Boolean algebra. But if we desire to have a non-distributive (though, of course, modular) example of the kind just mentioned, then it is sufficient to take the (simplest) five-element modular and non-distributive lattice instead of the above four-element Boolean algebra, and to perform further 4 symmetric adjunctions. Unlike these almost trivial answers to the first two questions of the problem Nr 75 of (L), an answer to the third question seems to be more difficult. This is closely related to problem Nr 56 of (L) of finding a non-Desarguesian plane projective geometry which admits orthocomplements. I could not find any satisfactory answer to this third question.

Measures in Almost Independent Fields.

By

Edward Marczewski (Wroclaw).

Introduction. This paper deals with the existence of common extensions of measures defined in given fields of sets to the smallest field containing all the given fields. This problem taken in such a general way might have more than one solution. But we propose a restrictive condition viz., that the extension in question be multiplicative, i.e. that all given fields be stochastically independent with respect to it. Then, as it is easy to prove, if the required extension exists, it is unique (Lemma 3).

Section 1 contains all the definitions and a few examples. Section 2 contains the complete solution of the problem in the case of the finite additivity of fields and measures: the almost-independence is a necessary and sufficient condition for the existence of the multiplicative extension (Theorem 1).

In Section 3 the same problem is considered for the denumerable additivity. Banach [2] has proved that the \( \sigma \)-independence is here a sufficient condition but it is easy to see that it is not a necessary condition. On the other hand Helson [1] has established that, even in the case of two fields, the almost-independence (which is in this case identical with the almost-\( \sigma \)-independence) is not a sufficient condition. An analogous necessary and sufficient condition is not known so far. In this paper only a sufficient condition is given (Theorem III and IV), formulated thanks to some ideas of Kakutani [1]: it is namely the almost-\( \sigma \)-independence under the additional condition (obviously a very restrictive one)

1) Stochastic extension in the terminology of Helson [4] and Sikorski [10].

2) I proved this theorem and I presented it to the Polish Mathematical Society, Warsaw Section, in spring 1939; cf. Marczewski [8], p. 127, Théorème III, and Banach [1], pp. 159-160.

3) Marczewski [8], p. 130.
that each of the given fields — except at most one — contains a finite number of sets $^4$.

It is interesting to note that Theorems I-IV continue to hold for Boolean algebras, whereas Banach's theorem, as recently proved by Sikorski [11], does not.

Theorem IV can be applied to problems of extensions of Lebesgue measure. Invariant (or, more exactly, invariant with respect to the congruence) $\sigma$-extensions of Lebesgue measure are well-known (see e.g. Marczewski [6]). The existence of a non-separable $\sigma$-extension of this measure has been proved by Kakutani [6]. In 1935 I proposed the problem of the existence of $\sigma$-extension of Lebesgue measure which is simultaneously invariant and non-separable ([6], p. 558). In 1950, using my Theorem IV and some ideas of Banach [1], Sierpiński [9] and Kakutani [5] I proved, with the aid of the continuum hypothesis, that such a $\sigma$-extension exists $^5$; this proof is not given here, because quite recently a stronger result has been published by Kakutani, Kodaira, and Oxtoby $^6$.

The following related problem of Sierpiński is not yet solved ([6], p. 558): Does there exist for each invariant $\sigma$-extension of Lebesgue measure its proper invariant $\sigma$-extension?

1. Fields and their independence. Measures and their extensions. By field $\mathfrak{F}$ of subsets of a fixed set $X$ we understand any class $\mathfrak{F}$ of subsets of $X$ which is additive (i.e. such that if $E_i, E_j \in \mathfrak{F}$, then $E_i \cap E_j \in \mathfrak{F}$) and complementative (i.e. such that if $E \in \mathfrak{F}$, then $X - E \in \mathfrak{F}$). Any $\sigma$-additive field $\mathfrak{K}$ (i.e. such that if $E_j \in \mathfrak{K}$ for $j = 1, 2, \ldots$, then $E = \bigcup_{j=1}^{\infty} E_j \in \mathfrak{K}$) is called a $\sigma$-field. The smallest field and the smallest field containing a class $\mathcal{Q}$ of subsets of $X$ will be denoted by $\mathcal{Q}$ and $\mathcal{Q}$ respectively. A trivial but important example of a $\sigma$-field is the four-element field $\mathfrak{K} = \{X, \emptyset, E, X - E\}$. For each set $E \in \mathfrak{F}$ we put $E^0 = X - E$ and $E^0 = E$. We denote by $A - B$ the symmetric difference of $A$ and $B$.

