finite intersections constitute a basis for the

[2]) Alexandroff A. D., *Additive Set Functions in Abstract Spaces*, Rec. Math. (Mat. Sbornik) N. S. **8** (1940), p. 307.

neighborhoods of φ_0. Let $\mathcal{O}(\varphi_0,O,a)$ be any unitary neighborhood of φ_0. Let r be a positive rational number $<a/2$. O is the limit of sets of ω and therefore there is a set $A\in\omega$ for which $A\subset O$ and $\varphi_0(O)-\varphi_0(A)<r$. We shall prove that $\mathcal{O}_1(\varphi_0,A,r)\subset\mathcal{O}(\varphi_0,O,a)$. Let $\varphi\in\mathcal{O}_1$. Then $\varphi_0(A)<\varphi(A)+r$. Thus $\varphi_0(O)<\varphi_0(A)+r<\varphi(A)+2r<\varphi(O)+a$. Taking finite intersections completes the proof.

Corollary: *If R is separable, then the O-convergent sequences completely determine the topology of M.*

Equivalent systems of neighborhoods. To establish the connection between our topology and ordinary weak convergence, we introduce W-neighborhoods. Also A-neighborhoods are introduced for convenience in proofs.

Definition: A set A is called *nbφ* (null boundary for φ) if it is open and there exist a finite number of non-intersecting open sets $A_i\subset R-A$ such that $\sum_{i=1}^{k}\varphi(A_i)=\varphi(R-A)$. (If R is a topological space, then an open set A is *nbφ* if and only if $\varphi(A)=\varphi(\bar{A})$).

Definition: A unitary A-neighborhood $\mathcal{A}(\varphi_0,A,a)$ of φ_0 is the set of all measures φ for which $|\varphi(A)-\varphi_0(A)|<a$, where A is *nbφ_0* and $a>0$.

Definition: A function f on R is called *continuous* if for each pair of numbers a, b, the set $(a<f<b)$ is open or vacuous.

Definition: A unitary W-neighborhood $\mathcal{W}(\varphi_0,f,a)$ of φ_0 is the set of all measures φ for which

$$\left|\int_R f\,d\varphi-\int_R f\,d\varphi_0\right|<a,$$

where f is a bounded, non-negative [3]), continuous function, and $a>0$.

Finite products are then allowed in each definition. M has been given two new topologies, whose relation will now be discussed.

Theorem 4: *Every unitary A-neighborhood of φ_0 contains an O-neighborhood of φ_0.*

[3]) Henceforth all continuous functions on R will be understood to be bounded and non-negative, unless the contrary possibility is stated.

Proof: Let $\mathcal{A}(\varphi_0, A, a)$ be any unitary A-neighborhood of φ_0. We assert that the intersection of the $k+1$ neighborhoods $\mathcal{O}(\varphi_0, A, a)$ and $\mathcal{O}(\varphi_0, A_i, a/k)$ $(i=1,\dots,k)$ is contained in $\mathcal{A}$. For let φ be in this intersection. Then

$$\varphi(A) \leqslant \varphi(R-\sum_{i=1}^{k} A_i) = 1-\sum_{i=1}^{k}\varphi(A_i) < 1-\sum_{i=1}^{k}\varphi_0(A_i)+a =$$
$$= 1-\varphi_0(R-A)+a = \varphi_0(A)+a.$$

On the other hand, $\varphi_0(A) < \varphi(A)+a$. Hence $|\varphi(A)-\varphi_0(A)| < a$ and $\varphi \in \mathcal{A}$.

Observe that the only assumption concerning the class of open sets is that R is open. Despite the appearance of sums in the foregoing proof, it is not assumed that the sum of two open sets is open. If this assumption were made, then the proof could be slightly simplified.

