tinues $\{f_n(\xi)\}_{n<\omega}$, où $f_n(\xi)=\omega$ pour $\xi\leqslant n$ et pour $\xi>\omega$ et $f_n(\xi)=1$ pour $n\leqslant\xi\leqslant\omega$.

Il est à remarquer que pour les fonctions réelles d'une variable réelle on a la proposition suivante:

La limite d'une suite infinie non décroissante d'une variable réelle continues partout du côté gauche est une fonction continue partout du côté gauche.

En effet, soit $f(x)=\lim_{n=\infty}f_n(x)$ la limite d'une suite infinie non décroissante de fonctions non décroissantes d'une variable réelle continues du côté gauche. Soit x_0 un nombre réel quelconque et ε un nombre positif donné. Comme $f(x_0)=\lim_{n=\infty}f_n(x_0)$, il existe un nombre naturel m tel que $f(x_0)-\varepsilon<f_m(x_0)$. On a donc $\eta=f_m(x_0)-$ $-f(x_0)+\varepsilon>0$. La fonction $f_m(x)$ étant continue au point x_0 du côté gauche, il existe un nombre $\delta>0$, tel que $f_m(x)>f_m(x_0)-\eta$, c'est-à-dire $f_m(x)>f(x_0)-\varepsilon$ pour $x_0-\delta<x<x_0$. La suite $\{f_n(x)\}_{n<\omega}$ étant non décroissante, on a d'après $\lim_{n=\infty}f_n(x)=f(x)$, $f(x)\geqslant f_m(x)$, donc $f(x)>f(x_0)-\varepsilon$ pour $x_0-\delta<x<x_0$ et, la fonction $f(x)$ étant non décroissante (en tant que limite d'une suite infinie de fonctions non décroissantes) on a $f(x_0)\geqslant f(x)$ pour $x<x_0$. On a ainsi

$$f(x_0)-\varepsilon<f(x)\leqslant f(x_0) \quad \text{pour} \quad x_0-\delta<x<x_0,$$

ce qui prouve que la fonction $f(x)$ est continue pour $x=x_0$ du côté gauche.

Some Remarks on Automorphisms of Boolean Algebras [1]).

By

Ladislav Rieger (Praha).

The main subject of the present paper is the construction of an algebra $\mathfrak{B}$ admitting no proper homomorphic mapping onto itself [2]).

Especially, $\mathfrak{B}$ has no proper automorphism, i. e. we get a negative solution of the known problem (see (L), problem Nr 74, p. 162 and the problem listed at the end of the first edition (1939) of this book) as to whether each algebra must have a proper automorphism[3]).

The elementary construction of $\mathfrak{B}$ is essentially a topological one, i. e. we solve an equivalent topological problem in disproving the known hypothesis (see e. g. (L), p. 174) that every zero-dimensional bicompact space should admit some proper homeomorphic transformation onto a suitable subspace of it. Actually, we have an ordered zero-dimensional bicompact space without such homeomorphic mappings.

A remark is added concerning the construction of algebras with rather singular automorphism-groups, which may be of some interest from the point of view of abstract ergodic theories [4]). Some consideration of a part of problem Nr 75 of (L) (of whether a certain dual-automorphism-property is typical in Boolean algebras) concludes the paper.

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[1]) For basic notions of the theory of Boolean algebras (in the sequel, the attribute „Boolean" will often be omitted) see G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. XXV, Sec. Ed. (1948), quoted as (L).

[2]) The result was communicated by the author at the session of the Polish Mathematical Society in Warsaw on January 26-th 1951.

[3]) Recently M. Katětov (Praha) has given an elegant solution in which he uses the theory of Čech bicompactification. Katětov's result will be published in Coll. Math. (1951) and has priority.

[4]) Cf. P. R. Halmos, Trans. Am. Math. Soc. **55** (1944), p. 1-18.

Theorem 1. There exists a zero-dimensional bicompact ordered (consequently hereditarily normal) space Q^* without proper homeomorphic transformations onto any subspace Q_1 of Q^*.

Proof. Let P_1 be the set of pairs (α,n) such that

(i) $n=0,1,2,\ldots,$ (ii) α are ordinals with $1\leqslant\alpha\leqslant\omega_n$,

where ω_n is the initial ordinal of the $(n+1)$-st Cantor class (i. e. card $(\omega_n)=\aleph_n$).

P_1 is well-ordered in the following (lexicographical) sense:

$$(\alpha,n)<(\alpha',n')$$

if and only if either $n<n'$ or $n=n'$, $\alpha<\alpha'$.

