A Proof of the Skolem-Löwenheim Theorem.

By

H. Rasiowa and R. Sikorski (Warszawa).

This paper is a supplement to our paper *A Proof of the Completeness Theorem of Gödel* [1]) cited hereafter as [TG]. The subject of this paper is a simple proof of the Skolem-Löwenheim theorem [2]):

Every consistent set A of formulae of the classical (first order) functional calculus is simultaneously satisfiable in the domain I of positive integers.

The method of the proof is the same as the method used in [TG]. The knowledge of [TG] is here assumed. The terminology and notation are in this paper the same as in [TG].

Proof. Let A_0 be the set of all consequences of A.

For any formula α, let $\mathcal{E}(\alpha)$ be the class of all formulae γ such that $(\alpha\to\gamma)\in A_0$ and $(\gamma\to\alpha)\in A_0$. The set $\mathcal{B}$ of all classes (cosets) $\mathcal{E}(\alpha)$ is a Boolean algebra with the following definition of Boolean operations

$$\mathcal{E}(\alpha)+\mathcal{E}(\beta)=\mathcal{E}(\alpha+\beta);\quad \mathcal{E}(\alpha)'=\mathcal{E}(\alpha').$$

[1]) Fund. Math. **37** (1950), pp. 193-200.

[2]) See L. Löwenheim, *Über Möglichkeiten im Relativkalkül*, Math. Annalen **76** (1915), pp. 447-470; Th. Skolem, *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen*, Skrifter utgit ar kindeskapsselskapet i Kristania, I. Mat. nat. Klasse, 1919, nº 3; Th. Skolem, *Über einige Grundlagenfragen der Mathematik*, ibid. 1929, nº 4; Th. Skolem, *Sur la portée du théorème de Skolem-Löwenheim*, Les Entretiens de Zürich sur les Fondements des Sciences Mathématiques, Zürich 1941, pp. 25-47; Leon Henkin, *The completeness of the first order functional calculus*, The Journal of Symbolic Logic **14**, Nr 3 (1949), pp. 159-166.

Clearly [3])

(i) $\mathcal{E}(\alpha)\subset\mathcal{E}(\beta)$ if and only if $(\alpha\to\beta)\in A_0$,

(ii) The unit element of $\mathcal{B}$ is the class A_0.

Using the notation from [TG], p. 198, we have [4])

$$\text{(iii)}\qquad \sum_{p\in I}\mathcal{E}\left(\beta\binom{x_p}{x_k}\right)=\mathcal{E}\left(\sum_{x_k}\beta\right).$$

In fact, since $\left(\beta\binom{x_p}{x_k}\to\sum_{x_k}\beta\right)\in A_0$ we have by (i)

$$\text{(iv)}\qquad \mathcal{E}\left(\beta\binom{x_p}{x_k}\right)\subset\mathcal{E}\left(\sum_{x_k}\beta\right)\quad\text{for every } p\in I.$$

Suppose now that

$$\text{(v)}\qquad \mathcal{E}\left(\beta\binom{x_p}{x_k}\right)\subset\mathcal{E}(\gamma)\qquad\text{for every } p\in I.$$

By (i), we have $\left(\beta\binom{x_p}{x_k}\to\gamma\right)\in A_0$. Let p be such an integer that x_p is free neither in β nor in γ. We infer that $\left(\sum_{x_p}\beta\binom{x_p}{x_k}\to\gamma\right)\in A_0$. Hence, by (i),

$$\text{(vi)}\qquad \mathcal{E}\left(\sum_{x_k}\beta\right)=\mathcal{E}\left(\sum_{x_p}\beta\binom{x_p}{x_k}\right)\subset\mathcal{E}(\gamma).$$

The equation (iii) follows from (iv), (v) and (vi).

Since A is consistent, the algebra $\mathcal{B}$ contains at least two elements. By [TG] (iv) there is a prime ideal $\mathfrak{p}$ which preserves all the sums (iii). The algebra $B_0=\mathcal{B}/\mathfrak{p}$ has only two elements 0 and 1.

As in [TG], pp. 199-200, we obtain that, for every $\alpha\in A$, the (I,B_0) functional Φ_α assumes the values $1\in B_0$ for the following values of its arguments:

$$x_n=n\quad\text{and}\quad F_n^m=\varphi_n^m,\quad n,m=1,2,\ldots,$$

[3]) $\subset$ denotes here the Boolean inclusion in $\mathcal{B}$ (not the set-theoretical inclusion between the cosets $\mathcal{E}(\alpha)$ and $\mathcal{E}(\beta)$).

[4]) The proof of (iii) is analogous to that of [TG] (vii).

where $\varphi_n^m(p_1,\dots,p_m)=[\mathcal{E}(F_n^m(x_{p_1},\dots,x_{p_m}))]$ [5]. Clearly the two-valued function φ_n^m may be interpreted as the characteristic function of a set K_n^m, the elements of which are m-element sequences of positive integers.

This means (see the proof of [TG] (i)) that the sequences

$$\{n\}_{n=1,2,\dots} \quad \text{and} \quad \{K_n^m\}_{m,n=1,2,\dots}$$

satisfy simultaneously all the formulae $\alpha \in A$.

[5]) According to [TG], § 5, the symbol $[a]$ denotes the element (in $\mathfrak{B}/\mathfrak{p}$) determined by an element $a \in \mathfrak{B}$.

Państwowy Instytut Matematyczny.

On the application of Tychonoff's theorem in mathematical proofs.

By

J. Łoś and C. Ryll-Nardzewski (Wrocław).

In this paper [1]) we shall give two theorems: one, effective [2]), on finite properties (Th. 1), the other non effective, on consistent choices (Th. 2). Both those theorems are closely connected with Tychonoff's theorem on the product of bicompact spaces [3]), according to which [4])

(T) *the product of as many bicompact spaces is bicompact in product topology.*

This theorem is often used in existence proofs; the theorems presented here are to some extent a scheme of such proofs.

The proofs given in this paper are effective, with exception of two cases in which the theorem of Tychonoff (T) is used. All proofs of Tychonoff's theorem are non effective, i. e. all in its proofs the principle of choice is used. In another paper we shall prove that no effective proof of this theorem exists [5]).

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on October 27, 1950.

[2]) Effective, i. e. no transfinite methods are used in the proof of this theorem.

[3]) By a *topological space* we understand in this paper a Hausdorff's space.

[4]) Cf. e. g. N. Bourbaki [2], p. 63.

[5]) In the previous volume of this Journal J. L. Kelley [3] has shown that the theorem of Tychonoff for Kuratowski's closure spaces implies the axiom of choice. In the proof of this implication unfortunately there is a mistake (the set Z_a considered on p. 76 is open and non closed), which however can be easily corrected.