**Lemma 1.** For each finite sequence $E_0, E_1, \ldots, E_n$ of sets belonging to a field $\mathfrak{K}$, there exists a sequence $F_1, F_2, \ldots, F_n$ of disjoint sets belonging to $\mathfrak{K}$, such that each set $F_k$ $(k = 1, 2, \ldots, n)$ is the sum of some sets $E_i$.

In fact, all atoms of $E_0, E_1, \ldots, E_n$ (i.e. the sets of the form $E_1^0 E_2^0 \ldots E_n^0$, where $i_k = 0, 1$) form the required sequence $(F_k)$.

**Lemma 2.** If $E_0, E_1, \ldots, E_n$ is a finite sequence of sets of subsets of $X$, then each set $E$ belonging to $\mathfrak{K} = \{X, E_0, E_1, \ldots, E_n\}$ has the form

$$E = \bigcup_{i=1}^{n} E_{i},$$

where $E_i \in \mathfrak{K}$ for $i = 1, 2, \ldots, n$.

In fact, the class of all sets of the form (*) is a field containing all the fields $E_0, E_1, \ldots, E_n$.

**Lemma 3.** We can suppose that the sets $\bigcup_{i=1}^{n} E_i$ in the formula (*) are disjoint.

To prove this we apply Lemma 1 to each sequence $E_0, E_1, \ldots, E_n$ $(i = 1, 2, \ldots)$ in the formula (*) and we transform the obtained expression so as to give it the usual "polynomial" form.

A non-negative set function $\mu$ defined in a field $\mathfrak{M}$ of subsets of $X$ is called a measure, if $\mu(\emptyset) = 1$ and if it is additive (i.e. if $\mu(E_1 + E_2) = \mu(E_1) + \mu(E_2)$ whenever $E_1 \cap E_2 = \emptyset$). A measure $\mu$ is called a $\sigma$-measure if it is defined in a $\sigma$-field and if it is $\sigma$-additive (i.e. if $\mu(E_1 + E_2 + \ldots) = \mu(E_1) + \mu(E_2) + \ldots$ whenever the sets $E_i$ are disjoint). By an extension [or $\sigma$-extension] of a measure $\mu$ defined in a field $\mathfrak{M}$ we understand any measure $[\sigma$-measure $]$ $\nu$ defined on a field containing $\mathfrak{M}$ and such that $\nu(\emptyset) = \mu(\emptyset)$ for $\emptyset \in \mathfrak{M}$.

In the sequel we consider families $(E_1), (M_0), (\mu_0)$ of sets, of fields, and of measures. The index $i$ runs over the arbitrary set $T$.

The sets belonging to a family $(E_i)$ (where $E_i \in \mathfrak{K}$ are called independent [or $\sigma$-independent], if for each finite sequence [finite or infinite sequence] of different indices $t_i \in T$ and for each sequence $(i_k)$ of numbers $0$ and $1$ we have

$$E_{i_0}^0 E_{i_1}^0 \ldots E_{i_n}^0 = \emptyset.$$
the Lebesgue exterior measure of \( Z \) and \( I - Z \) is 1. Let us define a measure \( \nu \) in \( \mathcal{N} \) by putting \( \nu(Z) = \nu(I - Z) = \frac{1}{4} \). The \( \sigma \)-fields \( \mathcal{L} \) and \( \mathcal{N} \) are obviously not independent but they are almost independent with respect to the Lebesgue measure and the measure \( \nu \).

3. Let \( I \) denote the unit interval and \( X = I^2 \) the unit square. Let \( \mathcal{M}_I \) and \( \mathcal{M}_I \) denote the class of all sets of the forms \( I \times I \) and \( I \times L \), respectively, where \( L \) is a Lebesgue-measurable subset of \( I \). The \( \sigma \)-fields \( \mathcal{M}_I \) and \( \mathcal{M}_I \) are independent and, stochastically independent with respect to the Lebesgue plane measure \( \mu \). Let us denote by \( \mu_1 \) and \( \mu_2 \) the measure \( \mu \) restricted to the fields \( \mathcal{M}_I \) and \( \mathcal{M}_I \). The set functions \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-measures, and \( \mu \) is their multiplicative \( \sigma \)-extension.