Let the well-ordered set P_n be defined for $n\geqslant 1$ so, that $P_{n-1}\subset P_n$.

We introduce the following notations:

$x,y,z,\ldots$ are arbitrary elements of any P_k $(k=1,2,\ldots)$,

(small Greek letters denote ordinals), $\mathrm{ord}_n(x)$ is the ordinal of the place of the element $x\in P_n$ in the well-ordering of P_n. ${}^n\omega(x)=\omega_\xi$ is the initial (i. e. the lowest) ordinal of the $(\xi+1)$-th Cantor class of ordinals (i. e. card $(\omega_\xi)=\aleph_\xi$), where $x\in P_n$ and

$$\xi=\sup_{y\in P_{n-1}}(\mathrm{ord}_{n-1}(y))+\mathrm{ord}_n(x)$$

(the first summand being 0 for $n=1$).

Now let us form the set

$$(1,x),(2,x),\ldots,(\alpha,x),\ldots,\big({}^n\omega(x),x\big)$$

of all the ordered pairs whose first members are ordinals with $1\leqslant\alpha\leqslant{}^n\omega(x)$ and whose second members are $x\in P_n$ with non-limit $\mathrm{ord}_n(x)\neq 1$.

Then we get $P_{n+1}\supset P_n$ by adjoining this set of pairs to P_n.

Determine the well-ordering in P_{n+1} as follows:

a) The ordering relation between elements of P_n in P_{n+1} remains the same as before.

b) $(\beta,x)<(\gamma,y)$ if either $x<y$ in P_n or $x=y$ yet $\beta<\gamma\leqslant{}^n\omega(x)$.

c) If $x\in P_n$ and $z=(\beta,y)\in P_{n+1}-P_n$, then $x<(\beta,y)$ whenever $x<y$ in P_n, and $x>(\beta,y)$ whenever $x\geqslant y$ in P_n.

By induction, the P_n are well-ordered, as it is easy to see.

Next, put $P=\sum_{n=1}^{\infty}P_n$.

Then P clearly is a densely ordered set if we define $x<y$ in P_n whenever $x<y$ in the first P_k with $x\in P_k$, $y\in P_k$.

Consider the subset Q of P consisting of all the elements $x\in P$ of the form $x=\big({}^n\omega(y),y\big)$ (with a suitable $y\in P_{n+1}$).

Clearly Q is a densely ordered set. Between any two different elements x and y of Q there is an element $z\in P-Q$, whence the gaps of Q lie densely in Q.

Consider Q as an ordered space. Then, as can easily be seen, any two different points[5]) $u=\big({}^n\omega(x),x\big)$ and $v=\big({}^m\omega(y),y\big)$ have different topological characters, i. e. the cardinals $\aleph_\xi$ and $\aleph_\eta$, where ${}^n\omega(x)=\omega_\xi$, ${}^m\omega(y)=\omega_\eta$.

Finally, consider the gaps $\Gamma=(Q_1,Q_2)$ of Q (where Q_1,Q_2 are the lower and the upper classes of the corresponding proper Dedekind cut).

To any $\Gamma=(Q_1,Q_2)$ let correspond the following ideal points:

(1) To the gap $\Gamma=(0,Q)$ there corresponds only one ideal point, i. e. the left end-point, say 0.

(2) To the gap $\Gamma=(Q,0)$ there corresponds only one ideal point, i. e. the right end-point, say ∞.

(3) To any $\Gamma=(Q_1,Q_2)$ with $Q_1\neq 0\neq Q_2$ there correspond exactly two ideal points, i. e. the left ideal point, say Γ_l, and the right ideal point, say Γ_r.

Adjoining all these ideal points to the real points of Q we get the set Q^*.

Determine the ordering-relation in Q^* as follows:

(a) The ordering between real points remains the same in Q^* as in Q.

(b) If Λ, Λ', are two ideal points corresponding to the gaps Γ,Γ', then $\Lambda<\Lambda'$ if either $\Gamma<\Gamma'$ (in the obvious sense of the ordering of the gaps of Q) or $\Gamma=\Gamma'$, $\Lambda=\Gamma_l$, $\Lambda'=\Gamma_r$.

(c) If x is a real point and Λ an ideal point corresponding to the gap $\Gamma=(Q_1,Q_2)$, then $x<\Lambda$ if $x\in Q_1$, and $x>\Lambda$ if $x\in Q_2$.

[5]) For basic notions of General Topology see e. g. Kuratowski, *Topologie I*. The topological character of a point x of the topological space T is known to be the lowest power of a system $\mathfrak{S}_x$ of open sets $G\in\mathfrak{S}_x$ containing x, so that any open U containing x contains some $G\in\mathfrak{S}_x$. Clearly, the topological character of a point is a topological invariant (under homeomorphic transformations).

