where $\varphi_n^m(p_1, ..., p_m) = [\mathcal{E}(F_n^m(x_{p_1}, ..., x_{p_m}))]$ [5]. Clearly the two-valued function φ_n^m may be interpreted as the characteristic function of a set K_n^m, the elements of which are m-element sequences of positive integers.

This means (see the proof of [TG] (i)) that the sequences

$$\{n\}_{n=1,2,...} \quad \text{and} \quad \{K_n^m\}_{m,n=1,2,...}$$

satisfy simultaneously all the formulae $\alpha \in A$.

[5]) According to [TG], § 5, the symbol $[a]$ denotes the element (in $\mathcal{B}/\mathfrak{p}$) determined by an element $a \in \mathcal{B}$.

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On the application of Tychonoff's theorem in mathematical proofs.

By

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In this paper [1]) we shall give two theorems: one, effective [2]), on finite properties (Th. 1), the other non effective, on consistent choices (Th. 2). Both those theorems are closely connected with Tychonoff's theorem on the product of bicompact spaces [3]), according to which [4])

(T) *the product of as many bicompact spaces is bicompact in product topology.*

This theorem is often used in existence proofs; the theorems presented here are to some extent a scheme of such proofs.

The proofs given in this paper are effective, with exception of two cases in which the theorem of Tychonoff (T) is used. All proofs of Tychonoff's theorem are non effective, i. e. all in its proofs the principle of choice is used. In another paper we shall prove that no effective proof of this theorem exists [5]).

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on October 27, 1950.

[2]) Effective, i. e. no transfinite methods are used in the proof of this theorem.

[3]) By a *topological space* we understand in this paper a Hausdorff's space.

[4]) Cf. e. g. N. Bourbaki [2], p. 63.

[5]) In the previous volume of this Journal J. L. Kelley [3] has shown that the theorem of Tychonoff for Kuratowski's closure spaces implies the axiom of choice. In the proof of this implication unfortunately there is a mistake (the set Z_a considered on p. 76 is open and non closed), which however can be easily corrected.

1. Let X_0 be a set of arbitrary elements, $\mathfrak{W}$ a property defined for subset of X_0; $\mathfrak{W}(E)$ means: E has the property $\mathfrak{W}$.

(1.1) $\mathfrak{W}$ is called a *finite property* if it fullfils two conditions:

(1.1.1) $E_1\subset E_2\subset X_0$ and $\mathfrak{W}(E_2)$ implies $\mathfrak{W}(E_1)$,

(1.1.2) If $\mathfrak{W}(E)$ for every finite $E\subset E_0\subset X_0$, then $\mathfrak{W}(E_0)$.

(1.2) Let Y_0 be a set, and $\mathfrak{W}_y$, for $y\in Y_0$, a property defined for subset of X_0. The family $\{\mathfrak{W}_y\}_{y\in Y_0}$ is called a *compact family* of finite properties of X_0, if every property of this family is finite and

(1.2.1) for every finite $E\subset X_0$ there exists such a $y\in Y_0$ that $\mathfrak{W}_y(E)$,

(1.2.2) Y_0 is a bicompact space and for every finite $E\subset X_0$ the set

$$T(E)=\underset{y\in Y_0}{E}[\mathfrak{W}_y(E)]$$

is closed in Y_0.

Theorem 1 (on finite properties). *If $\{\mathfrak{W}_y\}_{y\in Y_0}$ is a compact family of finite properties of X_0, then $\mathfrak{W}_{y_0}(X_0)$ for some $y_0\in Y_0$.*

Proof. From (1.1.1) and (1.2.1) we have for every natural n and for finite subset $E_1,E_2,\dots,E_n$ of X_0

$$\prod_{i=1}^{n} T(E_i)\supset T\Big(\sum_{i=1}^{n} E_i\Big)\neq 0.$$

Since Y_0 is compact and $T(E)$, for finite E, closed in Y_0, we have

$$T=\prod_{\substack{E<X_0\\ \bar{\bar{E}}<\aleph_0}} T(E)\neq 0.$$

Let $y_0\in T$; then obviously $\mathfrak{W}_{y_0}(E)$ for every finite $E\subset X_0$, and by (1.1.2) $\mathfrak{W}_{y_0}(X_0)$.

2. *Application.* We shall show how theorem 1 may be applied to prove the following well-known theorem [6]):

(2.1) If $\rightarrow$ is a partial ordering relation of the set X_0, then there exists a (simply) ordering relation $\prec$ which is an extension of $\rightarrow$.

[6]) Cf. E. Szpilrajn-Marczewski [4].

Every binary relation R defined for elements of X_0 may be considered as a function on $X_0^2=X_0\times X_0$, such that $f(\langle x_1,x_2\rangle)=1$, if x_1Rx_2, and $=0$ in the contrary case. Let Y_0 be the set of all binary relations definite for elements of X_0, in view of the above statement, Y_0 is the product $(1,0)^{X_0^2}$. The set $(0,1)$, as finite, can be considered as a bicompact space, and finally, in view of Tychonoff's theorem, Y_0 is a bicompact space. For $y\in Y_0$ and $E\subset X_0$, denote by $\mathfrak{W}_y(E)$ the property: the relation y is an extension of $\rightarrow$ and orders E. It is easy to see that $\{\mathfrak{W}_y\}_{y\in Y_0}$ is a compact family of finite properties of X_0, and so in virtue of theorem 1 one $y_0=\prec\in Y_0$ exists, so that $\mathfrak{W}_{\prec}(X_0)$, which is precisely the thesis of theorem (2.1).