2. Multiplicative extension of measures.

**Theorem 1.** Let \( \{\mu_i\} \) be a family of measures defined respectively in the fields \( \mathcal{M}_I \) of subsets of a set \( X \). There exists a multiplicative extension of the measures \( \mu_i \) if and only if the fields \( \mathcal{M}_I \) are almost independent with respect to \( \mu \).

Let us put \( \mathcal{M} = \bigoplus \mathcal{M}_I \).

**Lemma 4.** If \( \mathcal{M}_I \) are fields of subsets of \( X \), then
\[
\bigoplus_{I \in \mathcal{P}_I} \mathcal{M}_I = \bigoplus_{I \in \mathcal{P}_I} (\mathcal{M}_I + \mathcal{M}_I + \cdots + \mathcal{M}_I),
\]
where \( I \in \mathcal{P}_I \) runs over the set of all finite sequences of elements of \( T \).

To prove this it is sufficient to show that the right-hand side \( R \) of the formula is a field. Obviously the class \( R \) is complementative.

If \( A \) \( (\mathcal{M}_I + \mathcal{M}_I + \cdots + \mathcal{M}_I) \) and \( B \) \( (\mathcal{M}_I + \mathcal{M}_I + \cdots + \mathcal{M}_I) \), then \( A \) and \( B \) also belong to the field
\[
\bigoplus_{I \in \mathcal{P}_I} (\mathcal{M}_I + \mathcal{M}_I + \cdots + \mathcal{M}_I),
\]
and consequently \( A + B \in R \). Thus the class \( R \) is additive, q. e. d.

**Lemma 5.** Let \( \mu_1, \mu_2, \cdots, \mu_{n+1} \) be measures in the fields \( \mathcal{M}_I, \mathcal{M}_I, \cdots, \mathcal{M}_I \), of subsets of \( X \). If these fields are almost independent with respect to the considered measure, and if the measure \( \mu \) is a common extension of \( \mu_1, \mu_2, \cdots, \mu_{n+1} \) to the field \( \mathcal{M} = (\mathcal{M}_I + \mathcal{M}_I + \cdots + \mathcal{M}_I) \), then \( \mathcal{M} \) and \( \mathcal{M} \) are almost independent with respect to \( \mu \) and \( \mu_{n+1} \).
Measures in Almost Independent Fields

whence we obtain, in virtue of (3):

\[ \sum_{0 \leq j \leq k} \mu(\tilde{F}_i) = \sum_{0 \leq j \leq k} \mu(\tilde{B}_j). \]

Consequently, since the sets \( G_j \) and \( H_j \) are disjoint whenever \( (i,j) \neq (k,l) \), we see that if \( (i,j) \in \mathbb{R} \times \mathbb{R}^* \), then \( G_j H_j = \emptyset \). Further, since \( M \) and \( N \) are almost independent, \( \mu(G_j) = 0 \) or \( \nu(H_j) = 0 \). Therefore

\[ \sum_{0 \leq j \leq k} \mu(G_j) \cdot \nu(H_j) = \sum_{0 \leq j \leq k} \mu(G_j) \cdot \nu(H_j). \]

Since the terms of the sums (5) are disjoint, we have

\[ \mu(G_i \setminus A) = \sum_{i \in \tilde{A}} \mu(G_i) \quad \text{and} \quad \nu(H_i \setminus B) = \sum_{i \in \tilde{B}} \nu(H_i). \]

If a pair of indices \((i,j)\) recurs in the last sum, i.e. if for some \( i, j, k, l \) (where \( k+l \)) we have \( i \in \tilde{F}_k \tilde{F}_l \) and \( j \in \tilde{Q}_k \tilde{Q}_l \), then by (5)

\[ G_j H_j \cap A \setminus B \quad \text{and} \quad G_j H_j \cap A \setminus B. \]

Since \( A_k B_k \) and \( A_k B_k \) are disjoint, \( G_j H_j = \emptyset \), whence by the almost-independence of \( M \) and \( N \), we have \( \mu(G_j) \cdot \nu(H_j) = 0 \).