Theorem 5: *Every unitary W-neighborhood of φ_0 contains an A-neighborhood of φ_0.*

Proof: Let $\mathcal{W}(\varphi_0, f, a)$ be any unitary W-neighborhood of φ_0. The set $(f=y)$, being φ_0-measurable, has φ_0-measure zero for a dense class of values of y. We choose from this class numbers y_i such that

$$y_0 < LBf < y_1 < \dots < y_{r-1} < M = UBf < y_r$$

and $y_i - y_{i-1} < a/4$ $(i=1,\dots,r)$. Let $A_i = (y_{i-1} < f < y_i)$. The A_i are $nb\varphi_0$.

We shall prove that the intersection of the r neighborhoods $\mathcal{A}(\varphi_0, A_i, a/4rM)$ is contained in $\mathcal{W}(\varphi_0, f, a)$.

Let $A = \sum_{i=1}^{r} A_i$ and $E = R-A$.

Let φ be *any* measure (including possibly φ_0).

$$\left|\int_A f\,d\varphi - \sum_{i=1}^{r} y_i\varphi(A_i)\right| < (a/4)\,\varphi(R) = a/4. \tag{1}$$

Now let φ be in the given intersection. $\varphi(E) = 1-\varphi(A) = \varphi_0(A)-\varphi(A)$. Thus

$$\varphi(E) = \sum_{i=1}^{r}\varphi_0(A_i) - \sum_{i=1}^{r}\varphi(A_i) \leqslant \sum_{i=1}^{r}|\varphi_0(A_i)-\varphi(A_i)| < a/4M.$$

Hence

$$\left|\int_E f\,d\varphi - \int_E f\,d\varphi_0\right| = \int_E f\,d\varphi \leqslant M\varphi(E) < a/4. \tag{2}$$

Also

$$\left|\sum_{i=1}^{r} y_i\varphi(A_i) - \sum_{i=1}^{r} y_i\varphi_0(A_i)\right| \leqslant M\sum_{i=1}^{r}|\varphi(A_i)-\varphi_0(A_i)| < a/4. \tag{3}$$

Adding the four inequalities consisting of (1) as written, (1) with $\varphi=\varphi_0$, (2) and (3), we obtain

$$\left|\int_R f\,d\varphi - \int_R f\,d\varphi_0\right| < a.$$

Theorem 6: *If the open sets of R are such that R is a normal topological space, then every unitary O-neighborhood of φ_0 contains a W-neighborhood of φ_0.*

Proof: Let $\mathcal{O}(\varphi_0, O, a)$ be any unitary O-neighborhood of φ_0. There exists a closed set $F \subset O$ for which $\varphi_0(O-F) < a/2$.

We shall say, after A. D. Alexandroff, that a continuous function f *joins* the closed sets F_1 and F_2 if $f=0$ on F_1, $f=1$ on F_2, and $0 \leqslant f \leqslant 1$. In a normal space any two non-intersecting closed sets can be joined, according to Urysohn's Lemma.

Let f join $R-O$ to F. We assert that $\mathcal{W}(\varphi_0, f, a/2) \subset \mathcal{O}$. For let $\varphi \in \mathcal{W}$. Then

$$\varphi_0(O) < \varphi_0(F)+a/2 \leqslant \int_R f\,d\varphi_0 + a/2 < \int_R f\,d\varphi + a \leqslant \varphi(O)+a.$$

Theorems 4, 5, and 6, together, imply the equivalence of the neighborhood systems O, A, W, if R is normal.

We shall now state the usual definition of weak convergence.

Definition: $\varphi_n \xrightarrow{W} \varphi$ if $\int_R f\,d\varphi_n \to \int_R f\,d\varphi$ for every bounded, possibly negative, continuous function f.

By decomposing f into its positive and negative parts, it is easy to see that a sequence φ_n converges to φ in the W-topology if and only if $\varphi_n \xrightarrow{W} \varphi$. If R is normal, O-convergence is equivalent to W-convergence. This was proved by A. D. Alexandroff, using results on functions of bounded variation. The proof of Theorem 6 is essentially his proof that W-convergence implies O-convergence, transcribed into neighborhood terminology.

Theorem 7: *If R is a normal topological space, then M is a Hausdorff space.*

Proof: Since R is normal, we may use A-neighborhoods instead of O-neighborhoods.