3. *The theorem of consistent choice.*

(3.1) If P_1 and P_2 are topological spaces, σ a binary relation defined for $p_1\in P_1$ and $p_2\in P_2$, then we say that *σ is closed between P_1 and P_2*, if the set $\underset{\langle p_1,p_2\rangle}{E}[p_1\sigma p_2]$ is closed in the product space $P_1\times P_2$.

(3.2) If $\{P_x\}_{x\in X_0}$ is a family of compact spaces, then every symmetrical relation σ defined for $p_1,p_2\in\sum'_{x\in X_0} P_x$ shall be called *relation of consistency* if it is closed between each pair of different spaces of the above-mentioned family.

(3.3) The function $y(x)$ defined for $x\in X_0$ and such that $y(x)\in P_x$ is called a *choice* from the family $\{P_x\}_{x\in X_0}$. If σ is a relation of consistency for the family $\{P_x\}_{x\in X_0}$, the choice y is called *σ-consistent on* $E\subset X_0$, if $y(x_1)\sigma y(x_2)$ for $x_1,x_2\in E$ and $x_1\neq x_2$.

Theorem 2 (on consistent choice). *If $\{P_x\}_{x\in X_0}$ is a family of bicompact spaces, σ a relation of consistency for this family, and if for every finite $E\in X_0$ there exists a choice from $\{P_x\}_{x\in X_0}$ σ-consistent on E, then there also exists a σ-consistent choice on X_0.*

Proof. Let Y_0 denote the product $\mathbf{P}_{x\in X_0} P_x$ and consider Y_0 as topological space in the product topology. It follows from the theorem of Tychonoff that Y_0 is a bicompact space. Every $y\in Y_0$ is evidently a choice from $\{P_x\}_{x\in X_0}$. Let for $E\subset X_0$, $\mathfrak{W}_y(E)$ denote the property: y is a σ-consistent choice on E. The family $\{\mathfrak{W}_y\}_{y\in Y_0}$ is obviously a compact family of finite properties of X_0 (from the continuity of σ it follows that $T(E)=\underset{y\in Y_0}{E}[\mathfrak{W}_y(E)]$ is closed); therefore the theorem 1 asserts that for some $y_0\in Y_0$ we have $\mathfrak{W}_{y_0}(X_0)$, which expresses that y_0 is a σ-consistent choice on X_0, q. e. d.

4. ***The case of finite spaces.*** If every space of the family $\{P_x\}_{x\in X_0}$ is finite, then every symmetrical relation σ defined for elements of $\sum_{x\in X_0} P_x$ is a relation of consistency for this family. In view of this we can formulate the theorem 2 as follows:

Theorem 2*. *If $\{P_x\}_{x\in X_0}$ is a family of finite sets, σ a symmetrical relation defined for elements of $\sum_{x\in X_0} P_x$, and if for every finite $E\subset X_0$ there exists a choice from this family σ-consistent on E, there also exists a σ-consistent choice on the whole X_0.*

In some of the proofs it is convenient to use the theorem 2* instead of theorem 2. It is interesting to note that both theorems are effectively equivalent.

5. ***Application.*** Let W be a vectorial space and $G\subset W$ its subspace; let $g(w)$ be a functional defined on W fulfilling the conditions ($w_1, w_2 \in W$, λ a real number):

(5.1) $$g(w_1+w_2)\leqslant g(w_1)+g(w_2); \quad \lambda g(w_1)=g(\lambda w_1), \quad \lambda\geqslant 0,$$

and $f(w)$ a homogeneous and additive functional on G and such that

(5.2) $$f(w)\leqslant g(w), \quad \text{for} \quad w\in G.$$

Let X_0 denote the family of all finite subsets of W and for $X\in X_0$ let P_x be the n-dimensional closed interval $\underset{w\in X}{\boldsymbol{P}}[-g(-w), g(w)]$ (where n is the power of X). Every $p\in P_x$ may be considered as a function $p(w)$ defined for $w\in X$ and such that

(5.3) $$-g(-w)\leqslant p(w)\leqslant g(w).$$

We shall say that $p\in P_x$ is a *partial extension* of the functional f, if

(5.4) $p(w)=f(w)$ for $w\in G\cdot X$,

(5.5) for $w_1, w_2, w_1+w_2\in X$ we have $p(w_1+w_2)=p(w_1)+p(w_2)$,

(5.6) for w, $\lambda w\in X$ we have $\lambda p(w)=p(\lambda w)$.

We set, for $X_1, X_2\in X_0$ and $p_1\in P_{x_1}$, $p_2\in P_{x_2}$,

(5.7) $p_1\sigma p_2$ if and only if both p_1 and p_2 are partial extensions of the functional f and, besides, are equal on $X_1\cdot X_2$.

It is easy to see that σ is a relation of consistency for $\{P_x\}_{x\in X_0}$ and that moreover for every finite subset of X_0 a σ-consistent choice

exists. From theorem 2 we see that there exists a function $\varphi(X)$ which is a σ-consistent choice on X_0. By setting for $w\in W$

(5.8) $$F(w)=[\varphi(X)](w), \quad \text{where} \quad w\in X\in X_0.$$

We obtain a functional which fulfils in the whole space W the conditions (5.3)-(5.6) (for $p=F$, and $X=W$).

Thus we have proved the well-known theorem of Banach on the extension of functional [7]).

The theorems 1 and 2 may be used in proofs of many other theorems [8]) which deal with the extension of some finite properties from a subset of the given set on the whole set. The application of the theorem 1 generally requires the use of Tychonoff's theorem.

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[7]) Cf. S. Banach [1], p. 27-29.

[8]) Other applications of the theorem 2 and a discussion on the effectivity of the theorem of Tychonoff for Hausdorff's spaces will be presented in our next paper *Effectivity of the theory of representations of Boolean algebras*.

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