Consequently, in the right-hand side of (8) only those products \( \mu(G_i) \cdot \nu(H_j) \) recur which are equal to zero; therefore we may write

\[ \sum_{i \leq j \leq k} \mu(\tilde{A}_i) \cdot \nu(\tilde{B}_j) = \sum_{0 \leq j \leq k} \mu(\tilde{A}_i) \cdot \nu(\tilde{B}_j). \]

and analogically

\[ \sum_{i \leq j \leq k} \nu(\tilde{A}_i) \cdot \mu(\tilde{B}_j) = \sum_{0 \leq j \leq k} \nu(\tilde{A}_i) \cdot \mu(\tilde{B}_j). \]

The formulae (7), (9), and (9*), give (4), q.e.d.

Proof of Theorem 1. 1° First we shall deduce the existence of a multiplicative extension of \( \mu_i \) from the almost-independence in the case of a finite number of fields and measures. By Lemma 5 the proof reduces to the case of two measures: \( \mu_1 \) and \( \mu_2 \), defined on two fields: \( M_1 \) and \( M_2 \). By Lemma 2 each set \( \tilde{F} \in \mathcal{M} = (M_1 \cup M_2) \) is of the form

\[ E = \sum_{i \leq j \leq k} A_i B_j, \]

where \( (A_i B_j : A_j B_j) = 0 \) for \( i \neq j \).
We put
\[ \mu(E) = \sum_{k=1}^{n} \mu_k(A_k) \mu(E_k); \]
it follows from Lemma 6 that this number does not depend on the choice of \(A_k \) and \(E_k \).

It follows directly from this definition that \( \mu \) is a multiplicative extension of \( \mu_k \) and \( \mu_n \) for \( N \).

Now we pass to the case of an infinite set \( T \) of indices. Let us put for each finite sequence \( U=(u_1, u_2, \ldots, u_a) \) of elements of \( T \)
\[ M_U = (M_{u_1} + M_{u_2} + \ldots + M_{u_a})_{\nu}. \]

Since \( U \) is finite, there exists a multiplicative extension \( \nu \) of all \( \mu_j \) \( (j=1, 2, \ldots, n) \) to \( M_U \).

We shall prove that all the measures so defined are compatible, i.e., that if \( E \in M_U \cap M_V \) (where \( U \) and \( V \) are finite sequences of elements of \( T \)), then
\[ \mu(E) = \mu(U) = \mu(V) = \mu(E). \]

Indeed, \( E \in M_U \cup M_V \), and it follows from the uniqueness of the multiplicative extension (Lemma 3) that \( \mu(E) \) coincides with \( \mu_U \) in \( M_U \) and with \( \mu_V \) in \( M_V \), whence the identity (10).

Consequently we may put for each \( E \in M_U \)
\[ \mu(E) = \mu(U). \]

It follows from Lemma 4 that the function \( \mu \) is defined in \( M \). By (11) we have \( \mu(0)=0 \) and \( \mu(X)=1 \).

In order to prove the additivity of \( \mu \), let us remark that for each \( A, B \in M \) there exists by Lemma 4 two finite sequences \( U \) and \( V \) of elements of \( T \), such that \( A \in M_U \) and \( B \in M_U \), and consequently \( A \in M_{U,V} \) and \( B \in M_{U,V} \). Since \( \mu(A) = \mu(U, V)(A) \) and \( \mu(B) = \mu(U, V)(B) \), we have
\[ \mu(A+B) = \mu(U, V)(A) + \mu(U, V)(B) = \mu(A) + \mu(B), \]
whenever \( A \) and \( B \) are disjoint.

It follows easily from the definition (11) that \( \mu \) is a common extension of \( \mu_k \) \( (t \in T) \) and that the fields \( M_t \) are stochastically independent with respect to \( \mu \).

**Theorem II.** Let \( \{M_k\} \) denote a family of fields of subsets of a set \( X \). Then the following statements are equivalent:
(a) The fields \( M_k \) are independent.
(b) There exists a multiplicative extension of \( \mu_k \) defined respectively in \( M_k \).
(c) There exists a common extension of each family of measures \( \mu_k \) defined respectively in \( M_k \).