Let $\varphi_1 \neq \varphi_2$ Then there is an open set O on which $\varphi_1 \neq \varphi_2$. Let $|\varphi_1(O)-\varphi_2(O)|=4a$. There is a closed set $F_i \subset O$ for which $\varphi_i(O-F_i)<a$ $(i=1,2)$. $F=F_1+F_2$ is closed and $\varphi_i(O-F)<a$ $(i=1,2)$. Join $R-O$ to F by a continuous function f. For some number y $(0 \leqslant y<1)$ we have $\varphi_1(f=y)=\varphi_2(f=y)=0$. Then $A=(f>y)$ is open and both $nb\varphi_1$ and $nb\varphi_2$. Since $F \subset A \subset O$, we have $\varphi_i(O-A)<a$ $(i=1,2)$. The two neighborhoods $\mathcal{A}(\varphi_i,A,a)$ $(i=1,2)$ have no common points.

Later we shall prove that if R is bicompact, then the converse of Theorem 7 is true.

In general, we shall not assume that R is normal, and we shall use the O-topology for M.

Point measures. Separability of M. Let p be a point of R. We shall define the so-called *point measure* φ_p associated with p by saying that $\varphi_p(E)=1$ or 0 according as E contains p or does not. It is easy to verify that φ_p satisfies the (Caratheodory) conditions 1, 2, 3, and that all sets are φ_p-measurable. φ_p is not always regular, and therefore not always admissible as an element of M. It is obvious that φ_p is regular if and only if the set $R-p$ is open. Since we shall make use of point measures frequently, and since $R-p$ open is the minimum condition for their admissibility, we shall introduce it as an axiom.

Axiom II: *For each point p, $R-p$ is open.*

If R is a topological space, then Axiom II is equivalent to the statement that R is T_1. We shall have no occasion to refer to Axiom II directly. Its only use is to insure that all point measures are actually measures in the sense of our definition.

Theorem 8: *If R is separable, then M is separable.*

Proof: Let ω be a finitely additive countable basis for the open sets of R. For each set A of ω choose a point in A and a point not in A (unless $A=R$), and call D the set of all these points.

Any finite linear combination of measures of M with non-negative coefficients is a measure of M, so long as the sum of the coefficients is 1. Denote by N the class of measures $t\varphi_p+(1-t)\varphi_q$, where t is rational, $0 \leqslant t \leqslant 1$, and p and q are in D. Let $\mathcal{N}$ denote the class of neighborhoods $\mathcal{O}(\varphi,A,r)$ and their finite intersections, where $\varphi \in N$, $A \in \omega$, r rational. Clearly $\mathcal{N}$ is countable. We shall prove that $\mathcal{N}$ is a basis for the open sets of M. We must prove that if φ_0 is any measure and $\mathcal{O}$ any unitary neighborhood of it, there exists an element $\mathcal{O}_1 \in \mathcal{N}$ such that $\varphi_0 \in \mathcal{O}_1 \subset \mathcal{O}$.

Let φ_0 be any measure and $\mathcal{O}(\varphi_0,O,a)$ any unitary neighborhood of it.

If $O=R$, then $\mathcal{O}=M$. Choose $A \in \omega$, $p \notin A$, $p \in D$, r rational. Then $\varphi_p(A)=0<\varphi_0(A)+r$, and therefore $\varphi_0 \in \mathcal{O}_1(\varphi_p,A,r) \in \mathcal{N}$.

Now let $O \neq R$. There is an A in ω which is contained in O and for which

$$\varphi_0(O)-\varphi_0(A)<a/4. \tag{1}$$

Let $p \in A$, $q \in R-A$, $p,q \in D$, $|\varphi_0(A)-t|<a/4$, t rational. Let $\varphi=t\varphi_p+(1-t)\varphi_q$; then $\varphi \in N$, and $\varphi(A)=t$. Thus

$$|\varphi(A)-\varphi_0(A)|<a/4. \tag{2}$$

Let $\mathcal{O}_1=\mathcal{O}_1(\varphi,A,r)$ where r is rational and $a/4<r<a/2$. Then

(A) $\varphi_0 \in \mathcal{O}_1$ because $\varphi(A)-\varphi_0(A)<a/4<r$ by (2).

