where \( \phi^m[p_1, \ldots, p_m] = [\mathcal{G}(\mathcal{F}^m_x(x_1, \ldots, x_m))] \). Clearly the two-valued function \( \phi^m_x \) may be interpreted as the characteristic function of a set \( \mathcal{F}^m_x \), the elements of which are \( m \)-element sequences of positive integers.

This means (see the proof of [TG] (i)) that the sequences

\[ \{a\}_{m=1,2, \ldots} \quad \text{and} \quad \{x^m_{a,m}, m=1,2, \ldots \} \]

satisfy simultaneously all the formulae \( a \in A \).

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\( ^{1} \) According to [TG], § 5, the symbol \( [x] \) denotes the element (in \( \mathbb{B}/\mathbb{P} \)) determined by an element \( x \in \mathbb{B} \).

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On the application of Tychonoff’s theorem in mathematical proofs.

By

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In this paper \(^{1}\) we shall give two theorems: one, effective \(^{2}\), on finite properties (Th. 1), the other non effective, on consistent choices (Th. 2). Both those theorems are closely connected with Tychonoff’s theorem on the product of bicom pact spaces \(^{3}\), according to which \(^{4}\)

\( (T) \) the product of as many bicom pact spaces is bicom pact in product topology.

This theorem is often used in existence proofs; the theorems presented here are to some extent a scheme of such proofs.

The proofs given in this paper are effective, with exception of two cases in which the theorem of Tychonoff (T) is used. All proofs of Tychonoff’s theorem are non effective, i. e. all in its proofs the principle of choice is used. In another paper we shall prove that no effective proof of this theorem exists \(^{5}\).

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\(^{1}\) Presented to the Polish Mathematical Society, Wrocław Section, on October 27, 1960.

\(^{2}\) Effective, i.e. no transfinite methods are used in the proof of this theorem.

\(^{3}\) By a topological space we understand in this paper a Hausdorff’s space.

\(^{4}\) Cf. e.g. N. Bourbaki [2], p. 83.

\(^{5}\) In the previous volume of this Journal J. I. Kelley [3] has shown that the theorem of Tychonoff for Kuratowski’s closure spaces implies the axioms of choice. In the proof of this implication unfortunately there is a mistake (the set \( \mathcal{E}_x \) considered on p. 76 is open and not closed), which however can be easily corrected.
1. Let \( X_0 \) be a set of arbitrary elements, \( B \) a property defined for subset of \( X_0 \); \( B(E) \) means \( E \) has the property \( B \).

(1.1) \( B \) is called a finite property if it fulfills two conditions: (1.1.1) \( E \subset F \subset X_0 \) and \( B(E) \) implies \( B(F) \), (1.1.2) if \( B(E) \) for every finite \( E \subset X_0 \), then \( B(F) \).

(1.2) Let \( X_0 \) be a set, and \( B_0 \) for \( y \in X_0 \), a property defined for subset of \( X_0 \). The family \( \{ B_0 \} \) is called a compact family of finite properties of \( X_0 \), if every property of this family is finite and

(1.2.1) for every finite \( E \subset X_0 \), there exists such a \( y \in X_0 \) that \( B_0(E) \),

(1.2.2) \( E \) is a bicomponent space and for every finite \( E \subset X_0 \) the set

\[
T(E) = \bigcup_{y \in E_0} \{ B_0(E) \}
\]

is closed in \( Y_0 \).

**Theorem 1** (on finite properties). If \( \{ B_0 \} \) is a compact family of finite properties of \( X_0 \), then \( B_0(X_0) \) for some \( y_0 \in X_0 \).

Proof. From (1.1.1) and (1.2.1) we have for every natural \( n \) and for finite subset \( E_1, E_2, \ldots, E_n \) of \( X_0 \)

\[
\prod_{i=1}^{n} T(E_i) \cap T(\bigcup_{i=1}^{n} E_i) = \emptyset.
\]

Since \( Y_0 \) is compact and \( T(E) \), for finite \( E \), closed in \( Y_0 \), we have

\[
T = \bigcap_{E \subset X_0} T(E) = \emptyset.
\]

Let \( y_0 \in T \); then obviously \( B_0(E) \) for every finite \( E \subset X_0 \) and by (1.1.2) \( B_0(X_0) \).

2. **Application.** We shall show how theorem 1 may be applied to prove the following well-known theorem 4:

(2.1) If \( \rightarrow \) is a partial ordering relation of the set \( X_0 \), then there exists a (simply) ordering relation \( \triangleright \) which is an extension of \( \rightarrow \).

4) C. E. Szpilrajn-Marczewski [4].

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**Application of Tychonoff's theorem**

Every binary relation \( R \) defined for elements of \( X_0 \) may be considered as a function on \( X_0^2 = X_0 \times X_0 \), such that \( f((x_1, x_2)) = 1 \), if \( x_1 \rightarrow x_2 \) and \( = 0 \) in the contrary case. Let \( Y_0 \) be the set of all binary relations definite for elements of \( X_0 \), viewed in the above statement, \( Y_0 \) is the product \((1,0)^X_0 \). The set \((0,1)\), as finite, can be considered as a bicomponent space, and finally, in view of Tychonoff's theorem, \( Y_0 \) is a bicomponent space. For \( y \in Y_0 \) and \( B \subset X_0 \), denote by \( B(y) \) the property: the relation \( y \) is an extension of \( \rightarrow \) and \( B \).

It is easy to see that \( \{ B(y) \} \) is a compact family of finite properties of \( X_0 \) and so in virtue of theorem 1 one \( y_0 \in Y_0 \) exists, so that \( B_0(X_0) \) which is precisely the thesis of theorem 2.1.