It is easily seen that Q^* has no gaps (and has the lowest and the highest element) in the ordering just defined. Hence the ordered space Q^* is bicompact [6]) and zero-dimensional as follows from its construction.

We also easily see that Q is dense in Q^* and that the topological characters of the real points $x \in Q$ remain the same in Q^* as in Q (i. e. different for two different real points). (Note that the open-and-closed intervals of the forms

$$\underset{t \in Q^*}{E}(\Gamma_l < t < \Gamma_r') = \underset{t \in Q^*}{E}(\Gamma_r \leqslant t \leqslant \Gamma_l') \quad (\text{where } \Gamma < \Gamma')$$

or

$$\underset{t \in Q^*}{E}(\Gamma_l < t) = \underset{t \in Q^*}{E}(\Gamma_r \leqslant t) \quad \text{or} \quad \underset{t \in Q^*}{E}(t < \Gamma_r) = \underset{t \in Q^*}{E}(t \leqslant \Gamma_l) \quad \ldots(\neq)$$

constitute an open basis of Q^*).

Now, the following statement (A) follows almost immediately from the construction of Q^* (ω_ξ denotes the initial ordinal of the $(\xi+1)$-th Cantor class):

(A) A point $x \in Q^*$ is real if and only if

(i) there exists such a countable sequence u_n that

$$\lim_{n<\omega_0} u_n = y, \quad u_n \neq y, \quad u_n \in Q^*$$

and simultaneously

(ii) there exists such a transfinite sequence v_β $(1 \leqslant \beta < \omega_\xi, \xi > 0)$ that

$$\lim_{\beta<\omega_\xi} v_\beta = y, \quad v_\beta \neq y, \quad v_\beta \in Q^*.$$

The cardinal $\aleph_\xi$ is then the topological character of x; thus the ordinals ω_ξ are different for different real points.

Now let us return to the proof [7]) that any homeomorphic transformation Φ of Q^* onto its subspace $\Phi(Q^*)$ is an identity mapping.

Let $x \in Q^*$ be a real point. Then $x = \lim_{n<\omega_0} a_n = \lim_{\beta<\omega_\xi} b_\beta$, $a_n \neq x \neq b_\beta$, in the sense of the statement (A). Therefore

$$y = \Phi(x) = \lim_{n<\omega_0} \Phi(a_n) = \lim_{\beta<\omega_\xi} \Phi(b_\beta), \quad \Phi(a_n) \neq y \neq \Phi(b_\beta).$$

[6]) Cf. Haar-König, Crelle Jour. **139** (1910), p. 16-28, or (L), p. 41.

[7]) I owe to R. Sikorski the present simplification of this proof.

Since the convergence in the subspace $\Phi(Q^*)$ obviously implies the convergence in Q^*, the statement (A) requires $x = y = \Phi(x)$.

But since the real points $x \in Q^*$ form a dense subset of Q^*, we get $\Phi(z) = z$ for each point $z \in Q^*$, q. e. d.

Theorem 2. *There exists a Boolean algebra $\mathfrak{B}$ without proper homomorphic mappings onto itself.*

Proof. Denote by $\mathfrak{B}$ the set-field (i. e. the algebra) of all the open-and-closed subsets of the space Q^* of theorem 1. (This algebra is generated by the open-and-closed segments ($\neq$) below).

Then, as it is well known (cf. (L), p. 174, Ex. 3), $\mathfrak{B}$ cannot have proper automorphisms (the automorphism-group of $\mathfrak{B}$ and the homeomorphism-group of Q^* are essentially the same).

Suppose there were a homomorphic mapping Φ of $\mathfrak{B}$ onto $\mathfrak{B}$ itself. Then (by the well known first lemma on isomorphism for algebras) we would get the isomorphism $\mathfrak{B}/\mathfrak{J} \cong \mathfrak{B}$, where $\mathfrak{J}$ is the ideal of all elements of $\mathfrak{B}$ mapped onto the zero of $\mathfrak{B}$.

From M. H. Stone's representation-theory for Boolean algebras [8]) we know that the ideal $\mathfrak{J}$ is topologically characterized as the set of all the open-and-closed subsets of Q^* contained in a certain open set $Q_\mathfrak{J} \subset Q^*$ ($Q_\mathfrak{J}$ being uniquely determined by $\mathfrak{J}$). Moreover, it is well known and easily seen that the above-mentioned isomorphism $\mathfrak{B} \cong \mathfrak{B}/\mathfrak{J}$ is topologically a homeomorphism of the space Q^* with its subspace $Q^* - Q_\mathfrak{J}$. Hence our theorem follows from the preceding one.