(B) $\mathcal{O}_1 \subset \mathcal{O}$. For let $\psi \in \mathcal{O}_1$. Then $\varphi_0(O)<\varphi_0(A)+a/4$ from (1), $\varphi_0(A)<\varphi(A)+a/4$ by (2), and $\varphi(A)<\psi(A)+r$ because $\psi \in \mathcal{O}_1$. Thus $\varphi_0(O)<\psi(A)+a/2+r<\psi(A)+a \leqslant \psi(O)+a$. Hence $\psi \in \mathcal{O}(\varphi_0,O,a)$. This completes the proof of Theorem 8.

It is of interest to note that although the finite intersections of the neighborhoods of measures in N form a basis for M, the set N itself is not necessarily dense in M. Counter exemples are furnished by the following lemma.

Lemma: Let M_2 denote the class of measures $s\varphi_p+(1-s)\varphi_q$ $(0 \leqslant s \leqslant 1)$. If R is separable, normal, and compact[4]), then M_2 is closed.

Proof: Let φ be a limit point of M_2. Since M is separable (Theorem 8), φ is the limit of a sequence of measures of M_2. Since R is compact, we can choose a subsequence $\varphi_n=s_n\varphi_{p_n}+(1-s_n)\varphi_{q_n}$ for which $p_n \to p$, $q_n \to q$, $s_n \to s$.

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[4]) The lemma is true even if R is not compact, but the proof is then much longer.

It is easy to prove that $\varphi_n \overset{\circ}{\to} s\varphi_p+(1-s)\varphi_q$. (See Theorem 10 for the method of proof). But also $\varphi_n \overset{\circ}{\to} \varphi$. The normality of R implies uniqueness of limits in M (Theorem 7). Hence $\varphi=s\varphi_p+(1-s)\varphi_q \in M_2$.

Since the set N is contained in the closed set M_2, it cannot be dense in M.

The following converse of a theorem proved later is given here because of the weak hypothesis needed.

Definition: R is called *compact* if every covering of R by a non-decreasing sequence of open sets is reducible to a finite covering.

Theorem 9: *If M is compact, then R is compact.*

Proof: Assume R is not compact. Let $\sum_{n=1}^{\infty} O_n=R$, $O_{n+1}\supset O_n$, but no $O_n=R$. Choose $p_n \in R-O_n$. The infinite set of point measures (φ_{p_n}) has a limit measure φ (not necessarily a point measure, so far as we know). Since M is a T_1-space, each neighborhood of φ contains infinitely many φ_{p_n}. Name $\varepsilon>0$. Infinitely many φ_{p_n} are in $\mathcal{O}(\varphi,O_k,\varepsilon)$, and hence $\varphi(O_k)<\varphi_{p_n}(O_k)+\varepsilon$ for these p_n. But $\varphi_{p_n}(O_k)=0$ for $n\geqslant k$. Thus $\varphi(O_k)<\varepsilon$. Hence $\varphi(O_k)=0$ for each k. But this contradicts $1=\varphi(R)\leqslant\sum_{k=1}^{\infty}\varphi(O_k)$. Hence R is compact.

Axiom III: *R is a topological space (i. e. the class of open sets is finitely multiplicative and unrestrictedly additive, and the null set is open).*

Theorem 10: *R is topologically equivalent to the subspace M_1 of point measures.*

Proof: For convenience of notation, we shall identify the point p with the corresponding point measure φ_p.

Let p_0 be any point in R, and O any open set containing p_0. Then $\mathcal{O}(\varphi_{p_0},O,1/2)\subset O$. For let $\varphi_p \in \mathcal{O}$. Then $\varphi_p(O)>\varphi_{p_0}(O)-1/2=$ $=1-1/2=1/2$, hence $=1$, and $p \in O$.