Proof. Theorem I implies the implication (a) \( \rightarrow \) (b). The implication (b) \( \rightarrow \) (a) is trivial. Finally, in order to prove (c) \( \rightarrow \) (a), let us consider a finite sequence of non void sets \( E_j \in M_j \) \( (j=1, 2, \ldots, n) \), where \( t_j \) is a sequence of different indices. Let \( \{p_j\} \) be a family of points of \( X \) such that \( \int X \) for \( j = 1, 2, \ldots, n \). We define in each field \( M_k \) a measure \( \mu_k \) by putting \( \mu_k(E) = 1 \) or 0 according as \( p_k \) belongs to \( E \) or not. Obviously \( \mu_k(E)=1 \) and consequently \( \mu_k(E_i)=1 \) for \( j=1, 2, \ldots, n \), whence \( E_1 E_2 \cdots E_n = 0 \), q. e. d.

**3. Multiplicative \( \sigma \)-extensions of \( \sigma \)-measures.** We prove now

**Theorem III.** Let \( \mu_t \) \( (t \in T) \) be a family of \( \sigma \)-measures defined respectively on \( \sigma \)-fields \( M_t \), of subsets of \( X \). Let us suppose all \( M_t \) except \( M_0 \) are finite. If \( M_t \) are almost-\( \sigma \)-independent with respect to \( \mu \), then there exists a multiplicative \( \sigma \)-extension \( \nu \) of all \( \mu_t \).

We denote by \( A^T \), where \( t=1, 2, \ldots, K_T \), the sequence of all non void atoms of the field \( M_t \) \( (t \in T) \).

Let us put \( M_0 = \{m \in M_0 \} \) and \( N_0 = \{m \in M_0 \} \); we have then \( N_0 = M_0 \).

**Lemma 6.** If \( E \in M_0 \), then
\[ E = \sum_{t} A^0_t \quad \mu_t \quad \cdots \quad A^0_{K_T} \quad \mu_t = \nu, \]
where \( B^0_t \in M_0 \) and where \( t = (t_1, t_2, \ldots, t_a) \) runs over all sequences consisting of a numbers \( t_1 \) such that \( t_1 \in (K_T)^{a-1} \).


Fundamenta Math. t. XXXVIII. 15.
This is an easy consequence of Lemma 4 and 2. Using an analogous notation we have

**Lemma 7.** If $E_{k} \subseteq E_{k+1} \subset M$ and

\[
E_{k} = \sum_{i} A_{i}^{(1)} \cdot A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}},
\]

then $E_{k}$ may be represented in the form

\[
E_{k} = \sum_{i} A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}},
\]

where $n_{i} \leq n_{k}$ and $B_{i_k}^{p_{i_k}} \in E_{k}$. By Lemma 6

\[
E_{k} = \sum_{i} A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}} \subset M_{k},
\]

we obtain easily

\[
E_{k} = \sum_{i} A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}} = E_{k+1},
\]

and since the set $A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)}$ is disjoint with $A_{i_k}^{(1)} A_{i_k}^{(2)} \cdot \ldots \cdot A_{i_k}^{(n)}$ whenever $(i_1, i_2, \ldots, i_k) = (i_2, i_2, \ldots, i_n)$, we have

\[
A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}} = A_{i_k}^{(1)} A_{i_k}^{(2)} \cdot \ldots \cdot A_{i_k}^{(n)} B_{i_k}^{p_{i_k}} \cdot \ldots \cdot B_{i_n}^{p_{i_n}}.
\]

Thus, putting

\[
B_{i_k}^{p_{i_k}} = B_{i_k}^{p_{i}} \cdot B_{i_n}^{p_{i_n}},
\]

we obtain

\[
A_{i}^{(1)} A_{i}^{(2)} \cdot \ldots \cdot A_{i}^{(n)} B_{i_k}^{p_{i_k}} = A_{i_k}^{(1)} A_{i_k}^{(2)} \cdot \ldots \cdot A_{i_k}^{(n)} B_{i_k}^{p_{i_k}} \cdot B_{i_k}^{p_{i_k}} = E_{k+1},
\]

which implies the required formula (*)

**Proof of Theorem III.** In view of Theorem I there is a multiplicative extension $\nu$ of $\mu$ to $M$.

On account of the well-known theorem on the $c$-extension of a measure, in order to prove Theorem III it is sufficient to prove that for each sequence $(M_{k})$ such that

\[
E_{k} \subseteq \bigcap_{k \geq 0} E_{k} 
\]

we have

\[
E_{0} \subseteq E_{k} \subseteq \bigcap_{k \geq 0} E_{k} 
\]

and consequently we may apply the following arithmetical proposition which is easy to prove:

(A) If

\[
eq \sum_{j \neq 1} a_{j} b_{j} \quad a_{j} \geq 0, \quad b_{j} \geq 0, \quad \sum_{j \neq 1} a_{j} = 1,
\]

then there is $i_{k}$ such that $c \leq b_{i_{k}}$ and $a_{k} = 0$.