3. **The theorem of consistent choice.**

(3.1) If \( P_1 \) and \( P_2 \) are topological spaces, \( \sigma \) a binary relation defined for \( p_1 \in P_1 \) and \( p_2 \in P_2 \), then we say that \( \sigma \) is closed between \( P_1 \) and \( P_2 \), if the set \( \prod_{p_1 \in P_1} \sigma(p_2) \) is closed in the product space \( P_1 \times P_2 \).

(3.2) If \( \{ P_0 \} \) is a family of compact spaces, then every symmetrical relation \( \sigma \) defined for \( p_0, p_1 \in \bigcap_{i=0}^{n} P_i \) shall be called a relation of consistency if it is closed between each pair of different spaces of the above-mentioned family.

(3.3) The function \( y(x) \) defined for \( x \in X_0 \) such that \( y(x) \in X_0 \) is called a choice from the family \( \{ P_0 \} \).

If \( \sigma \) is a relation of consistency for the family \( \{ P_0 \} \), the choice \( y \) is called a consistent on \( X_0 \), if \( y(0) \in y(1) \) for \( 0, 1 \in X_0 \), and\( \sigma \).

**Theorem 2** (on consistent choice). If \( \{ P_0 \} \) is a family of bicomponent spaces, \( \sigma \) a relation of consistency for this family, and if every finite \( E \subset X_0 \), there exists a choice from \( \{ P_0 \} \), consistent on \( E \), then there also exists a consistent choice on \( X_0 \).

Proof. Let \( Y_0 \) denote the product \( P_0 \), and consider \( Y_0 \) as topological space in the product topology. It follows from the theorem of Tychonoff that \( Y_0 \) is a bicomponent space. Every \( y \in Y_0 \) is evidently a choice from \( \{ P_0 \} \). Let for \( E \subset X_0 \) \( B_0(E) \) denote the property: \( y \) is a \( \sigma \)-consistent choice on \( E \). The family \( \{ B_0 \} \) is obviously a compact family of finite properties of \( X_0 \). Therefore, the continuity of \( \sigma \) follows that \( T(y) = \bigcap_{y \in (0,1)} \{ B_0(y) \} \) is closed; therefore the theorem 1 asserts that for some \( y_0 \in Y_0 \) we have \( B_0(Y_0) \), which expresses that \( y_0 \) is a \( \sigma \)-consistent choice on \( X_0 \), q. e. d.
4. The case of finite spaces. If every space of the family $\{P_x\}_{x \in X}$ is finite, then every symmetrical relation $\sigma$ defined for elements of $\Sigma P_x$ is a relation of consistency for this family. In view of this we can formulate the theorem 2 as follows:

**Theorem 2**. If $\{P_x\}_{x \in X}$ is a family of finite sets, $\sigma$ a symmetrical relation defined for elements of $\Sigma P_x$, and if for every finite $E \subset X$, there exists a choice from this family $\sigma$-consistent on $E$, there also exists a $\sigma$-consistent choice on the whole $X$.

In some of the proofs it is convenient to use the theorem 2 instead of theorem 2. It is interesting to note that both theorems are effectively equivalent.

5. Application. Let $W$ be a vectorial space and $G \subset W$ its subspace; let $g(w)$ be a functional defined on $W$ fulfilling the conditions: (w1, w2 \in W, \lambda \in \mathbb{R}):

$$
g(w_1 + w_2) \leq g(w_1) + g(w_2), \quad g(\lambda w_1) = \lambda g(w_1), \quad \lambda \geq 0,
$$

and $f(w)$ a homogeneous and additive functional on $G$ and such that

$$
f(w) \leq g(w), \text{ for } w \in G.
$$

Let $X$ denote the family of all finite subsets of $W$ and for $X \in X$, let $P_x$ be the $n$-dimensional closed interval $I^n = \{(-w, g(w)) \text{ for } w = x \}$ where $n$ is the power of $X$. Every $p \in P_x$ may be considered as a function $p(w)$ defined for $w \in X$ and such that

$$
g(-w) \leq g(w) \leq g(w).
$$

We shall say that $p \in P_x$ is a partial extension of the functional $f$, if

$$
p(w) = f(w) \text{ for } w \in G \cdot X,
$$

then for $w_1, w_2, w_3 \in X$ we have $p(w_1 + w_2) = p(w_1) + p(w_2)$.

We set, for $X_1, X_2 \in X$ and $p_1, p_2 \in P_x$, $p_1 \circ p_2$.

It is easy to see that $\sigma$ is a relation of consistency for $\{P_x\}_{x \in X}$ and that moreover for every finite subset of $X_1$ a $\sigma$-consistent choice exists. From theorem 2 we see that there exists a function $\phi(X)$ which is a $\sigma$-consistent choice on $X_1$. Setting for $w \in W$

$$
F(w) = (\phi(X))(w), \text{ where } w \in X \cdot X.
$$

We obtain a functional which fulfills in the whole space $W$ the conditions (5.3)-(5.6) for $p = F$, and $X = W$.

Thus we have proved the well-known theorem of Banach on the extension of functional. 1)

The theorems 1 and 2 may be used in proofs of many other theorems 2) which deal with the extension of some finite properties from a subset of the given set on the whole set. The application of the theorem 1 generally requires the use of Tychonoff's theorem.

**Bibliography.**


2) Other applications of the theorem 2 and a discussion on the effectivity of the theorem of Tychonoff for Hausdorff's spaces will be presented in our next paper Effectivity of the theory of representations of Boolean algebras.

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