Let $\mathcal{O}(\varphi_{p_0},O,a)$ be any unitary neighborhood of φ_{p_0} in M_1. Then $O\subset\mathcal{O}$. For let $p \in O$. Then $\varphi_{p_0}(O)\leqslant 1=\varphi_p(O)<\varphi_p(O)+a$, and therefore $\varphi_p \in \mathcal{O}$.

Some immediate corollaries are:

(1) *If M is a Hausdorff space, so is R.*

(2) If R is bicompact, then it is normal if and only if it is a Hausdorff space. Hence, combining (1) with Theorem 7, we have: *If R is bicompact, then M is a Hausdorff space if and only if R is normal.*

(3) *If M is separable, then R is separable.* Combining with Theorem 8, *M is separable if and only if R is separable.*

Compactness of *M*. ***Theorem 11.*** *If R is a compact, separable, Hausdorff space, then M is bicompact.*

(The hypotheses imply that R is metrizable. In a metric space our measures are seen to be equivalent to those of Kryloff and Bogoliouboff [5]). They proved the compactness of M. Our proof is shorter and purely topological in character, while theirs is strongly metric).

Proof: Let ω be a finitely additive countable basis. Every open set O is the sum of a non-decreasing sequence (A_i) of sets of ω. We shall call such a representation of O *admissible* if $\bar{A}_i\subset O$ for each i. The hypotheses of the theorem imply that R is regular, and therefore admissible representations exist for each open set. Also, if (A_i) is an admissible representation for O_1 and (B_i) is an admissible representation for O_2, then (A_i+B_i) is admissible for O_1+O_2.

Let a sequence of measures be given. By the diagonal process we may choose a subsequence (φ_n) convergent at each set of ω. Call the limit θ. Then $\varphi_n(A)\to\theta(A)$ for each A in ω.

We shall prove that if (A_i) and (B_i) are two admissible representations for O, then $\operatorname{Lim}\theta(A_i)=\operatorname{Lim}\theta(B_i)$. Let A_k be a definite A_i. $\bar{A}_k$ is closed and contained in $\sum_{i=1}^{\infty}B_i$, which are open. Since R is compact, $A_k\subset\sum_{i=1}^{N}B_i=B_N$. Hence $A_k\subset B_N$. It follows that $\operatorname{Lim}\theta(A_i)\leqslant\operatorname{Lim}\theta(B_i)$. Similarly $\operatorname{Lim}\theta(A_i)\geqslant\operatorname{Lim}\theta(B_i)$. Hence the two limits are equal.

Let O be any open set. Define $\varphi(O)=\operatorname{Lim}\theta(A_i)$, where (A_i) is any admissible representation for O. According to the preceding paragraph, this definition is independent of the choice of the admissible representation.

We now prove that the set function φ (defined on the open sets) has the following properties.

[5]) Kryloff N. and Bogoliouboff N., *La Théorie Générale de la Mesure dans son Application à l'Étude des Systémes Dynamiques de la Mécanique Non Linéaire*, Ann. of. Math. **38** (1937), p. 65.

(A) $\varphi(O) \geqslant 0, \quad \varphi(0)=0.$

Proof trivial.

(B) $O_1 \subset O_2 \Longrightarrow \varphi(O_1) \leqslant \varphi(O_2).$

Let (A_i) and (B_i) be admissible representations for O_1 and O_2 respectively. Then (A_i+B_i) is admissible for $O_1+O_2=O_2$.

$$\varphi(O_1)=\operatorname{Lim}\theta(A_i) \leqslant \operatorname{Lim}\theta(A_i+B_i)=\varphi(O_2).$$

(C) $\varphi(O_1+O_2) \leqslant \varphi(O_1)+\varphi(O_2).$

Let (A_i) and (B_i) be admissible representations for O_1 and O_2.
$\varphi(O_1+O_2)=\operatorname{Lim}\theta(A_i+B_i) \leqslant \operatorname{Lim}\theta(A_i)+\operatorname{Lim}\theta(B_i)=\varphi(O_1)+\varphi(O_2).$

(D) $\varphi\left(\sum_{n=1}^{N} O_n\right) \leqslant \sum_{n=1}^{N} \varphi(O_n).$

Proof by induction on N.