Remarks on theorem 2. The power of the space Q can easily be computed: it is the cardinal $\mathfrak{m} = \aleph_\omega + \aleph_{\omega_\omega} + \aleph_{\omega_{\omega_\omega}} + \ldots$

Hence the space Q^* and the algebra $\mathfrak{B}$ both have the unsatisfactorily high power $\mathfrak{n}$, where $\mathfrak{m} \leqslant \mathfrak{n} \leqslant 2^\mathfrak{m}$. This is, of course, a disadvantage of our elementary construction. The above-mentioned result of Katětov is much better in this sense, since his resulting algebra has the power of the continuum. On the other hand, more important questions, e. g. as to whether there exists complete, or, at least, a σ-complete algebra without proper automorphisms, seem to require constructions of spaces of a rather high power, similar to the present construction.

Let us return to the few remarks concerning the construction of algebras with rather singular automorphism-groups.

[8]) Cf. M. H. Stone, Trans. Amer. Math. Soc. **40** (1936), p. 37-111 and ibid. **41** (1937), pp. 375-481, or (L), p. 174.

There are two basic methods for this purpose:

(1) We form the space

$$Q' = Q_1^* + Q_2^* + \dots + Q_n^*$$

as a set-sum of disjoint spaces homeomorphic with the space Q^* of theorem 1. Obviously Q' is a zero-dimensional bicompact space and the summands Q_i^* are open-and-closed in Q'. The group $\mathfrak{G}$ of homeomorphic transformations of Q' onto itself, i. e. the automorphism-group $\mathfrak{G}$ of the algebra $\mathfrak{B}'$ of all open-and-closed subsets of Q', has the following properties:

(a) $\mathfrak{G}$ is an infinite group each element of which has a finite order not exceeding $n!$ (Remember that there are at most n different points in the sequence $t, \Phi(t), \Phi^2(t) \dots$ for any $\Phi \in \mathfrak{G}$).

(b) Let k be any positive integer. Then the direct product of k symmetric groups $\mathfrak{S}_{n,i}$ of degree n $(i=1,2,\dots,k)$ is contained as a subgroup in $\mathfrak{G}$. Indeed, let us decompose each Q_j^* in the same manner in a set-sum $\sum_{i=1}^{n} Q_{ij}^* = Q_j^*$ of non-void disjoint open-and-closed summands (i. e. the Q_{ij}^* with fixed i are mutually homeomorphic). Then the $\Phi \in \mathfrak{S}_{n,i}$ are the homeomorphic transformations of Q' onto itself which permute all the summands Q_{ij}^* (for fixed i and $j=1,2,\dots,n$) in an arbitrary way without changing the points of the rest of the space Q'.

(c) Especially put $n=2$. Then the elements (different from the identity-mapping) of $\mathfrak{G}$ obviously are all of the order 2, whence $\mathfrak{G}$ is an infinite Abelian group (for $(\Phi_1\Phi_2)^{-1} = \Phi_1\Phi_2 = \Phi_2\Phi_1$).

(1′) The above elementary construction-method of (1) can be considerably enriched by admitting that some of the summands Q_j^* have non-void open-and-closed intersections.

(2) The second basic method of constructing Boolean algebras for the given purpose is that of forming direct products. Especially, we easily see that to any power $\mathfrak{m} \neq 0$ there are various non-isomorphic types of algebras with the same symmetric group $\mathfrak{S}_{\mathfrak{m}} = \mathfrak{G}$ (of all permutations of $\mathfrak{m}$ elements) as the automorphism-group [9]).

So e. g. the finite algebra of all subsets of some aggregate of n elements $(n=1,2,3,\dots)$ has the same symmetric automorphism-group $\mathfrak{S}_n$ as the algebra $\mathfrak{B}_1 \times \mathfrak{B}_2 \times \dots \times \mathfrak{B}_n$, where $\mathfrak{B}_i \cong \mathfrak{B}$ and $\mathfrak{B}$ is the algebra of Theorem 2.

[9]) In the case of an infinite $\mathfrak{m}$ we also can use the known fact (see e. g. (L), p. 45) that the Cartesian product of bicompact spaces is bicompact too.

Combining (1′) with (2) we get further constructive possibilities. We do not enter here into a detailed and systematical discussion of these possibilities.