Since $\mu(E_{1}) > \delta$, it follows from (3) and (A) that for each natural $k$ there exists a sequence $(i_{1}, i_{2}, \ldots, i_{k})$ of numbers $i_{k} \in \mathbb{N}$ satisfying the following condition:

\[
\mu(A_{i}^{(1)}) > 0, \quad \mu(A_{i}^{(2)}) > 0, \quad \ldots, \quad \mu(A_{i_k}^{(n)}) > 0, \quad \mu(B_{i_k}^{p_{i_k}} \cdot B_{i_n}^{p_{i_n}}) > \delta.
\]

Obviously if $(i_{1}, i_{2}, \ldots, i_{k})$ satisfies (4), then $(i_{1}, i_{2}, \ldots, i_{k})$ satisfies (4) too.
Consequently it is easy to define by induction an infinite sequence \( i_1, i_2, \ldots \) such that \((i_1, \ldots, i_n)\) satisfies (4) for \( k = 1, 2, \ldots \).

Let us put

\[
B = B_{i_1}^{i_1} \cdots B_{i_n}^{i_n}
\]

Since \( \mu_k \) is a \( \sigma \)-measure in the field \( M_k \), we have \( B \in M_k \) and \( \mu(B) \geq \delta \). Consequently, the almost-independence of \( M_i \) implies

\[
E_1 \cdot E_2 \cdots \cdot B \cdot A_i^1 \cdot A_i^2 \cdots \neq 0.
\]

The relation (1) is thus proved.

**Theorem IV.** Let \( \mu \) be a \( \sigma \)-measure in a \( \sigma \)-field \( M \) of subsets of \( X \), and \( \varphi \) a real set function defined on a family \( F \) of subsets of \( X \), such that always \( 0 \leq \varphi(E) \leq 1 \). If for each sequence of sets \( A_n \in F \), each set \( B \in M \) such that \( \mu(B) > 0 \), and each sequence \( \{i_n\} \) of numbers 0 and 1 we have

\[
\mu(A_1 \cdot A_2 \cdots \cdot A_n \cdot B) = \mu(A_1) \cdot \mu(A_2) \cdot \cdots \cdot \mu(A_n) \cdot \mu(B)
\]

then there is a \( \sigma \)-measure \( \nu \) in \( N = (M + F)_\mu \) which is an extension of \( \mu \) and \( \varphi \), such that

\[
\mu(A_1 \cdot A_2 \cdots \cdot A_n \cdot B) = \nu(A_1) \cdot \nu(A_2) \cdot \cdots \cdot \nu(A_n) \cdot \nu(B)
\]

for each \( A_j \in F \) and each \( B \in M \).

In order to prove this theorem it is sufficient to consider for each set belonging to the family \( E = (A) \) the four-element field \( (X, 0, A, X - A) \) and the measure \( \mu_i \):

\[
\mu_i(X) = 1, \quad \mu_i(0) = 0, \quad \mu_i(A) = \varphi(A), \quad \mu_i(X - A) = 1 - \varphi(A)
\]

and to apply Theorem III.

We do not know a necessary and sufficient condition for the existence of multiplicative \( \sigma \)-extensions, analogous to that contained in Theorem I (cf. Introduction). Instead we may complete the theorem of Banach to the following one, analogous to Theorem II:

Let \( \{M_i\} \) denote a family of \( \sigma \)-fields of subsets of \( X \). Then the following statements are equivalent:

(a) The fields \( M_i \) are \( \sigma \)-independent.

(b) There exists a multiplicative \( \sigma \)-extension of each family \( \mu_i \) of \( \sigma \)-measures defined respectively in \( M_i \).

(c) There exists a common \( \sigma \)-extension of each family \( \mu_i \) of \( \sigma \)-measures defined respectively in \( M_i \).