(E) $\varphi\left(\sum_{n=1}^{\infty} O_n\right) \leqslant \sum_{n=1}^{\infty} \varphi(O_n).$

Let (A_i) be an admissible representation for $O=\sum_{n=1}^{\infty} O_n$. Let A_k be a definite A_i. Then $\bar{A}_k \subset O$. Hence $\bar{A}_k \subset \sum_{n=1}^{N} O_n$. Therefore A_k can be made to form part of an admissible representation for $\sum_{n=1}^{N} O_n$, and $\theta(A_k) \leqslant \varphi\left(\sum_{n=1}^{N} O_n\right) \leqslant \sum_{n=1}^{N} \varphi(O_n) \leqslant \sum_{n=1}^{\infty} \varphi(O_n)$. Since $\varphi(O)=\operatorname{Lim}\theta(A_i)$, inequality (E) follows.

(F) $O_1 O_2=0 \Longrightarrow \varphi(O_1+O_2)=\varphi(O_1)+\varphi(O_2).$

Let (A_i) and (B_i) be admissible representations for O_1 and O_2. For each i, $A_i B_i=0$. Hence $\varphi_n(A_i+B_i)=\varphi_n(A_i)+\varphi_n(B_i)$, from which follows $\theta(A_i+B_i)=\theta(A_i)+\theta(B_i)$. Letting i approach infinity, we obtain (F).

We now extend the set function φ from the open sets to all sets by $\varphi(E)=LB\varphi(O)$ for all $O \supset E$. It can be proved by standard methods, as a result of Properties A, B, and E, that φ is a regular outer measure (i. e. that it satisfies conditions 1-4 of our definition of measure stated in the first section).

We must prove next that open sets are φ-measurable. To do this we could merely quote Caratheodory's [6]) theorem (valid in any metric space) that open sets are measurable. But we may avoid all

[6]) Carathéodory C., *Vorlesungen über Reelle Funktionen* (1927).

reference to a metric by making a very slight modification in the Caratheodory proof. Our hypotheses are the same as his except that $\varphi(A+B)=\varphi(A)+\varphi(B)$ is implied by separation of A and B by non-intersecting open sets (Property F) rather than $d(A,B)>0$. Let O be open. Since R is compact, we can choose an admissible representation (A_i) for O with the additional property $\bar{A}_i \subset A_{i+1}$. Then proceed exactly as in Caratheodory.

This completes the proof that φ is a measure. Now we prove that $\varphi_n \overset{\circ}{\to} \varphi$. Let O be any open set, and (A_i) an admissible representation for O.

$$\varphi(O)=\operatorname*{Lim}_{i\to\infty}\theta(A_i)=\operatorname*{Lim}_{i\to\infty}\operatorname*{Lim}_{n\to\infty}\varphi_n(A_i) \leqslant \operatorname*{Lim}_{i\to\infty}\operatorname*{\underline{Lim}}_{n\to\infty}\varphi_n(O)=\operatorname*{\underline{Lim}}_{n\to\infty}\varphi_n(O).$$

Thus $\varphi_n \overset{\circ}{\to} \varphi$, and M is compact.

Finally, R separable implies M separable (Theorem 8), and therefore M is bicompact.

Theorem 9 implies the converse of Theorem 11. Combining them: *If R is separable and Hausdorff, then M is compact if and only if R is compact.*

Let R be a compact metrizable space.

R compact metrizable $\Longrightarrow R$ compact, separable, Hausdorff, normal $\Longrightarrow M$ bicompact Hausdorff $\Longrightarrow M$ normal. Also R separable $\Longrightarrow M$ separable. But M normal, separable $\Longrightarrow M$ metrizable. This and similar reasoning applied in the reverse direction yields.

Theorem 12: *M is compact and metrizable if and only if R is compact and metrizable.*

An explicit metric for M can be obtained as follows: Since R is compact and metrizable, the space of continuous functions on R is separable (in the uniform convergence topology). Let (f_k) be a dense sequence of functions in the unit sphere. Then

$$(\varphi,\psi)=\sum_{k=1}^{\infty} 1/2^k \left|\int_R f_k d\varphi-\int_R f_k d\psi\right|$$

is a metric for M.