Let us conclude with a small remark on problem Nr 75, p. 162 of (L). *Verbatim*, this problem (ascribed to M. Ward) is as follows:

Does there exist a (finite) lattice, not a Boolean algebra, which has a dual automorphism σ of period 2, permutable with every lattice automorphism? Must σ be unique? What about non-Desarguesian projective geometries?

The first question can be answered in the affirmative, by giving trivial example of a finite or of a suitable complete infinite chain, that is by complete distributive lattices. In the finite case $L = 1 < 2 < \dots < n$ we put, of course, $\sigma(k) = n - k$, whereas evidently there is only the identical automorphism. In the infinite case, put e. g. $L' = 1 < 2 < \dots < +\infty = -\infty < \dots -2 < -1$, as perhaps the simplest example of a complete distributive lattice of the desired kind, and set $\sigma(k) = -k$ for $k = \pm 1, \pm 2, \dots, \pm\infty$. By an easy inductive argument, it can be proved that also L' has only identical automorphism. In both these cases, of course, σ was unique.

As for the second question, note that any example of a dual automorphism σ desired in the first question furnishes at once a negative answer to the second one, whenever the lattice in question has an automorphism α of period 2 which lies in the centre of its automorphism group. Indeed, in this case each transformation $\sigma' = \sigma\alpha = \alpha\sigma$ is a further dual automorphism of the period 2, for

$$\sigma\alpha(a \overset{\cup}{\cap} b) = \sigma[\alpha(a) \overset{\cup}{\cap} \alpha(b)] = \sigma\alpha(a) \overset{\cap}{\cup} \sigma\alpha(b) \quad \text{and} \quad (\sigma\alpha)^2 = \sigma\alpha\sigma\alpha = \sigma^2\alpha^2 = 1.$$

Of course, $\sigma' = \sigma\alpha$ is permutable with any automorphism β (of the lattice in question) since (by hypothesis) $\sigma'\beta = (\sigma\alpha)\beta = \sigma\beta\alpha = \beta\sigma\alpha = \beta(\alpha\sigma) = \beta\sigma'$.

In order to have a simple concrete finite example of a (distributive) lattice of this kind, let us adjoin a new unit u and a new zero z to the four-element Boolean algebra $a, a', a\cup a', a\cap a'$, by setting $u \underset{\neq}{\supset} a\cup a'$ and $z \underset{\neq}{\subset} a\cap a'$. The obvious verification of the desired result may be left to the reader. If we wish to have an infinite distributive lattice with more than one dual automorphism of period 2 permutable with any automorphism, then we have simply to adjoin both the chain of positive and of negative integers instead of u and z respectively, to the above four-element

Boolean algebra. But if we desire to have a non-distributive (though, of course, modular) example of the kind just mentioned, then it is sufficient to take the (simplest) five-element modular and non-distributive lattice instead of the above four-element Boolean algebra, and to perform further 4 symmetric adjunctions. Unlike these almost trivial answers to the first two questions of the problem Nr 75 of (L), an answer to the third question seems to be more difficult. This is closely related to problem Nr 56 of (L) of finding a non-Desarguesian plane projective geometry which admits orthocomplements. I could not find any satisfactory answer to this third question.

Measures in Almost Independent Fields.

By

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Introduction. This paper deals with the existence of common extensions of measures defined in given fields of sets to the smallest field containing all the given fields. This problem taken in such a general way might have more than one solution. But we propose a restrictive condition viz., that the extension in question be multiplicative [1]), i. e. that all given fields be stochastically independent with respect to it. Then, as it is easy to prove, if the required extension exists, it is unique (Lemma 3).

Section 1 contains all the definitions and a few examples.

Section 2 contains the complete solution of the problem in the case of the finite additivity of fields and measures: the almost-independence is a necessary and sufficient condition for the existence of the multiplicative extension (Theorem I) [2]).

In Section 3 the same problem is considered for the denumerable additivity. Banach [2] has proved that the σ-independence is here a sufficient condition but it is easy to see that it is not a necessary condition [3]). On the other hand Helson [1] has established that, even in the case of two fields, the almost-independence (which is in this case identical with the almost-σ-independence) is not a sufficient condition. An analogous necessary and sufficient condition is not known so far. In this paper only a sufficient condition is given (Theorem III and IV), formulated thanks to some ideas of Kakutani [1]: it is namely the almost-σ-independence under the additional condition (obviously a very restrictive one)

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[1]) Stochastic extension in the terminology of Helson [4] and Sikorski [10].

[2]) I proved this theorem and I presented it to the Polish Mathematical Society, Warsaw Section, in spring 1939; cf. Marczewski [8], p. 127, Théorème III, and Banach [1], pp. 159-160.

[3]) Marczewski [8], p. 130.