When R is not Hausdorff and separable, the truth of R compact and $T_1 \Longrightarrow M$ compact is an open question. We have resolved the question for the following special class of spaces R.

Let L be a T_1-space, a a symbol, $R=L+a$. Let the closed sets of R be the finite subsets of L and sets $F+a$, where F is closed in L. R is a compact T_1-space. We have given a proof (omitted here) that M is compact for such spaces $R=L+\alpha$.

I) Let L be discrete and non-denumerable. Then R is *Hausdorff* and *not* separable, and M is compact.

A special case of $R=L+\alpha$ is the space R whose closed sets are R itself and its finite subsets.

II) Let R be non-denumerable. Then R is *not* Hausdorff, *not* separable, and M is compact.

III) Let R be denumerably infinite. Then R is *separable* and *not* Hausdorff, and M is compact.

Thus R Hausdorff and R separable are not necessary, either singly or together, for the compactness of M.

On Free $\aleph_\xi$-complete Boolean Algebras.

(With an Application to Logic).

By

Ladislav Rieger (Praha).

A Boolean algebra A is said to be $\aleph_\xi$-*complete* if any subset of elements of A the power of which does not exceed $\aleph_\xi$ has a g. l. b. and a l. u. b. in A. An $\aleph_\xi$-complete Boolean algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ is said to be *free with* $\mathfrak{m}$ *free* $\aleph_\xi$-*generators* (where $\mathfrak{m}$ is any cardinal number) if there exists a subset $G\subset A_{\mathfrak{m}}^{\aleph_\xi}$ the power of which is $\mathfrak{m}$ so that G has the following properties:

(i) The only $\aleph_\xi$-complete subalgebra of $A_{\mathfrak{m}}^{\aleph_\xi}$ containing G is $A_{\mathfrak{m}}^{\aleph_\xi}$ itself. (We say that the elements of G $\aleph_\xi$-generate $A_{\mathfrak{m}}^{\aleph_\xi}$).

(ii) If φ is any mapping of G into another $\aleph_\xi$-complete algebra B then φ can be extended to a $\aleph_\xi$-complete [1]) homomorphic mapping of the whole algebra $A_{\mathfrak{m}}^{\aleph_\xi}$ into B.

Familiarity with these and other (better known) basic notions of the theory of Boolean algebras will be assumed. I refer to a brief exposition of these notions in R. Sikorski's papers [1] and [2] (this Fund. Math. 1948 and 1949). For a more extensive treatise, the monograph of G. Birkhoff [1] on Lattice Theory (sec. ed. 1948) is recommended.

Note that by an *α-ideal* (the symbol due to M. H. Stone), I understand what sometimes is called a *dual ideal*, i. e. a (nonvoid) subset I of the algebra A in question so that if $a,b\in I$ then $a\cap b\in I$ and if $a\subseteq b, a\in I$ then $b\in I$.

Of course, to each of the theorems of the present paper there is a dual one. The dualisation is left to the reader.

1) Instead of $\aleph_\xi$-complete Boolean algebra and $\aleph_\xi$-complete homo(iso)-morphic(ism) and $\aleph_\xi$-complete ideal we simply say $\aleph_\xi$-algebra, $\aleph_\xi$-homo(iso)-morphic(ism), $\aleph_\xi$-ideal resp. Especially, a homomorphic mapping f is said to be $\aleph_\xi$-*homomorphic* if $f(\bigcup_{\iota\in I} x_\iota)=\bigcup_{\iota\in I} f(x_\iota)$ holds for any set I of indices with card $(I)\leqslant\aleph_\xi$. $\aleph_\xi$ is then said to be the *level of completeness*. Instead of the prefix $\aleph_0$- we use the more common symbol σ-($\aleph_0$-algebra $=\sigma$-algebra, $\aleph_0$-ideal $=\sigma$-ideal,...).